ELECTRICAL POWER TRANSMISSION

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PRINCIPLES OF DESIGN AND PERFORMANCE

BY

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PREFACE

The material of this volume represents the outgrowth of a course in Transmission Line Theory and Design given to senior and graduate students in electrical engineering at the University of Washington during the past ten years. Most of the material comprising the last half of the volume was developed and carefully tested in the field of practical line design. Throughout the book the emphasis is put upon the development, so far as possible, of a complete rational theory. Details of a practical nature, which can be better had from handbooks, from the manufacturer of items of equipment, or may be had only by experience, are for the most part omitted.

The first seven chapters are devoted largely to a discussion of the mathematical tools and the underlying circuit theory of the transmission line, including dielectric, magnetic, and electrical circuits. One brief chapter only is devoted to approximate circuits and their calculation, since adequate discussions of these are usually available in books on alternating-current theory. Throughout the book the emphasis is put upon the exact rather than approximate methods. There are a number of good reasons for this. This method is the only correct one; it is not particularly complex; it is easily applied where calculating machines are available, as is now quite generally the case; it serves as a basis for further study of both communication and power-circuit problems; and, finally, it offers a splendid opportunity to apply a little advanced circuit theory, with respect to which no self-respecting advanced student in electrical engineering can afford to remain in ignorance.

The last seven chapters are devoted to a discussion of the mechanical features of line design, a study of the economics of line design as influenced by both the mechanical and electrical features, and, finally, to the working out of an illustrative example in which the principles and theory previously developed are applied to the complete solution of a hypothetical design problem.

The mechanical features of transmission lines are often touched upon very lightly or are entirely neglected in books on electric

power transmission Yet it is recognized that the design of the span and the proper selection of towers are important features of the problem of transmission line design. While very little detailed discussion is given pertaining to the types of supports available or their proper design from the standpoint of strength, yet three chapters are devoted to the discussion of the theory and application of span design and their bearing upon the selection of the most economical line. A method is outlined and illustrated by means of which the most economical type of structure may be chosen.

The matter of economics as affecting engineering problems is a subject which is rarely mentioned to engineering students, and yet the practicing engineer fully realizes that in nearly every engineering problem it is a prime consideration. This phase of the problem of line design is strongly featured. Kelvin's law of maximum economy, modified to fit the conditions more completely, is made the basis upon which the size of conductor and the line voltage are chosen. By the use of empirical equations developed from appropriate engineering data, a mathematical statement of Kelvin's modified law is evolved, from which reliable design results are obtained. The method has been tested in a number of practical line designs with entire success.

The writer has drawn material and inspiration from many sources. Where the specific source is apparent it is usually acknowledged in a footnote. Much of our knowledge of a given subject, however, represents a gradual accumulation over a long period of time; its subject matter is the contribution of many workers in the field; its origin is obscure or unknown. Such material has been used without specific mention.

Much of the work of the last seven chapters is based on articles which originally appeared either in the *Journal* or the *Transactions* of the A.I.E.E., in bulletins of the University of Washington Engineering Experiment Station, or in the *Electrical World.* These papers, written for the most part by the author and his associates, Professor F. K. Kirsten and Assistant Professor G. S. Smith, are acknowledged in footnotes appearing in the text.

When this volume was originally conceived, it was planned to bring it out under the joint authorship of Professor Kirsten and the writer. This accounts in part for the fact that Chap. XII, particularly, follows Professor Kirsten's work very closely The writer regrets that the press of other work made it impossible for Professor Kirsten to participate as originally planned.

In acknowledging his indebtedness to the colleagues above mentioned, the author desires also to express his gratitude for their interest in the progress of his work, for permission to use their material, and for their helpful suggestions and criticisms. The author is likewise indebted to Mr. Roy E. Lindblom for carefully reading and checking the manuscript.

Credit is due Mr. R. A. Hopkins of Boston for much of the data underlying table 25, and for permission to use figures 81 to 84.

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University of Washington,
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CHAPTER I

COMPLEX QUANTITIES, CIRCULAR AND HYPERBOLIC FUNCTIONS

The Mathematics of Transmission-line Design.—In the electrical calculation of long transmission-line circuits as well as in the preparation of temperature-tension stringing charts required for the proper stringing of the line conductors, a rigorous solution requires the use of certain mathematical relations which are not usually well understood by the average engineer. This lack of understanding arises, perhaps, partly because much of the engineer's everyday work does not usually require mathematics beyond a fair working knowledge of algebra and trigonometry, and partly from the fact that these more powerful mathematical tools have come into more or less general use only during recent years, and are therefore unfamiliar to many engineers who received their college training before that time.

In special cases, as in the calculation of the performance curves for short lines or the stringing charts for short spans, approximate solutions involving better known and somewhat simpler mathematical relations may give results to a degree of accuracy well within the limits of practical requirements. For long lines and long spans, however, the approximate solutions will not give reliable results, and more accurate methods are essential. After all, the only final test of the accuracy of any approximate method is a method which is rigorously correct. For the sake of completeness it seems desirable to include in this chapter a brief discussion of complex numbers, even though these are usually better understood than some of the other mathematical relations included in transmission line problems.

Complex Quantities.—Algebraic numbers such as 2, 4, x, $-m$ are one-dimensional quantities and may be considered as locating

a point in a line with reference to a fixed starting point or origin (Fig. 1). Thus $+4$, $+7$, $+x$ locate points of corresponding numbers of units to the right of zero, while $-m$ indicates a point m units to the left of zero. Ordinary numbers of this type, however, will locate points on a line such as OX only. These numbers are called *real numbers,* and the X-axis is chosen as the axis of reals.

In a similar manner points may be located in the direction taken at right angles to OX by the use of ordinary numbers measured in the Y-direction. Numbers measured in the Y-

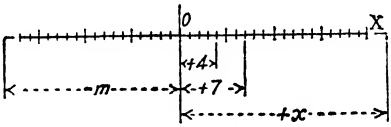

FIG. 1.—Real numbers locate points on a line.

direction are called *quadrature numbers* or *imaginaries,* and are written with the operator j prefixed to the magnitude, as $j4$, $-j5$, $+jy$, etc. Quadrature numbers are also one-dimensional and serve to locate points above or below the X-axis. Thus the Y-axis becomes the quadrature or imaginary axis.

By combining the real and the quadrature quantities into a single quantity as $a + jb$, $4 + j5$, $x + jy$, a *complex quantity* is obtained. A complex quantity is thus a two-dimensional quantity which serves to locate a point anywhere within a given plane. For example, the quantity

$$OP = 3 + j4 \qquad (1)$$

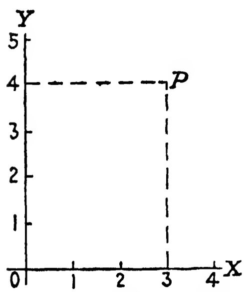

FIG. 2.—Complex numbers locate points in a plane.

locates the point P (Fig. 2) in the plane XY three units to the right of the line OY and four units above the line OX. The point P is then completely located in the plane XY by giving its two coordinates, namely, $(x = 3)$ and $(y = 4)$. Likewise, the vector quantity OP is determined by giving its x and y components ($ox = 3$ and $oy = 4$, respectively), and Eq. (1) may be interpreted to mean that the vector quantity OP is made up of a $+x$ component $= 3$, and a $+y$ component $= 4$. Thus the length of the vector, as well as its position in the plane, is fixed.

The Operator j.—The symbol j is a symbol of operation and, like other symbols of operation, denotes that a certain operation is to be performed on the quantity with which it is associated. Common symbols of operation are log, $\sqrt{\ }$, $\int$, as well as numbers written in the form $2 \cdot 3 \cdot 5 \cdot 1$. Each of these symbols commands us to do a certain thing to the quantity following the symbol. Thus $\sqrt{2}$ is interpreted to mean an operation which, when performed twice, is equivalent to doubling, that is, an operation such that $\sqrt{2} \times \sqrt{2} = 2$. The numbers $2 \cdot 3 \cdot 5 \cdot 1$ indicate that 1 is to be taken five times, the resulting 5 is to be taken three times, making 15, and the resulting 15 taken twice, making 30. Thus each of the numbers becomes a symbol of operation. The numerical operators 2, 3, 5, etc., serve to stretch or increase the magnitude of quantities with which they are associated and are sometimes called *tensors*. The operator -1, on the other hand, does not affect the magnitude of its associated quantity but simply reverses the sense of the magnitude, and this operator is therefore a *reversor*. Consistent with this idea 1×4 means four units to the right of the origin along the axis of reals, and -1×4 means four units to the left of the origin on the same axis. A reversal without change in magnitude may be conceived of as a rotation of a fixed magnitude through 180°, the counter-clockwise direction being taken as positive for convenience. Since such rotation is accomplished by the reversor -1, one may think of the operator $\sqrt{-1}$ as an operator which, when applied twice, will produce a rotation of 180° since $\sqrt{-1} \times \sqrt{-1} = -1$. Thus the expression $\sqrt{-1}$, for which the symbol j is used in engineering literature, may be defined as an operator which turns a fixed magnitude through 90° in a positive or counter-clockwise direction. This symbol is used to denote quadrature quantities, as already explained.

Powers of j—Powers of j higher than the first, which arise from performing the ordinary algebraic operations on vector or complex quantities, may be easily eliminated. This is readily understood from a consideration of the significance attached to the operator. A single application of the operator produces a positive rotation of 90°; two applications produce a rotation of $2 \times 90°$, or 180°, equivalent to a reversal; three applications produce a rotation of 270°, which is equivalent to a negative rotation of 90°; and four applications of the operator produce a complete revolution of the vector, equivalent to a rotation

of zero degrees. Repeated applications of $-j$ produce corresponding rotations in the reverse or negative direction. Accordingly, the higher powers of j carry the following meanings:

$$
\begin{aligned}
+j &= + 90^\circ \text{ rotation} = +j \\
+j^2 &= +180^\circ \text{ rotation} = -1 \\
+j^3 &= +270^\circ \text{ rotation} = -j \\
+j^4 &= +360^\circ \text{ rotation} = +1 \\
+j^5 &= +360^\circ + 90^\circ \text{ rotation} = +j, \text{ etc.}
\end{aligned}
$$

Also,

$$+j \times -j = -j^2 = 0^\circ \text{ rotation} = +1.$$

Rectangular and Polar Forms.—Complex quantities are very conveniently used to represent vectors such as forces, velocities, accelerations, voltages, currents, etc. Maximum and effective values of sinusoidal alternating currents and voltages of the same frequency are conveniently represented by such quantities, and their use in electrical calculations has become quite general. To illustrate, suppose the currents $\dot{I}_1$ and $\dot{I}_2$ of a branched circuit combine to form the current $\dot{I}_3$, and let

$$\dot{I}_1 = 20 + j30 \tag{2}$$
$$\dot{I}_2 = 40 - j10 \tag{3}$$

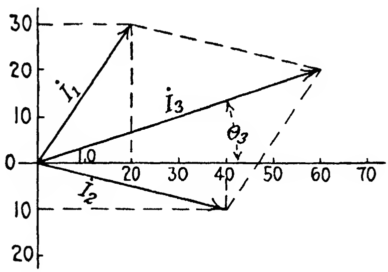

FIG. 3.—Addition of complex or vector numbers.

These currents are graphically represented in Fig. 3. The sum of the two currents is

$$
\begin{aligned}
\dot{I}_3 &= \dot{I}_1 + \dot{I}_2 \\
&= (20 + 40) + j(30 - 10) \\
&= 60 + j20
\end{aligned} \tag{4}
$$

and

$$
\begin{aligned}
\theta_3 &= \tan^{-1}\frac{20}{60} \\
&= \underline{/18^\circ\ 26'}.
\end{aligned}
$$

The vector current $\dot{I}_3$ is thus completely defined by giving the magnitudes of its real and quadrature components and the angle of inclination of the resulting vector as related to a chosen reference axis, usually the X-axis. The form of expression used in Eq. (4) is most commonly used to represent vector quantities in rectangular coordinates. Since the length of $\dot{I}_3$ is

$$\begin{aligned} I_3 &= \sqrt{(60)^2 + (20)^2} \\ &= 63.2 \text{ amp.} \end{aligned}$$

the complex quantity for $\dot{I}_3$ may also be written

$$\dot{I}_3 = I_3 (\cos 18° 26' + j \sin 18° 26') \tag{5}$$
$$= 63.2 \underline{/18° 26'}. \tag{6}$$

(Equations (5) and (6) are polar expressions for the complex quantity $\dot{I}_3$.)

The notation used above is that ordinarily followed; that is, the vector quantity is written with a dotted capital as

$$\dot{I}_3 = I_1 + jI_2$$

or

$$\dot{E} = E_1 + jE_2, \text{ etc.,}$$

while the magnitude or length of the vector is indicated by the capital letter without the dot, thus,

$$I_3 = \sqrt{I_1^2 + I_2^2}$$
$$E = \sqrt{E_1^2 + E_2^2}, \text{ etc.}$$

This method of differentiating between a vector and its length is helpful for beginners, but is an unnecessary refinement for the more advanced reader, since the form of the equation or other information available in the context will clearly indicate whether the vector quantity or its length is meant. Therefore, in the later chapters of this book the dot will not be used.

Equation (5) gives the current $\dot{I}_3$ in terms of its polar components, and this form is regarded as the polar expression for the complex quantity. The symbol $\underline{/\quad\quad}$ is used by some writers to indicate the quadrant in which the angle lies. Thus we have the following symbols:

First quadrant $\underline{/\quad\quad}$
Second quadrant $\underline{\backslash\quad\quad}$
Third quadrant $\overline{/\quad\quad}$
Fourth quadrant $\overline{\backslash\quad\quad}$

Frequently, however, only the symbol for the first quadrant is used, with the angle inserted, as, for example, $\underline{/-\delta}$, $\underline{/26^\circ\ 40'}$, $\underline{/-10^\circ\ 30'}$, etc.

Addition and Subtraction.—The advantage of using complex quantities in rectangular coordinates in the solution of problems involving vectors is, perhaps, most evident when the addition or subtraction of two or more vectors is required. In the illustration (Fig. 3) the resultant current $\dot{I}_3$ is found by adding separately the real and the quadrature components of $\dot{I}_1$ and $\dot{I}_2$.

Or, in general, if

$$\dot{I}_1 = a_1 + jb_1$$
$$\dot{I}_2 = a_2 + jb_2.$$

Adding,

$$\dot{I}_3 = (a_1 + a_2) + j(b_1 + b_2) \tag{7}$$

or

$$I_3 = \sqrt{(a_1 + a_2)^2 + (b_1 + b_2)^2}$$

and

$$\theta_3 = \tan^{-1}\frac{b_1 + b_2}{a_1 + a_2}.$$

The corresponding polar form is

$$\dot{I}_3 = I_3 \underline{/\tan^{-1}\frac{b_1 + b_2}{a_1 + a_2}}.$$

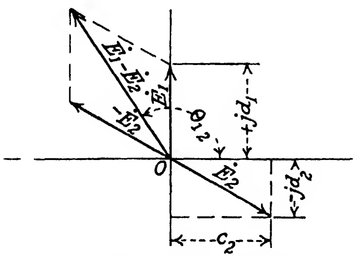

Fig. 4.—Subtraction of complex or vector numbers.

Subtraction of vectors is accomplished by reversing the sign of one of the vectors and adding as before. In Fig. 4 let the voltages, $\dot{E}_1$ and $\dot{E}_2$ be two of the three leg voltages of a three-phase circuit. Then

$$\dot{E}_1 = 0 + jd_1$$
$$\dot{E}_2 = c_2 - jd_2.$$

Subtracting,

$$\dot{E}_{1-2} = \dot{E}_1 - \dot{E}_2 \tag{8}$$
$$= -c_2 + j(d_1 + d_2) \tag{9}$$
$$E_{1-2} = \sqrt{c^2 + (d_1 + d_2)^2}$$
$$\theta_{1-2} = \tan^{-1}\frac{d_1 + d_2}{-c_2}$$

or

$$\dot{E}_{1-2} = E_{1-2} \angle \tan^{-1}\frac{d_1 + d_2}{-c_2} \text{ vector volts.} \tag{10}$$

$\dot{E}_{1-2}$ is the potential difference between the lines 1 and 2. The length of this vector is E_{1-2} and its angle of inclination referred to the X-axis is θ_{1-2}.

Multiplication.—Let it be required to multiply the two vectors A and B of Fig. 5, where

$$\dot{A} = 8 + j4 \qquad A = 8.94 \qquad \theta_A = 26°\,34'$$
$$\dot{B} = 4 + j3 \qquad B = 5.0 \qquad \theta_B = 36°\,52'.$$

Multiplying,

$$\dot{C} = \dot{A}\dot{B} = (32 + j16 + j24 + j^2 12) \tag{11}$$

and since

$$+j^2 = -1$$
$$\dot{C} = 20 + j40$$
$$C = \sqrt{20^2 + 40^2}$$
$$= \sqrt{2000} = 44.7$$

and

$$\dot{C} = 44.7 \angle \tan^{-1}\frac{40}{20}$$
$$= 44.7 \angle 63°\,26'. \tag{12}$$

The magnitude of the resultant vector $\dot{C}$ thus obtained is the same as that obtained by taking the product of the magnitudes of A and B.

For

$$C = A \times B$$
$$= 5 \times 8.94$$
$$= 44.7.$$

It will also be observed that

$$\theta_C = \theta_a + \theta_b$$
$$= 26°\,34' + 36°\,52'$$
$$= 63°\,26'.$$

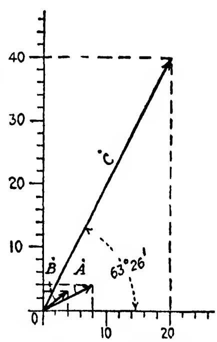

FIG. 5.—Multiplication of complex or vector numbers.

In the above is presented an illustration of the law of multiplication of complex quantities. This law may be stated as follows: *The product of two complex quantities is a complex quantity whose*

length is equal to the product of the lengths of the original quantities, and whose angle of inclination is the sum of the original angles.

The proof of this law is readily established by the use of the polar notation.

For, if

$$\dot{A} = A\ (\cos\theta_1 + j\sin\theta_1) \tag{13}$$

and

$$\dot{B} = B(\cos\theta_2 + j\sin\theta_2) \tag{14}$$

then

$$\begin{aligned}\dot{C} = \dot{A}\dot{B} &= AB(\cos\theta_1\cos\theta_2 + j\sin\theta_1\cos\theta_2 \\ &\qquad + j\cos\theta_1\sin\theta_2 + j^2\sin\theta_1\sin\theta_2) \\ &= AB\ (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + j(\sin\theta_1\cos\theta_2 \\ &\qquad + \cos\theta_1\sin\theta_2) \\ &= AB\,[\cos(\theta_1+\theta_2) + j\sin(\theta_1+\theta_2)]. \end{aligned} \tag{15}$$

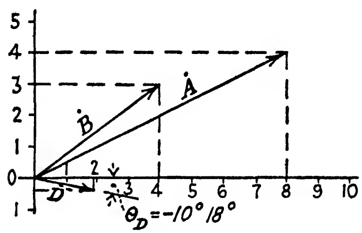

FIG. 6.—Division of complex or vector numbers.

Division.—If it be required to divide $\dot{A}$ by $\dot{B}$ the division (Fig. 6) is stated thus,

$$\dot{D} = \frac{\dot{A}}{\dot{B}}$$
$$= \frac{8 + j4}{4 + j3}.$$

In order to carry out the operation here indicated it is necessary to rationalize the denominator. This is done by multiplying both numerator and denominator of the fraction by the conjugate[1] of the denominator.

Carrying out this operation,

$$\dot{D} = \frac{8 + j4}{4 + j3} \times \frac{4 - j3}{4 - j3} \tag{16}$$
$$= \frac{32 + j16 - j24 - j^2 12}{16 + j12 - j12 - j^2 9}$$

and since $-j^2 = +1$

$$\dot{D} = \frac{44 - j8}{25}$$
$$= 1.76 - j0.32 \tag{17}$$

[1] Two complex quantities are said to be conjugate if they differ only in the sign of the quadrature term. For example, the complex numbers $a + jb$ and $a - jb$ are conjugate.

$$
\begin{aligned}
D &= \sqrt{1.76^2 + 0.32^2} \\
&= 1.79\,\overline{\backslash \tan^{-1}\frac{0.32}{1.76}} \\
&= 1.79\,\overline{\backslash 10^\circ\ 18'}. \qquad (18)
\end{aligned}
$$

The length of the vector resulting from the division is length A $\div$ length B;
for

$$
\begin{aligned}
\frac{A}{B} &= \frac{8.94}{5} \\
&= 1.79
\end{aligned}
$$

and its angle of inclination is

$$
\begin{aligned}
\theta_D &= \theta_A - \theta_B \\
&= (26^\circ\ 34') - (36^\circ\ 52') \\
&= \overline{\backslash 10^\circ\ 18'} \text{ or } \underline{\backslash -(10^\circ\ 18')}.
\end{aligned}
$$

The law for the division of complex quantities here illustrated is: *The division of two complex quantities yields a complex quantity whose magnitude is the quotient derived by dividing the magnitudes of the original quantities, and whose angle of inclination is the angle of the dividend less the angle of the divisor.*

The proof of this law is similar to that for the law of multiplication. Writing the polar equations in the general form;

$$\dot{A} = A\,(\cos\theta_1 + j\sin\theta_1) \qquad (19)$$

$$\dot{B} = B\,(\cos\theta_2 + j\sin\theta_2) \qquad (20)$$

$$
\begin{aligned}
\dot{D} &= \frac{\dot{A}}{\dot{B}} \\
&= \frac{A}{B}\left[\frac{\cos\theta_1 + j\sin\theta_1}{\cos\theta_2 + j\sin\theta_2}\right] \\
&= \frac{A}{B}\left[\frac{\cos\theta_1 + j\sin\theta_1}{\cos\theta_2 + j\sin\theta_2} \times \frac{\cos\theta_2 - j\sin\theta_2}{\cos\theta_2 - j\sin\theta_2}\right] \\
&= \\
&\frac{A}{B}\left[\frac{(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) - j(\cos\theta_1\sin\theta_2 - \sin\theta_1\cos\theta_2)}{\cos^2\theta_2 + \sin^2\theta_2}\right] \\
&= \frac{A}{B}[\cos(\theta_1 - \theta_2) - j\sin(\theta_1 - \theta_2)] \\
&= \frac{A}{B}\underline{/\theta_1 - \theta_2}. \qquad (21)
\end{aligned}
$$

Powers.—The nth power of a complex quantity is found by taking the quantity n times as a factor. Raising a given complex quantity to any power is in reality a special case of multiplication.

Let it be required to find the nth power of the quantity

$$\dot{A} = A\ (\cos\theta_1 + j\sin\theta_1) \tag{22}$$

Then

$$\dot{A}^n = A^n\ (\cos\theta_1 + j\sin\theta_1)^n \tag{23}$$

and, by the law for the multiplication of complex numbers,

$$\dot{A}^n = A^n\ [\cos(\theta_1 + \theta_1 + \theta_1 + \ .\ .\ .\ ) + j\sin(\theta_1 + \theta_1 + \theta_1 + \ .\ .\ .\ )] \tag{24}$$

to n terms

or

$$\begin{aligned}\dot{A}^n &= A^n\ (\cos n\theta_1 + j\sin n\theta_1) \\ &= A^n\ \underline{/n\theta_1}\end{aligned} \tag{25}$$

since

$$\theta_n = n\theta_1 \tag{26}$$

from which it follows that

$$(\cos\theta_1 + j\sin\theta_1)^n = (\cos n\theta_1 + j\sin n\theta_1). \tag{27}$$

This relation is known as De Moivre's theorem.

Thus, *when raising a complex quantity to any power n, the result is a complex quantity whose magnitude is the nth power of the original quantity, and whose angle of inclination is n times the original angle.* To illustrate, let it be required to find the third power of the complex number $5 + j2$. In polar form (Fig. 7),

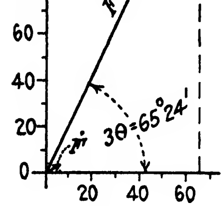

FIG. 7.—Powers of complex or vector numbers.

$$\dot{F} = (5 + j2) = 5.38\ (\cos 21°\ 48' + j\sin 21°\ 48') \tag{28}$$

and

$$\begin{aligned}\dot{F}^3 &= (5 + j2)^3 \\ &= 5.38^3[\cos 3(21°\ 48') + j\sin 3\ (21°\ 48')] \\ &= 5.38^3\ (\cos 65°\ 24' + j\sin 65°\ 24') \\ &= 155.7(0.416 + j0.909) \\ &= 64.8 + j141.5 \\ &= 155.6\underline{/65°\ 24'}.\end{aligned} \tag{29}$$

Roots.—It may be shown that De Moivre's theorem holds for both integral and fractional values of n. Using fractional values of n, this theorem furnishes a simple tool for finding the roots of complex numbers.

If

$$\dot{A} = A(\cos\theta_1 + j\sin\theta_1)$$

then the nth root of this expression is

$$\sqrt[n]{\dot{A}} = A^{\frac{1}{n}}(\cos\theta_1 + j\sin\theta_1)^{\frac{1}{n}} \tag{30}$$

$$= A^{\frac{1}{n}}\left(\cos\frac{\theta_1}{n} + j\sin\frac{\theta_1}{n}\right). \tag{31}$$

Thus, *the nth root of a complex quantity is a complex quantity whose magnitude is the nth root of the original quantity, and whose angle of inclination is* $\frac{1}{n}$*th of the original angle.*

For example, let it be required to evaluate the expression $\sqrt{4 + j3}$.

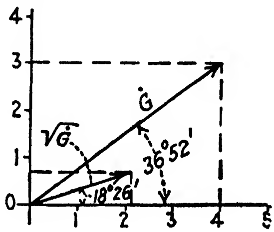

Fig. 8.—Roots of complex or vector numbers.

Let

$$\dot{G} = 4 + j3$$

$$= 5\underline{/\tan^{-1}\tfrac{3}{4}}$$

$$= 5\underline{/36°\ 52'})$$

Then

$$\sqrt{\dot{G}} = \sqrt{5}\,\underline{\Big/\tfrac{1}{2}(36°\ 52')}$$

$$= 2.23\underline{/18°\ 26'}$$

$$= 2.12 + j0.705 \text{ (Fig. 8).}$$

Circular Functions.—In rectangular coordinates, the equation of an ellipse whose center is at the origin of the coordinate axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{32}$$

where a and b represent the major and minor half axes. If, in this equation, a is made equal to b, there results the special case of the ellipse in which the two axes are equal, that is, the circle. Writing $a = b = r$, the equation of the circle is

$$x^2 + y^2 = r^2 \tag{33}$$

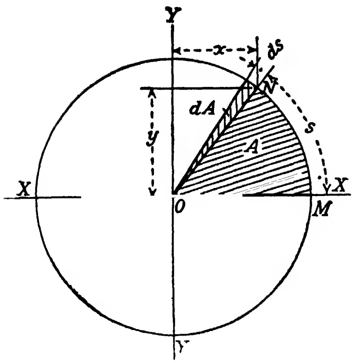

FIG. 9.—The circular angle θ

The radius r is of constant length, and, as it sweeps around from the position OM (Fig. 9), the point N traces out arc S of the circle, while the radius itself sweeps out the area A of a sector. The length ds of the arc S is proportional to the area of the sector dA, and the angle $d\theta$ through which the radius moves may therefore be measured either in terms of the length of arc or in terms of the area of the sector, *i.e.*,

$$\text{length of arc} = ds = rd\theta$$

or

$$d\theta = \frac{ds}{r}$$

and

$$\theta = \frac{1}{r}\int_0^S ds = \frac{s}{r}. \tag{34}$$

If the radius be chosen as the unit of measure, the arc and the angle become numerically equal, or

$$d\theta = ds$$

and

$$\begin{aligned}\theta &= \int_0^S ds \\ &= S.\end{aligned}$$

Also, the area of the sector is

$$\begin{aligned}dA &= \frac{rds}{2} \\ &= \frac{r^2 d\theta}{2}\end{aligned}$$

and

$$d\theta = \frac{2dA}{r^2} \tag{35}$$

or, for $r = 1$,

$$d\theta_1 = 2dA$$

$$\begin{aligned}\theta_1 &= 2\int_0^A dA \\ &= 2A.\end{aligned} \tag{36}$$

Thus a circular angle θ is measured either by its arc or by twice the area of the corresponding sector.

The relation

$$\theta = \frac{\text{length of arc}}{\text{length of radius}} = \frac{S}{r} \tag{37}$$

is called the circular measure of an angle and, for this reason, trigonometric functions of an angle are often called circular functions. The ratio may be looked upon as a percentage. The angle then becomes the percentage of the length of the arc in terms of the constant-length radius taken as unity. The principal circular functions are

$$\left.\begin{aligned} \frac{y}{r} &= \sin\frac{S}{r} = \sin\theta \\ \frac{x}{r} &= \cos\frac{S}{r} = \cos\theta \\ \frac{y}{x} &= \tan\frac{S}{r} = \tan\theta \end{aligned}\right\} \tag{38}$$

Hyperbolic Functions.—In rectangular coordinates the equation of an hyperbola whose center is at the origin of the coordinate axes is

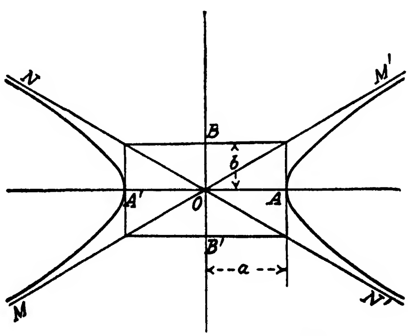

FIG. 10.—The hyperbola.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{39}$$

where a and b are equal to OA and OB of Fig. 10, respectively.

The lines MM' and NN' are asymptotes to the two branches of the hyperbola. If, in Eq. (39), a be put equal to b, there results the rectangular hyperbola whose equation is

$$x^2 - y^2 = a^2. \tag{40}$$

One branch of this hyperbola is pictured in Fig. 11.

If ds represents a small length of arc such that when x changes by an amount dx, y changes a corresponding amount dy and s by an equivalent amount ds, then

$$(ds)^2 = (dx)^2 + (dy)^2$$

or

$$ds = dx\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}$$

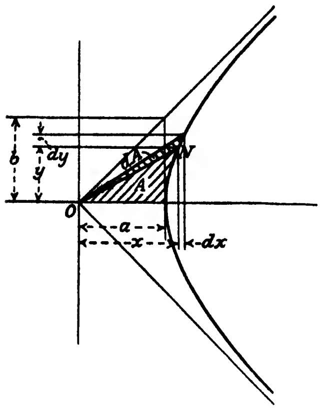

FIG. 11.—The hyperbolic angle u.

If ρ is the radius vector ON, then writing

$$\frac{ds}{\rho} = du,$$

$$du = \frac{dx}{\rho}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1} \tag{41}$$

and

$$u = \int_a^x \frac{dx}{\rho}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}. \tag{42}$$

If a be taken as the unit of measure, Eq. (42) becomes

$$u = \int_1^x \frac{dx}{\rho}\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}. \tag{43}$$

The function u is the integrated ratio of the length of an infinitesimal arc to the length of the radius vector ρ at any point P. Therefore, since

$$u = \int \frac{\text{length of arc}}{\text{length of radius}}$$

by analogy to the circular angle θ, the function u is called an *hyperbolic angle*. It is to be noted that, while in the circle r is

constant, here the radius vector ρ is a variable. Also, analogous to the circle, it may be shown that the function u is measured by twice the area of the sector.

$$\rho = \sqrt{x^2 + y^2}$$

and from Eq. (40),

$$y^2 = x^2 - a^2. \tag{44}$$

Differentiating Eq. (44),

$$2x \cdot dx = 2y \cdot dy$$

or

$$\frac{dy}{dx} = \frac{x}{y}. \tag{45}$$

Substituting these values in Eq. (42),

$$\begin{aligned} u &= \int_a^x \frac{dx}{\sqrt{x^2 - a^2}} \qquad (46) \\ &= \log_\epsilon[x + \sqrt{x^2 - a^2}] - \log_\epsilon a \\ &= \log_\epsilon \frac{x + \sqrt{x^2 - a^2}}{a} \end{aligned}$$

or

$$a\epsilon^u = x + \sqrt{x^2 - a^2}. \tag{47}$$

Solving,

$$x = \frac{a(\epsilon^u + \epsilon^{-u})}{2}$$

and

$$\frac{x}{a} = \frac{\epsilon^u + \epsilon^{-u}}{2}. \tag{48}$$

Since

$$y^2 = x^2 - a^2$$

by substituting the value of x from Eq. (48), it follows that

$$\begin{aligned} y^2 &= \frac{a^2}{4}(\epsilon^{2u} + 2\epsilon^u\epsilon^{-u} + \epsilon^{-2u} - 4) \\ &= \frac{a^2}{4}(\epsilon^u - \epsilon^{-u})^2 \end{aligned}$$

and

$$\frac{y}{a} = \frac{\epsilon^u - \epsilon^{-u}}{2}. \tag{49}$$

By dividing Eq. (49) by Eq. (48), there results

$$\frac{y}{x} = \frac{\epsilon^u - \epsilon^{-u}}{\epsilon^u + \epsilon^{-u}}. \tag{50}$$

In the case of the circle, the ratios $\frac{x}{a}$, $\frac{y}{a}$ and $\frac{y}{x}$ are called the cosine, sine, and tangent of θ, respectively. For the rectangular hyperbola these ratios are called respectively the cosine, sine, and tangent of the hyperbolic angle u. In abbreviated form they are written,

$$\sinh u = \frac{y}{a} = \frac{\epsilon^u - \epsilon^{-u}}{2} \tag{51}$$

$$\cosh u = \frac{x}{a} = \frac{\epsilon^u + \epsilon^{-u}}{2} \tag{52}$$

$$\tanh u = \frac{y}{x} = \frac{\epsilon^u - \epsilon^{-u}}{\epsilon^u + \epsilon^{-u}} \tag{53}$$

Sine, Cosine and Exponential Series.—By Maclaurin's theorem a function of a single variable may be expanded into a series of ascending powers of the variable. If y is a function of x, only, then, by Maclaurin's theorem,

$$y = f(x) = A + Bx + Cx^2 + Dx^3 + Ex^4 + \ . \ . \tag{54}$$

where A, B, C, D, etc., are constants whose values are independent of x but dependent upon the constants contained in the original expression of $f(x)$. In order to make the expression on the right-hand side of Eq. (54) definite, the values of the constants must be determined. This is done by successive differentiation.

By successive differentiation of Eq. (54) the following equations are obtained:

$$\frac{dy}{dx} = \frac{d}{dx} \cdot f(x) = 0 + B + 2Cx + 3Dx^2 + 4Ex^3 + \ . \ . \ . \tag{55}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx} \cdot f'(x) = 0 + 0 + 1 \cdot 2C + 2 \cdot 3Dx + 3 \cdot 4Ex^2 + . \ . \tag{56}$$

$$\frac{d^3y}{dx^3} = \frac{d}{dx} \cdot f''(x) = 0 + 0 + 0 + 1 \cdot 2 \cdot 3 \cdot D + 2 \cdot 3 \cdot 4 \cdot Ex + \ . \ . \ . \tag{57}$$

$$\frac{d^4y}{dx^4} = \frac{d}{dx} \cdot f'''(x) = 0 + 0 + 0 + 0 + 1 \cdot 2 \cdot 3 \cdot 4E + \ . \ . \ . \tag{58}$$

Since these equations hold for all values of x, they hold for $x = 0$, whence, substituting $x = 0$ in each equation and solving,

$$
\begin{aligned}
y &= f(0) = A & A &= y \\
\frac{dy}{dx} &= f'(0) = 1 \cdot B & B &= \frac{dy}{dx} \\
\frac{d^2y}{dx^2} &= f''(0) = 1 \cdot 2C & C &= \frac{1}{1 \cdot 2} \cdot \frac{d^2y}{dx^2} \\
\frac{d^3y}{dx^3} &= f'''(0) = 1 \cdot 2 \cdot 3 \cdot D & D &= \frac{1}{1 \cdot 2 \cdot 3} \cdot \frac{d^3y}{dx^3} \\
\frac{d^4y}{dx^4} &= f''''(0) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot E & E &= \frac{1}{1 \cdot 2 \cdot 3 \cdot 4} \cdot \frac{d^4y}{dx^4}
\end{aligned}
$$

Applying this theorem to the expansion of cos x,

$$y = \cos x = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots \tag{59}$$

and

$$
\begin{aligned}
A &= y = \cos 0 = 1 \\
B &= \frac{dy}{dx} = -\sin 0 = 0 \\
C &= \frac{d^2y}{dx^2} = \frac{-\cos 0}{\underline{|2}} = \frac{-1}{\underline{|2}} \\
D &= \frac{d^3y}{dx^3} = \frac{+\sin 0}{\underline{|3}} = 0 \\
E &= \frac{d^4y}{dx^4} = \frac{+\cos 0}{\underline{|4}} = \frac{1}{\underline{|4}}, \text{ etc.}
\end{aligned}
$$

Substituting these values in Eq. (59),

$$\cos x = 1 - \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} - \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} - \cdots \tag{60}$$

By similar methods, it is found that

$$\sin x = x - \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} - \frac{x^7}{\underline{|7}} + \frac{x^9}{\underline{|9}} - \cdots \tag{61}$$

and

$$\epsilon^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{x^5}{\underline{|5}} + \frac{x^6}{\underline{|6}} + \cdots \tag{62}$$

From Eq. (62), upon substituting $-x$ for $+x$ there results Eq. (63). If jx be substituted for x in Eq. (62), Eq. (64) is the result, while the substitution of $-jx$ for x gives rise to Eq. (65). Accordingly,

$$\epsilon^{-x} = 1 - x + \frac{x^2}{\underline{|2}} - \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - \frac{x^5}{\underline{|5}} + \frac{x^6}{\underline{|6}} - \cdots \tag{63}$$

$$\epsilon^{jx} = 1 + jx - \frac{x^2}{\underline{|2}} - \frac{jx^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{jx^5}{\underline{|5}} - \frac{x^6}{\underline{|6}} - \frac{jx^7}{\underline{|7}} + \ . \ . \ . \qquad (64)$$

$$\epsilon^{-jx} = 1 - jx - \frac{x^2}{\underline{|2}} + \frac{jx^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - \frac{jx^5}{\underline{|5}} - \frac{x^6}{\underline{|6}} + \frac{jx^7}{\underline{|7}} + \ . \ . \ . \qquad (65)$$

Multiplying Eq. (61) by j and adding to Eq. (60),

$$\cos x + j \sin x = 1 + jx - \frac{x^2}{\underline{|2}} - \frac{jx^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} + \frac{jx^5}{\underline{|5}} - \frac{x^6}{\underline{|6}} - \frac{jx^7}{\underline{|7}} + \qquad (66)$$

The right-hand member of Eq. (66) is seen to be identical with the right-hand member of Eq. (64), from which follows the familiar relation

$$\epsilon^{jx} = \cos x + j \sin x \qquad (67)$$

Similarly, multiplying Eq. (61) by j and subtracting from Eq. (60) yield results identical with the right hand member of Eq. (65), whence

$$\epsilon^{-jx} = \cos x - j \sin x. \qquad (68)$$

The series for the hyperbolic sines and cosines of a variable may also be readily derived from Eqs. (62) and (63); for, since

$$\cosh x = \frac{\epsilon^x + \epsilon^{-x}}{2}$$

and

$$\sinh x = \frac{\epsilon^x - \epsilon^{-}}{2}$$

the series for cosh x is obtained by adding Eqs (62) and (63) and dividing the result by 2, while the series for sinh x results from taking one-half the difference of Eqs. (62) and (63).

Thus

$$\cosh x = 1 + \frac{x^2}{\underline{|2}} + \frac{x^4}{\underline{|4}} + \frac{x^6}{\underline{|6}} + \frac{x^8}{\underline{|8}} + \qquad (69)$$

and

$$\sinh x = x + \frac{x^3}{\underline{|3}} + \frac{x^5}{\underline{|5}} + \frac{x^7}{\underline{|7}} + \frac{x^9}{\underline{|9}} + \quad . \qquad (70)$$

These series are rapidly convergent and are useful for computing tables of hyperbolic sines and cosines of real angles. Abbreviated tables of sinh $\frac{x}{c}$ and cosh $\frac{x}{c}$ are found in Appendix B.

Exponentials ϵ^{jx} and ϵ^{-jx} as Operators.—It has already been shown by Eqs. (67) and (68) that

$$\epsilon^{jx} = \cos x + j \sin x$$

and

$$\epsilon^{-jx} = \cos x - j \sin x.$$

Herefrom it is apparent that exponentials with imaginary exponents represent unit length vectors whose phase positions shift with the variable circular angle x. The truth of this statement is clearly evident from the polar expressions for the vectors. Thus,

$$\begin{aligned} \epsilon^{jx} &= \sqrt{\cos^2 x + \sin^2 x}\,\underline{/x} \\ &= 1\underline{/x}. \end{aligned}$$

Similiarly,

$$\epsilon^{-jx} = 1\underline{/-x}.$$

These unit vectors are shown in Fig. 12.

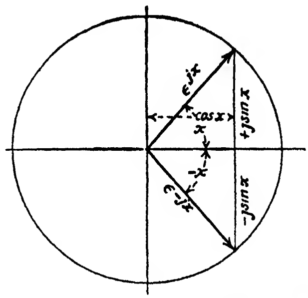

FIG. 12.—The unit vectors ϵ^{jx} and ϵ^{-jx}.

The length of the vector is constant regardless of the value of the angle, but, with increasing values of x, its slope increases in a counter-clockwise direction for the exponential ϵ^{jx} and in a clockwise direction for ϵ^{-jx}. If the angle x be made a function of time, such that $x = \omega t$, where ω is the angular velocity in radians per second, and t is the elapsed time in seconds, the unit vector will rotate at the constant angular velocity ω.

Considered as operators, then, the exponentials ϵ^{jx} and ϵ^{-jx} do not alter the size of the operand but simply change its slope. For the former, the shift is positive and, for the latter, negative through an angle of x circular radians.

Exponentials ϵ^{x} and ϵ^{-x} as Operators.—The curves

$$y = \epsilon^{x}$$

and

$$y' = \epsilon^{-x}$$

are called exponential curves. When put in the form

$$x = \log_\epsilon y = \int \frac{dy}{y}$$

and

$$-x = \log_\epsilon y' = \int \frac{dy'}{y'}$$

they are called logarithmic curves, although the distinction is of little value. It should be noted, however, that these equations are similar in form to those given on page 15, that x is a real

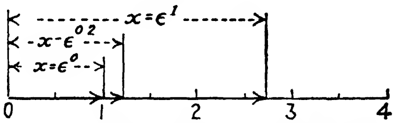

FIG. 13.—The exponentials ϵ^x and ϵ^{-x}

hyperbolic angle, and that the functions are unity for $x = 0$. The functions y and y' are vectors whose phase angles are always zero, but whose lengths vary with the hyperbolic angle x. Since

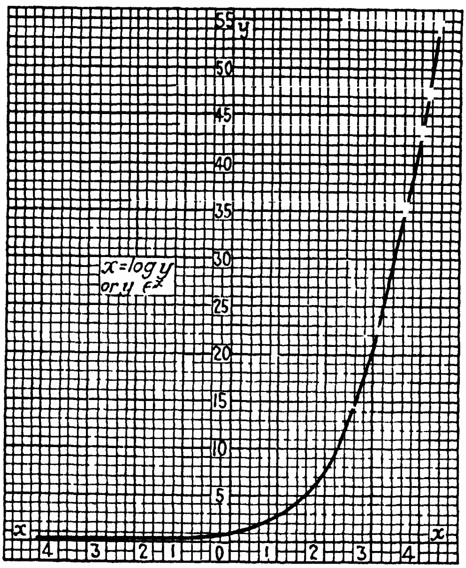

FIG. 14.—The graph of $y = \epsilon^x$.

the angle has no imaginary component, the vector does not shift its phase position as the angle increases, but can only stretch or shrink; that is, the end of the radius vector travels at all times in a straight-line path in the direction of its length, as in Fig. 13.

For positive values of the angle beginning with zero angle, the function ϵ^x increases from unity towards infinity at an accelerating rate or in geometric progression, as the angle increases. On the other hand, the function ϵ^{-x} starts at unity and, as the angle increases, diminishes or shrinks towards zero at a negatively accelerated rate. The graphs of these functions are shown in Fig. 14.

Considered as operators, the effect of the exponential ϵ^x is to stretch or increase the operand through the hyperbolic angle x without shifting its phase position, while that of the operator ϵ^{-x} is to cause a shrinking of the operand, but also without shifting its phase position.

Exponential, Trigonometric and Hyperbolic Functions Related. That exponential, circular and hyperbolic functions are intimately related is evident from the foregoing Any function of a variable expressed in terms of one of these may be converted into an equivalent expression in which only one or both of the other two appear. Some of the simpler of these relations are obtained as follows:

By the addition of Eqs. (67) and (68),

$$2 \cos x = \epsilon^{jx} + \epsilon^{-jx}$$

or

$$\cos x = \frac{\epsilon^{jx} + \epsilon^{-jx}}{2}. \tag{71}$$

By subtraction of Eq. (68) from Eq. (67),

$$2\,j \sin x = \epsilon^{jx} - \epsilon^{-jx}$$

or

$$\sin x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{2j}. \tag{72}$$

By division of Eq. (72) by Eq. (71),

$$\tan x = \frac{\epsilon^{jx} - \epsilon^{-jx}}{j(\epsilon^{jx} + \epsilon^{-jx})}. \tag{73}$$

Likewise, by substituting $x = jy$ in Eqs (67) and (68), there results:

$$\epsilon^{-y} = \cos jy + j \sin jy \tag{74}$$

$$\epsilon^{y} = \cos jy - j \sin jy. \tag{75}$$

By adding Eqs. (74) and (75), it follows that

$$2 \cos jy = \epsilon^{y} + \epsilon^{-y}$$

or

$$\cos jy = \frac{\epsilon^{y} + \epsilon^{-y}}{2}. \tag{76}$$

Similarly, by subtracting Eqs. (74) and (75),

$$\sin jy = j\frac{(\epsilon^{y} - \epsilon^{-y})}{2} \tag{77}$$

and, by dividing Eq. (77) by Eq. (76),

$$\tan jy = j\left(\frac{\epsilon^{y} - \epsilon^{-y}}{\epsilon^{y} + \epsilon^{-y}}\right). \tag{78}$$

From the above, it is seen that exponential functions with imaginary angles may be expressed as trigonometric functions with real angles, or as hyperbolic functions with imaginary angles, while exponential functions with real angles will yield trigonometric functions with imaginary angles or hyperbolic functions with real angles. For convenient reference the relations here developed are assembled in Table 1.

TABLE 1

Exponentials with imaginary angles	Exponentials with real angles
$\epsilon^{jx} = \cos x + j\sin x$	$\epsilon^{y} = \cos jy - j\sin jy$
$\epsilon^{-jx} = \cos x - j\sin x$	$\epsilon^{-y} = \cos jy + j\sin jy$
$\sin x = \dfrac{\epsilon^{jx} - \epsilon^{-jx}}{j2} = -j\sinh jx$	$\sin jy = j\dfrac{(\epsilon^{y} - \epsilon^{-y})}{2} = j\sinh y$
$\cos x = \dfrac{\epsilon^{jx} + \epsilon^{-jx}}{2} = \cosh jx$	$\cos jy = \dfrac{\epsilon^{y} + \epsilon^{-y}}{2} = \cosh y$
$\tan x = \dfrac{\epsilon^{jx} - \epsilon^{-jx}}{j(\epsilon^{jx} + \epsilon^{-jx})} = -j\tanh jx$	$\tan jy = j\dfrac{(\epsilon^{y} - \epsilon^{-y})}{(\epsilon^{y} + \epsilon^{-y})} = j\tanh y$

These relations among circular and hyperbolic functions make it possible to develop the formulas of hyperbolic trigonometry from those of circular trigonometry by simple substitution. For each one of the former there is one of the latter. For example, in circular trigonometry,

$$\sin^2 x + \cos^2 x = 1.$$

Then, if $y = jx$, substituting $j\sinh y = \sin x$ and $\cosh y = \cos x$ there follows:

$$j^2 \sinh^2 y + \cosh^2 y = 1$$

or

$$\cosh^2 y - \sinh^2 y = 1 \tag{79}$$

which is the corresponding identity in hyperbolic trigonometry. Again, from the trigonometry of circular functions,

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta.$$

By substituting

$$\begin{aligned} -j\sinh\gamma &= \sin\alpha \\ \cosh\gamma &= \cos\alpha \\ -j\sinh\delta &= \sin\beta \\ \cosh\delta &= \cos\beta \end{aligned}$$

there follows the equivalent hyperbolic form, namely,

$$\begin{aligned} \cosh(\gamma \pm \delta) &= \cosh\gamma\cosh\delta \mp j^2\sinh\gamma\sinh\delta \\ &= \cosh\gamma\cosh\delta \pm \sinh\gamma\sinh\delta. \end{aligned} \tag{80}$$

In a similar manner,

$$\sinh(\gamma \pm \delta) = \sinh\gamma\cosh\delta \pm \cosh\gamma\sinh\delta. \tag{81}$$

By the same general method of substitution, any relation of circular trigonometry will yield the corresponding relation of hyperbolic trigonometry. In Appendix A some of the more commonly used hyperbolic equations are given.

Complex Angles.—Consistent with the definition of a circular angle as given in Eq. (37), the general or complex angle is defined as the length of arc per unit of radius, *i.e.*,

$$d\dot{\theta} = \frac{ds}{r}$$

or

$$\dot{\theta} = \int \frac{ds}{r}.$$

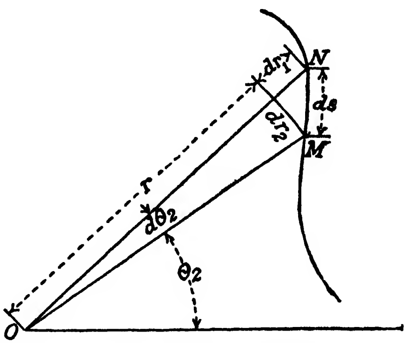

FIG. 15.—The complex angle.

then, considering the angular change produced when the radius vector traces along any curve such as MN (Fig. 15), thereby sweeping out a circular sector corresponding to a circular angle $d\theta_2$ circular radians, it may be noted that: (*a*) The radius vector changed its position with respect to the X-axis from θ_2 to $\theta_2 + d\theta_2$, and had it remained fixed in length during this change, it would have traced the arc $r d\theta_2 = dr_2$; (*b*) The radius vector changed in length

or stretched from r to $r + dr_1$, and, in so doing, had the angle θ_2 not changed in value, the former would have traced along the line dr_1.

The total change $\dot{d}s$, in arc from M to N, may therefore be considered as having been accomplished by two steps in quadrature with each other. Considering the direction of the radius vector as the axis of reference, *i.e.*, the axis of reals, and taking the original length as the unit of measure, the vector change of arc may be denoted by the equation

$$d\dot{s} = dr_1 + jdr_2 \tag{82}$$

or, dividing both sides of Eq. (82) by r,

$$d\dot{\theta} = \frac{dr_1}{r} + j\frac{dr_2}{r}$$

and

$$\dot{\theta} = \int_1^r \frac{dr_1}{r} + j\int_0^s \frac{dr_2}{r}$$

or

$$\dot{\theta} = \theta_1 + j\theta_2 \tag{83}$$
$$= \text{(hyperbolic angle)} + j \text{ (circular angle)},$$

in conformity with the definitions already given for these angles.[1]

The angle $\dot{\theta}$ is a general or complex angle. Its real component is an hyperbolic angle θ_1, and its quadrature component the circular angle θ_2. The hyperbolic component measures the integrated percentage of the stretch of the radius vector while elongating or contracting, while the circular component measures the percentage of the arc or swing of the radius vector of fixed length.

Functions of Complex Angles.—Hyperbolic functions of complex angles may be taken from charts[2] or they may be computed from appropriate trigonometric relations. It will be sufficient for the object in view to show how to derive the values of hyperbolic sines and cosines of complex angles, since the other functions may readily be obtained if these are known.

[1] For a more complete discussion of complex angles see: BOYAJIAN, ARAM, "Complex Angles," *Jour.*, A. I. E. E., February, 1923; KENNELY, A. E., "Hyperbolic Functions Applied to Electrical Engineering."

[2] KENNELLY, A. E., "Chart Atlas of Complex Hyperbolic and Circular Functions," Cambridge University Press, 1914.

By Eq. (80), the cosh of a complex angle is the complex quantity

$$\dot{A} = \cosh(x + jy) = \cosh x \cosh jy \pm \sinh x \sinh jy \tag{84}$$

$$= \cosh x \cos y + j \sinh x \sin y \tag{85}$$

$$A = \sqrt{\cosh^2 x \cos^2 y + \sinh^2 x \sin^2 y}$$

$$= \sqrt{\sinh^2 x + \cos^2 y}$$

or

$$= \sqrt{\cosh^2 x - \sin^2 y} \tag{86}$$

and

$$\theta_A = \tan^{-1}\left(\frac{\sinh x \sin y}{\cosh x \cos y}\right)$$

$$= \tan^{-1}(\tanh x \tan y). \tag{87}$$

Likewise, by Eq. (81), the sinh of a complex angle is the complex quantity

$$\dot{B} = \sinh(x \pm jy) = \sinh x \cosh jy + \cosh x \sinh jy \tag{88}$$

$$= \sinh x \cos y \pm j \cosh x \sin y \tag{89}$$

$$B = \sqrt{\sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y}$$

$$= \sqrt{\sinh^2 x + \sin^2 y} \tag{90}$$

or

$$= \sqrt{\cosh^2 x - \cos^2 y} \tag{91}$$

and

$$\theta_B = \tan^{-1}\left(\frac{\cosh x \sin y}{\sinh x \cos y}\right)$$

$$= \tan^{-1}(\coth x \tan y). \tag{92}$$

Dividing Eq. (89) by Eq. (85) and simplifying yields

$$\dot{C} = \tanh(x \pm jy) = \frac{\sinh 2x \pm j \sin 2y}{\cosh 2x + \cos 2y} \tag{93}$$

With the help of Eqs. (86), (87), (91) and (92), together with tables of hyperbolic and circular functions of real angles, the hyperbolic functions of complex angles may be readily calculated.[1] Circular functions of complex angles may be calculated by the same general methods.

[1] The modern computing machine makes it very easy to perform lengthy calculations such as are required here, at the same time giving results to almost any desired accuracy. Such machines are now usually available in engineering offices. When they are available, tables of natural hyperbolic and circular functions are conveniently used. When not available, tables containing the logs of these functions will be found far more convenient. Very satisfactory tables, containing both the natural functions and their

Thus, if

$$\dot{P} = \sin (x + jy) = \sin x \cos jy + \cos x \sin jy \qquad (94)$$

by substituting

$$\cos jy = \cosh y$$
$$\sin jy = j \sinh y$$

in Eq. (94), it follows that

$$\dot{P} = \sin x \cosh y + j \cos x \sinh y \qquad (95)$$

and

$$P = \sqrt{\sinh^2 y + \sin^2 x} \underline{/\tan^{-1} \cot x \tanh y} \qquad (96)$$

or

$$= \sqrt{\cosh^2 y - \cos^2 x} \underline{/\tan^{-1} \cot x \tanh y}. \qquad (97)$$

Likewise, let

$$\dot{Q} = \cos (x + jy) = \cos x \cos jy \mp \sin x \sin jy. \qquad (98)$$

Then, by transformation, as above,

$$\dot{Q} = \cos x \cosh y \mp j \sin x \sinh y \qquad (99)$$
$$Q = \sqrt{\sinh^2 y + \cos^2 x} \underline{/\tan^{-1}(\tan x \tanh y)} \qquad (100)$$
$$= \sqrt{\cosh^2 y - \sin^2 x} \underline{/\tan^{-1}(\tan x \tanh y)}. \qquad (101)$$

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logs, are the Smithsonian Mathematical Tables published in 1909 by the Smithsonian Institute of Washington, D. C. A very excellent table of the logs of hyperbolic functions is contained in the appendix to PERNOT'S "Electrical Phenomena in Parallel Conductors," vol. 1, John Wiley & Sons, Inc.

CHAPTER II

PROPERTIES OF TRANSMISSION-LINE CONDUCTORS

The important characteristics of transmission-line conductors are high electrical conductivity, high tensile strength, low density and low cost. The three metals that possess these characteristics to such a degree as to make their use economical in transmission-line construction are copper, aluminum and iron (steel). They are used alone or in various combinations. Of these, copper has the highest conductivity, is second in tensile strength and has the greatest density. Aluminum is second in conductivity, has the lowest tensile strength and the lowest density. Steel has the greatest strength, the lowest conductivity and is second in density. It has the additional disadvantage of a high internal reactance, is subject to hysteresis and eddy-current loss, and corrodes more readily than either of the other two.

Steel is used to particularly good advantage in combination with aluminum. Cables are built up with a steel core of either solid wire or stranded cable, depending upon the size of conductor. Thus the advantages of the high strength of steel and the light weight and high conductivity of aluminum are combined in a single cable. The use of steel reinforced cable makes it possible to use much longer and more economical spans than would be permissible with all-aluminum conductors.

Steel is also used in combination with copper in the so-called copper-clad steel conductors. These conductors are made from mild steel billets around which copper has been cast. In the casting process the steel billet is heated to a yellow heat so that the molten copper is welded firmly to it when poured around it in the mold. In the subsequent rolling and drawing processes the original relative thickness of the two metals is approximately maintained. The finished conductors are made in two commercial grades as 30 and 40 per cent of the volume conductivity of copper. Unless the steel core becomes exposed the conductor

does not corrode. Its strength depends upon the quality of steel used, and runs from $\frac{1}{3}$ to $\frac{2}{3}$ greater than for copper.

Steel cable has been used in transmission-line work chiefly as guard cables and for guying and anchoring in wood-pole construction. To a very limited extent it has also been used for short, relatively unimportant, lightly loaded transmission circuits, and, in at least one case, as a long span link in an important line. The Narrows Crossing on the city of Tacoma's Cushman line is a span of 6,244.5 ft., at present the longest transmission span in the world. The six cables of this span are made of 1¼ in. double-galvanized plow steel, having a tensile strength of about four times that of mild steel.

Steel cables and wires, however used in transmission-line work, whether alone or as cores in bi-metallic conductors, are heavily galvanized to prevent corrosion. Galvanizing softens the conductor somewhat and thereby reduces its tensile strength.

Density.—The density of a material is its weight per unit of volume. The densities of copper and aluminum vary only very slightly from the values given below. The density of iron in its various forms changes somewhat more depending upon its degree of chemical purity and its physical state.

Aluminum	2 70 g. per cubic centimeter =	168 5 lb. per cubic foot
Copper ..	8 89 g. per cubic centimeter =	555 lb. per cubic foot
Steel and iron.	7 86 g. per cubic centimeter =	490 lb per cubic foot

Conductivity and Resistivity.—The standard (100 per cent conductivity), to which electrical conductivities are compared, is known as the International Annealed Copper Standard. This standard copper has a density of 8.89 g. per cubic centimeter and a resistivity of 10.371 ohms per mil foot at 20° C. The conductivity of commercial grades of copper vary with the degree of hardness to which the conductors are drawn, and therefore with the size of the wire. In the sizes and in the degree of hardness used in medium hard-drawn, stranded cables, the conductivity of copper at 20° C. is about 97 per cent. The conductivity of hard-drawn aluminum is about 61 per cent, whence its resistivity is 17.01 ohms per mil foot at 20° C. The resistivity of steel wire varies considerably with its degree of hardness. Roughly, for hard, high-strength steel wires, it is ten times that of copper, while for iron wire it is from 6.5 to 8.5 times as great.

The resistances of stranded conductors are 2 per cent greater than the resistances of the equivalent solid conductors. This allowance is made to correct for the added length due to the "lay" of the cables, as recommended by the A.I.E.E.

Temperature Coefficient of Resistance.—This coefficient represents the change in resistance of a conductor in ohms per degree Centigrade per ohm of resistance. For 100 per cent conductivity copper, this constant is 0.00393 ohm per degree Centigrade at 20° C., while, for 97 per cent conductivity and the same temperature, it is 0.00381. The constant changes with both the temperature of reference and with the percentage of conductivity, it being proportional to the latter. For aluminum the average value of the coefficient is given by the Bureau of Standards as 0.00390 per degree Centigrade at 20°. This is also the value quoted by the Aluminum Company of America.

For 40 per cent copper-clad steel wire the coefficient varies from 0.004 to 0.005.

For iron and steel wires the coefficient varies considerably depending upon the grade of iron or steel and upon its hardness. An idea of the values to be expected in stranded steel cable may be had from the following figures published by the Indiana Steel and Wire Company.

TABLE 2

Designation	Temperature coefficient per degree Centigrade at 20°
"High-strength strand"	0.00338
$\frac{1}{2}$-in. Siemens-Martin strand	0.00338
$\frac{3}{8}$-in. Siemens-Martin strand	0.00348
$\frac{1}{4}$-in. Siemens-Martin strand	0.00309
$\frac{3}{8}$-in. Standard strand	0.00570
No. 8 B.W.G. "3-ply twisted guy wire"	0.00445
No. 6 B.W.G. "B.B. Telephone and Telegraph wire"	0.00496

The resistivity of metallic conductors at any temperature other than the base temperature is given by the equation

$$\rho_t = \rho(1 + \alpha t) \tag{102}$$

where

ρ_t = resistivity at temperature $t°$ C.
ρ = resistivity at base temperature
α = temperature coefficient of resistance in degrees C.
t = the rise in temperature above the base in degrees C. (sign of α is + for rise and − for fall).

Coefficient of Linear Expansion.—This coefficient is a measure of the change in length which accompanies a change of 1° in temperature, expressed as a decimal fraction. The following are average values, per degree Fahrenheit.

TABLE 3

Conductor material	Temperature coefficient of expansion, degrees Fahrenheit
Aluminum	12.8×10^{-6}
Aluminum-steel cable	10 to 11×10^{-6} (depending upon the ratio of steel to aluminum)
Copper	9.22×10^{-6}
40 per cent copper-clad steel	12.9×10^{-6}
Steel	6.4×10^{-6}

The corresponding values of the coefficients per degree Centigrade are obtained by multiplying the values in the table by the ratio $9 \div 5$.

Tensile Strength.—The tensile strengths of aluminum, copper and steel depend considerably upon the degree of hardness to which they are drawn and upon the heat treatment received. Strength increases with the hardness, and, accordingly, is greater for the smaller sizes of wire than for the larger ones owing to the greater number of passes required in drawing.

The strength of concentric-strand cables is not equal to the sum of the strengths of its component strands, but is usually estimated to be 90 per cent of this value for copper and from 85 to 90 per cent of it for aluminum.

Tensile strengths of commercial grades of wire are approximately as given in Table 4.

TABLE 4

Conductor material and grade	Ultimate tensile strength (pounds per square inch)
Aluminum wire, hard drawn:	
a. Sizes used as solid line conductors	22,000 to 24,000
b. Sizes used in stranded cables. .	24,000 to 27,000
Copper wire:	
a. Annealed	
Large sizes..................	32,000 to 34,000
Small sizes..................	Up to 40,000
b. Medium hard drawn:	
Large sizes............... ...	40,000 to 50,000
Small sizes..................	Up to 60,000
c. Hard drawn:	
Large sizes.	50,000
Small sizes....	65,000
Steel wire.........................	Varies greatly depending upon chemical constituency and hardness. The range covered by the different available grades is from 50,000 to about 400,000

Elastic Limit.—A body is said to be elastic when, if subjected to a *strain*, internal *stresses* are set up which will bring the body back to its original form when the strain is removed. A cable under a moderate tension will stretch slightly, but, upon removing the tension, the original length of the cable will be resumed. If the tension on the cable is gradually increased, some value of tension will finally be reached beyond which the cable will not resume its original length when the strain is removed; instead, the cable will be permanently deformed. The lowest tension at which such deformation occurs is called the "elastic limit" of the cable.

The elastic limit is not a very definite point in certain soft forms of conductor materials such as annealed copper and aluminum, since these materials begin to stretch at relatively low tensions. The term "elastic limit," as applied to these materials, has a somewhat special meaning not strictly in accord with the above definition.

The elastic limit of hard-drawn aluminum wire runs from 50 to 60 per cent of its ultimate tensile strength or from about 12,000 to 16,000 lb. per square inch. The value quoted by the Aluminum

Company of America is 14,000 lb. Steel-reinforced aluminum cables have an elastic limit equal to about 65 per cent of the ultimate strength of the cable.

Hard-drawn copper has an elastic limit of from 50 to 65 per cent of its ultimate strength, depending upon the degree of hardness; 50 per cent is a conservative figure that is much used.

The elastic limit of commercial steel wire is about 50 per cent of its ultimate strength, while for special high-strength steels it is much higher. For the steels used in reinforced-aluminum cables, the Aluminum Company of America gives the ultimate strength as 160,000 and the elastic limit as 130,000 lb. per square inch, respectively.

In the design of transmission line spans the question arises as to what maximum allowable tension is permissible for a given conductor. Here engineering practice has usually been to assume this to be from 75 to 100 per cent of the elastic limit of the cable. Any load slightly in excess of the elastic limit will cause the cable to stretch slightly, thus relieving the tension and increasing the sag. The length and sag at any given temperature will be permanently increased, but the strength of the cable will be unimpaired.

Modulus of Elasticity.—So long as the elastic limit of a material is not exceeded, the strain or deformation produced is proportional to the stress applied. This is known as Hooke's law. Under these conditions, the ratio of stress per unit area to the deformation per unit of length is a constant amount E for a given piece of material. This ratio is called Young's modulus of elasticity.

Thus, consider a wire or cable having A sq. in. of cross-sectional area and a length of L_0 in. when subjected to an initial tension of T_0 lb. If the wire is subsequently stressed to a slightly greater tension T lb., thereby producing an additional elongation of dl in. by Hooke's law (assuming the change in area due to the elongation to be a negligible amount),

$$E = \frac{T - T_0}{A} \div \frac{dl}{L_0} = \frac{(T - T_0)L_0}{A \cdot dl}. \tag{103}$$

The elongation is

$$dl = \frac{(T - T_0)L_0}{EA}$$

and the final length of the cable is

$$L = L_0 + dl$$
$$= L_0\left[1 + \frac{(T - T_0)}{EA}\right] \qquad (104)$$

It is difficult to determine experimentally the exact values of the modulus of elasticity for materials like copper and aluminum because they have no clearly defined elastic limits. These materials do not obey Hooke's law closely within the elastic limit, as a perfectly elastic material does. Furthermore, since stressing such materials tends to increase their tensile strengths, the modulus is usually higher after stressing than before. The modulus is less for stranded cables than for solid wires, and less for annealed than for hard-drawn wires. The values given below are average values of E in pounds per square inch for the various materials and grades:

TABLE 5

Material	Young's modulus, pounds per square inch
Aluminum wires and cables	9×10^6
Aluminum (steel reinforced)	11.4×10^6 to 13×10^6
Copper wire (annealed)	12×10^6
Copper wire (hard-drawn)	16×10^6
Copper concentric-strand cable	About 75 per cent of the corresponding solid wire
Copper-clad steel cable	16×10^6 to 20×10^6
Iron wire	24×10^6
Steel wire	27×10^6
Steel concentric lay cable	22×10^6

CHAPTER III

THE MAGNETIC CIRCUIT AND INDUCTANCE

Any space in which a magnet pole experiences magnetic forces is a magnetic field. When dealing with these fields and the magnetic forces associated with them, our thinking is greatly facilitated by the use of Faraday's concept of *magnetic lines of force.* These lines are so drawn that their directions at all points represent the directions of the resultant magnetic forces, and their number per unit area normal to their direction—that is, their density, in a field of unit permeability—represents the intensity of the forces acting.

Since the basic theory of the transmission-line circuit is intimately concerned with the magnetic circuit calculations, it will be advantageous, at this point, to discuss briefly the fundamental relations and the units involved.

Concepts, Definitions and Units.—Unless otherwise stated, the units used in the following definitions are the *absolute electromagnetic units.*

Measurements of magnetic field intensity are based upon the concept of an ideal *unit point magnetic pole.* Such a unit pole is a pole which in a medium of air, when separated from an equal pole of like sign by a distance of one centimeter, is repelled with a force of one dyne.

The *magnetic potential* of a point is the number of ergs of work done in bringing a unit point pole from the edge of the field to the point in question.

The *intensity* of a magnetic field is the force in dynes which unit point north pole experiences in the field, and is designated by the symbol H. Since the direction of a magnetic line of force is also the direction of the magnetic field intensity, if distances along a path be l, then the *magnetic potential difference* between any two points in a field of force is the line integral of the magnetic field intensity; that is,

$$dF = H \quad dl \cdot \cos \theta$$

and

$$F = \int_{(L)} H \cdot dl \cdot \cos \theta \tag{105}$$

where

H = the magnetic potential gradient in the direction of the field
dl = an elementary length of path

and

θ = the angle between the direction of the intensity H and dl.

The c.g.s. unit of field intensity is the *gilbert per centimeter.* *The ampere-turn per inch* is the common practical unit. The two units are related by the equation

$$NI(\text{per inch}) = 2.022H \text{ (per centimeter)} \tag{106}$$

where I is the current measured in amperes.

The drop of magnetic potential between any two points on a line of force in a magnetic circuit is the *magnetomotive force F*, absorbed in that portion of the circuit, as given by Eq. (107) below. Here again the current is given in amperes.

$$F = Hl = 0.4\pi NI \tag{107}$$

The c.g.s. unit of m.m.f. is the gilbert and the practical unit is the *ampere-turn.*

The unit of magnetic flux is a *line of force* called the "maxwell." The total number of maxwells in a given sectional area is represented by the symbol ϕ, while the *flux density* is $B = \frac{\phi}{A}$, where A is the sectional area of the path in square centimeters, normal to the direction of the flux. The unit of flux density, the *gauss*, represents 1 line per square centimeter.

In a medium of air a field intensity of 1 gilbert per centimeter will establish a flux density of 1 gauss, since the magnetic conductivity of air per cubic centimeter is unity. Magnetic materials have very much higher magnetic conductivities. In them the same field intensity will establish a flux having a density of many gausses. The ratio of flux density to field intensity in a material, $\mu = \frac{B}{H}$, is called the "permeability" of the material.

The law of the magnetic circuit, analogous to Ohm's law, and expressing the relation of cause and effect for the magnetic circuit, is

$$\phi = \frac{F}{R} = \frac{4\pi NI}{\dfrac{l}{\mu A}}. \tag{108}$$

The quantity designated by R is called the *reluctance* of the magnetic circuit.

When a magnetic field is produced by a current, variations in the current are accompanied by variations in the magnetic flux. The varying flux induces in the conductor a voltage which tends to prevent the change in current that causes it. These relations are expressed symbolically in c.g.s. units by the equation

$$e = -L\frac{di}{dt} = -N\frac{d\phi}{dt} \tag{109}$$

where L is the proportionality factor between the time rate of change of current and the induced potential difference. From Eq. (109) it follows that

$$L = N\frac{d\phi}{di}$$

or, if the permeability of the magnetic circuit is constant,

$$L = \frac{N\phi}{I}. \tag{110}$$

The product $N\phi$ represents the total number of *magnetic linkages* which the current I produces in the circuit. Thus, the c.g.s. unit of L, called the *coefficient of self-inductance* of a circuit, is the number of magnetic linkages per absampere of current. This unit is the *abhenry.* The practical unit, called the "henry," is 10^9 times as large as the abhenry.

By multiplying Eq. (109) through by idt and integrating, the energy stored in the magnetic field when the current in the circuit is I units is found to be

$$E = \frac{LI^2}{2} \text{ joules} \tag{111}$$

when L is given in henries, and I in amperes.

The Magnetic Field about a Long, Straight Cylinder.—Consider an elementary length dl of a long, straight, round conductor (Fig. 16). Let the conductor be of infinitesimal cross-section, and let the current in it be I absamperes. A unit north magnet pole at P will set up a flux density at O of $B = \frac{1}{x^2}$ gausses. The component of this flux density normal to the direction of the current is $B \cos \theta$, and is the component which reacts with the current in dl to produce a force. Due to it, the length dl of the conductor is pushed upwards with a force of $BI \cos \theta \cdot dl$ dynes. The unit pole at P is pushed downward with an equal force, due to the field at P caused by the current in dl. If the field intensity at P, due to dl, is dH, the force on the pole is dH.

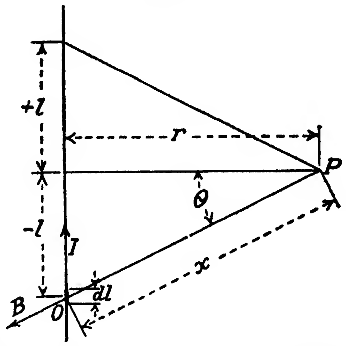


FIG. 16.—The magnetic field intensity near a long straight wire.

Expressing the equality of the forces on dl and on the pole at P, and substituting for B the value $\frac{1}{x^2}$, there results the equation,

$$dH = \frac{I \cos \theta \cdot dl}{x^2} \text{ gilberts per centimeter.} \tag{112}$$

From the figure, $x = r \sec \theta$ and $l = r \tan \theta$, whence $dl = r \sec^2 \theta d\theta$. By substituting the values of dl and x^2 in Eq. (112), this equation is transformed to

$$dH = \frac{I \cos \theta \cdot d\theta}{r}. \tag{113}$$

For a very long, straight wire, since θ approaches the value $\frac{\pi}{2}$, the field intensity due to the entire length of conductor, and at a distance of r cm. from it, is

$$H = \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \frac{I \cos \theta \cdot d\theta}{r}$$

$$= \frac{2I}{r} \text{ gilberts per centimeter.} \tag{114}$$

Thus, the field intensity at any point without a line conductor is inversely proportional to the distance from the center of the conductor. The same law holds for circular conductors, and practically also for conductors of other shapes when the distance of the point from the conductor is large as compared with the dimensions of the conductor area.

Magnetic Lines of Force about a Straight, Round Conductor.—It is to be remembered that field intensity is a vector quantity. Its magnitude is given by Eq. (114), while its direction and sense are found by the familiar right-hand rule. Lines of constant field intensities drawn about a straight, round conductor, in planes perpendicular to its length, are the familiar lines of force. By Eq. (114) these are concentric circles.

The circular lines of force about a single isolated conductor (in a medium of constant permeability) are illustrated in Fig. 17. The lines are drawn close together near the conductor, but, as distances from the conductor increase, they are separated by increasing spaces to represent the diminishing field intensities. The total drop of magnetic potential along a line of force is the same for all lines, and is equal to the magnetomotive force of the current in the conductor; that is, for any line of force the line integral of the field intensity ($\int_L H \cdot dl \cdot \cos \theta$), taken completely around the circuit, is equal to $4\pi I$. For the line of force distant r cm. from a straight wire,

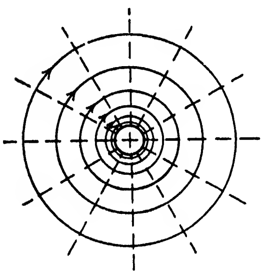


Fig. 17.—The magnetic field about an isolated current.

$$4\pi I = \int_0^{2\pi r} H \cdot dl. \tag{115}$$

This is essentially a statement of Kirchoff's law as applied to the magnetic circuit about the conductor.

It should be noted that in Fig. 17 the full lines are the circular lines of force. The resultant field intensity is tangent to these circles at every point. The broken lines are lines of constant magnetic potentials. (Dielectric lines of force as will be shown later.)

Magnetic Field Intensity within a Round, Straight Conductor. The field intensity at any point within a straight, round conductor having a uniform current density, is proportional to the distance from the center of the conductor. The proof of this

statement is apparent from the following: In Fig. 18, consider the field intensity at a point P distant x cm. from the center of a circular conductor of radius r cm. Let the current I in the conductor be uniformly distributed over its entire area. The field intensity at P is uninfluenced by the current in that part of the conductor lying without the circle of radius x. The current I_x lying within the inner circle is

$$I_x = \frac{Ix^2}{r^2}$$

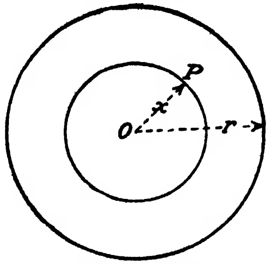


FIG 18—The magnetic field intensity within a straight, round wire

and its magnetizing effect at P is the same as though it were all flowing along the axis of the conductor. The field intensity at P may then be found from Eq. (114). It is

$$H_p = \frac{2Ix}{r^2}. \tag{116}$$

In Fig. 19 the field intensities both inside and outside the conductor are shown as functions of distance from the axis of the conductor.

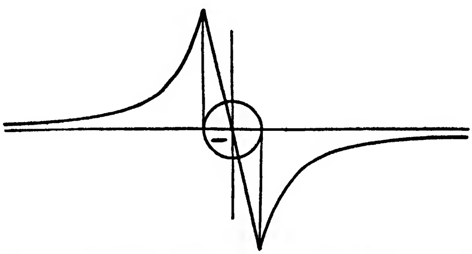


FIG 19—Curve of magnetic field intensity due to the current in a wire.

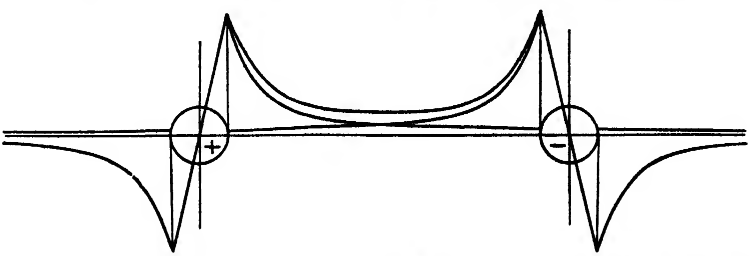


FIG 20—The curve of magnetic field intensity due to the current in a parallel sided, return loop

Figure 20 shows the separate field intensity curves due to each of two parallel, round conductors, carrying equal currents of

opposite sign, together with the resultant curve for the two conductors. The permeabilities of the conductors and the medium in which they are suspended are assumed to be unity.

Calculation of Magnetic Flux about a Straight, Round Conductor.—In transmission-line calculations, problems frequently arise in which it is necessary to compute the amount of magnetic flux passing between the two parallel sides of a loop, the sides of which are parallel to the conductors of the transmission line. The method employed in making these calculations may be illustrated by the following problem:

In Fig. 21, let a straight, round conductor at A be suspended in a medium of constant permeability μ. Assume the current of $+I$ amp. flowing in the conductor to be the only magnetizing force present. Let the current be uniformly distributed over the cross-section of the conductor. At all points without the conductor the magnetic effect is then the same as that produced by a current filament of $+I$ amp. coincident with the axis of the conductor. The problem is to find the number of magnetic lines of force per unit length of conductor, passing through any parallel-sided loop such as PQ, the sides of which are also parallel to the conductor A.

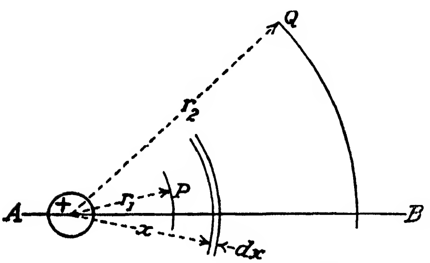


FIG. 21.—Calculation of the magnetic flux through a loop.

It has already been shown that the magnetic lines of force due to the current are concentric circles in this case. Accordingly, between two unit length cylinders having radii r_1 and r_2, whose axes are coincident with the axis of the conductor, there will be included the lines of force whose total number we desire to know. Equation (114) shows that at any point on any one of the circular lines of force, such as the one of radius x cm., for example, if the current is measured in amperes, the field intensity has the constant value,

$$H = \frac{2I}{10x} \text{ gilberts per centimeter}$$

and the flux density is $B = \mu H$.

Since radial lines such as AB are lines of constant magnetic potential, the flux crosses them at right angles. The flux per

unit length of conductor, passing between the two concentric cylinders of radii x and $x + dx$, is, therefore,

$$d\phi = Bdx = \frac{2I\mu dx}{10x} \text{ lines} \tag{117}$$

and that passing between the concentric cylinders of radii r_1 and r_2 is

$$\phi_{PQ} = \frac{2I\mu}{10}\int_{r_1}^{r_2}\frac{dx}{x}$$

$$= \frac{2I\mu}{10} ln\frac{r_2}{r_1}. \tag{118}$$

The Magnetic Lines of Force about Two Parallel, Round Wires are Circles.—The magnetic lines of force due to the combined, equal magnetomotive forces of a parallel-sided return loop in a

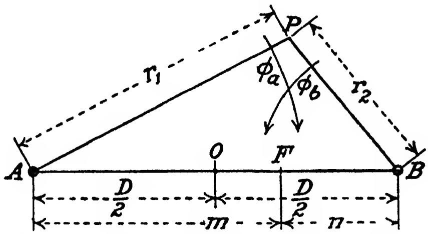


FIG 22.—The magnetic lines of force about the two round, parallel wires of a return loop circuit are circles

medium of constant permeability are circles. This fact may be proved as follows: In Fig. 22, let A and B be the two sides of such a loop carrying the equal currents $+I$ and $-I$ amp. as indicated. Let the point P move in the direction of the resultant magnetic field, that is, in a manner so that at every position of the point the drop of magnetic potential along a line at right angles to the direction of its motion is zero, and let some fixed point F on the line AB be on the locus of P. Since the locus of P is along the direction of the resultant field at every point, the resultant flux through the loop PF must be zero.

By the same line of reasoning as that used in deriving Eq. (118), the fluxes through this loop, due to the magnetomotive forces of the currents in A and B, are

$$\phi_a = \frac{2\mu I}{10} ln\frac{r_1}{m} \tag{119}$$

$$\phi_b = \frac{2\mu I}{10} ln\frac{r_2}{n}.$$

The resultant flux is

$$\phi_c = \phi_a + \phi_b$$
$$= \frac{2\mu I}{10}\left[ln\frac{r_1}{m} - ln\frac{r_2}{n}\right] = 0$$

whence

$$\frac{r_1}{m} = \frac{r_2}{n}$$

and

$$\frac{r_1}{r_2} = \frac{m}{n} = \text{a constant.} \tag{121}$$

Thus, since the point P moves so that the ratio $r_1 \div r_2$ of its distances from the two fixed points A and B is a constant, the locus of P is a circle by the theorem of the inverse points of a circle.

The Theorem of the Inverse Points of a Circle.—This theorem will now be demonstrated. In Fig. 23 let P be the point whose

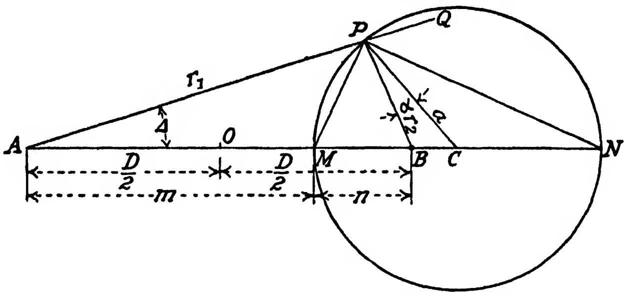


FIG. 23.—The inverse points with respect to a circle

distances r_1 and r_2 from the fixed points A and B, respectively, are in the constant ratio $\frac{m}{n}$. Bisect the angles APB and BPQ by the lines PM and PN. Since the bisector of any angle of a triangle divides the base into segments proportional to the adjacent sides, $\frac{AM}{MB} = \frac{m}{n} = \frac{NA}{NB}$. For any position of P the angle MPN is a right angle, since the external and internal bisectors of any angle of a triangle are perpendicular to each other. It follows that the locus of P is a circle, since the right triangle MPN has the fixed hypothenuse MN. The points A and B are called the inverse points with respect to the circle.

The following relations may also be noted: It is easily proven that angle BPC = angle PAB. The triangles CAP and CPB are

therefore similar, since the angles BPC and PAB are equal, and the angle PCB is common to both triangles. Hence their corresponding sides are proportional and

$$\frac{AC}{a} = \frac{r_1}{r_2} = \frac{a}{BC} = \frac{m}{n} = \text{a constant.} \tag{122}$$

Also

$$BC = \frac{a^2}{AC}. \tag{123}$$

Dipolar Circles.—In Fig. 24, the circles drawn in full lines are the magnetic lines of force, due to two parallel current filaments.

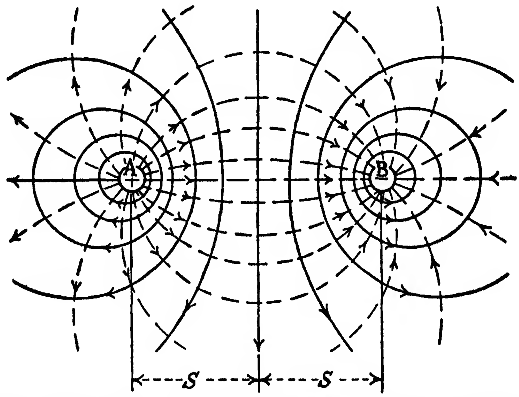


Fig. 24.—The magnetic and dielectric fields about the two round wires of a parallel-sided return loop.

These circles belong to two families, one described about A and the other about B, known as dipolar circles. The dotted circles are the corresponding lines of constant magnetic potentials. The latter are discussed in a later article.

Certain relations which are useful in drawing the circles are given below. These may readily be proved from Fig. 23. Let the constant ratio $\frac{m}{n} = e$, and let the spacing be $D = 2s$. Then, in Fig. 23,

$$\left.\begin{aligned} OM &= \frac{s(1-e)}{1+e} & MN &= \frac{4se}{1-e^2} \\ MB &= \frac{2se}{1+e} & MC &= \frac{2se}{1-e^2} \\ NB &= \frac{2se}{1-e} & OC &= \frac{s(1+e^2)}{1-e^2} \end{aligned}\right\}$$

Equation of the Circles.—Using rectangular coordinates with the origin at O, the equation of these circles is derived as follows: From Fig. 22, taking the distance between current filaments equal to $2s$, and letting x and y be the coordinates of P,

$$r_1^2 = (s + x)^2 + y^2 = s^2 + 2sx + x^2 + y^2$$
$$r_2^2 = (s - x)^2 + y^2 = s^2 - 2sx + x^2 + y^2$$

and, since

$$\frac{r_1^2}{r_2^2} = \frac{m^2}{n^2}$$

$$m^2(s^2 - 2sx + x^2 + y^2) = n^2(s^2 + 2sx + x^2 + y^2)$$

or

$$y^2(m^2 - n^2) + x^2(m^2 - n^2) - 2sx(m^2 + n^2) = -s^2(m^2 - n^2)$$

whence

$$y^2 + \left[x - \frac{m^2 + n^2}{2(m - n)}\right]^2 = \frac{m^2n^2}{(m - n)^2}. \tag{124}$$

Equation (124) is the equation of a circle whose center is at

$$x = \frac{m^2 + n^2}{2(m - n)},$$
$$y = 0$$

and whose radius is

$$\rho = \frac{mn}{m - n}.$$

Lines of Equal, Magnetic Potentials of a Parallel-sided, Return Loop are Circles.—To prove this, let A and B of Fig. 25 be the two sides of a parallel-sided loop carrying the current filaments of strength $+I$ and $-I$. Let Q be a point which moves along a line of constant magnetic potential P, and let the fixed point F, lying on the bisector of AB, be a point on the locus of Q.

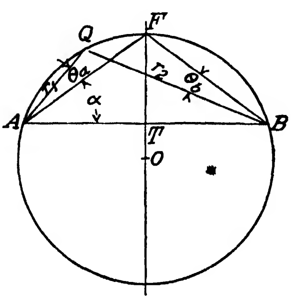


FIG 25.—The lines of equal magnetic potentials of a parallel sided return loop

Due to the currents at A and B each acting alone, the corresponding field intensities at Q, due to A and B, are

$$H_a = +\frac{2I}{r_1}$$
$$H_b = -\frac{2I}{r_2}.$$

The corresponding differences of magnetic potential between Q and F are

$$P_a = H_a r_1 \theta_a = 2I\theta_a$$
$$P_b = H_b r_2 \theta_b = -2I\theta_b.$$

Since Q and F are on an equipotential line,

$$P_a + P_b = 0 = 2I(\theta_a - \theta_b)$$

or

$$\theta_a = \theta_b. \tag{125}$$

If a circle be passed through the three points, A, F and B, the point Q will lie on this circle, for, since $\theta_a = \theta_b$ and both are angles inscribed in a given circle, they must intercept equal arcs. The circles of constant magnetic potentials and the circles representing magnetic lines of force are mutually perpendicular.

In the following chapter it will be shown that the circular magnetic lines of force are also the equipotential circles of the dielectric field, while the circles of equimagnetic potentials are the circular dielectric lines of force.

The Magnetic Flux-linkages of a Parallel-sided Return Loop. Coefficient of Self-inductance.—In Fig. 26, A and B represent the two sides of a parallel-sided return loop, carrying the currents of strength $+I$ and $-I$ absamperes, respectively. The medium is assumed to be air, having constant permeability equal to unity. Since the permeability is constant, the principle of superposition may be applied; that is, the flux through the loop AB may be found by adding the separate fluxes in the loop due to A and B, each calculated separately as though the other were not present.

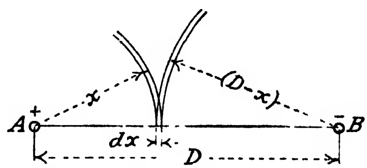


FIG. 26.—Calculation of the magnetic flux linking a parallel-sided return loop.

The flux through the loop is evidently that which crosses the line AB. At a point on AB distant x cm. from A, the field intensities due to A and B have the values

$$\left.\begin{aligned} H_a &= \frac{2I}{x} \\ H_b &= \frac{2I}{D - x} \end{aligned}\right\} \tag{126}$$

Both of these intensities act downward through the loop at right angles to AB, and may therefore be added algebraically to find

the resultant intensity at the point considered. Also, since $\mu = 1$, $H = B$, and

$$B = 2I\left(\frac{1}{x} + \frac{1}{D - x}\right),$$

the flux between conductors crossing AB per centimeter length of line is

$$\begin{aligned} d\phi &= Bdx \\ &= 2I\left(\frac{1}{x} + \frac{1}{D - x}\right)\cdot dx \text{ lines.} \end{aligned}$$

The corresponding flux through the loop between conductors is

$$\begin{aligned} \phi &= 2I\int_r^{D-r}\left(\frac{1}{x} + \frac{1}{D - x}\right)dx \\ &= 4Iln\frac{D - r}{r} \text{ lines,} \end{aligned} \tag{127}$$

where r is the radius of the conductor in the units used to measure D. By definition, the coefficient of self-inductance is

$$L = \frac{N\phi}{I}$$

and, since $N = 1$, the inductance per centimeter of loop (not including the linkages within the conductors), is, from Eq. (127),

$$L_1 = 4ln\frac{D - r}{r} \text{ abhenries per centimeter of loop.} \tag{128}$$

Since only half of the flux of Eq. (127) links each conductor, the value of L_1 per centimeter length of conductor is

$$L_1 = 2ln\frac{D - r}{r} \text{ abhenries per centimeter of conductor.} \tag{129}$$

For all practical transmission lines no appreciable error is made by substituting D for $D - r$. Making this substitution, using the mile as the unit of length, $\log_{10}$ and the practical unit of inductance, the value of L_1, per mile of one conductor, is

$$L_1 = 741.13 \times 10^{-6}\log_{10}\frac{D}{r} \text{ henries per mile.} \tag{130}$$

On the assumption of uniform distribution of current over the cross-section of the conductor, the coefficient of self-inductance L_2, due to the flux-linkages within the conductor, is found as follows:

By Eq. (116) (Fig. 18), the field intensity at a point within the conductor and distant x cm. from the center, is

$$H_x = \frac{2Ix}{r^2}.$$

Accordingly, if the permeability of the conductor material is μ, the flux density corresponding to H_x is

$$B_x = \frac{2I\mu x}{r^2}$$

and the flux, per unit of length of conductor in a circular strip of width dx, becomes

$$d\phi = B_x dA = \frac{2I\mu x dx}{r^2}.$$

This flux links the current lying within the circle of radius x; that is, each flux line represents the fraction $\frac{x^2}{r^2}$ of a complete linkage with all the current. Thus,

$$d(N\phi) = \frac{2I\mu x}{r^2} \cdot \frac{x^2}{r^2} \cdot dx$$

whence

$$\begin{aligned} N\phi &= \frac{2I\mu}{r^4} \int_0^r x^3 dx \\ &= \frac{I\mu}{2} \text{ linkage} \end{aligned}$$

and

$$L_2 = \frac{N\phi}{I} = \frac{\mu}{2} \text{ abhenries per centimeter of conductor.} \quad (131)$$

Converted to practical units per mile of one conductor,

$$L_2 = 80.47 \times 10^{-6}\mu \text{ henries per mile} \quad (132)$$

The final expression for the total inductance, per mile of a single conductor, is the sum of Eq. (130) and (132), or

$$L = \left(741.13 \log_{10} \frac{D}{r} + 80.47\mu\right)10^{-6} \text{ henries per mile of one conductor.} \quad (133)$$

Inductance of Split-conductor, Single-phase Circuits.—Consider a single-phase circuit as illustrated in Fig. 27, in which the outgoing current is divided between the equal paralleled conductors a and b. The same current returns through the corresponding equal paralleled conductors a' and b'. The radius of each conductor is r, and the spacings are D and kD, as shown, where k is a constant. The radii and spacings are measured in the same units.

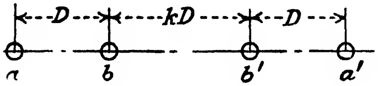


FIG. 27.—The split-conductor, single-phase circuit

If the total complex current in the circuit is I_0, and since from symmetry a' and b' are the return conductors for a and b, respectively, then

$$I_0 = I_a + I_b = -(I_a' + I_b')$$

and

$$I_a = -I_a'$$
$$I_b = -I_b'.$$

Furthermore, since the outgoing conductors and the return conductors each constitute a pair of paralleled conductors, and since their cross-sections are equal and uniform,

$$Z_aI_a = Z_bI_b$$

and

$$Z_a'I_a' = Z_b'I_b'$$

where Z_a, Z_b, Z_a' and Z_b' are the complex impedances per centimeter of the conductors indicated by the subscripts That is, in general,

$$ZI = (R + j\omega L)I$$

where R and ωL are respectively the resistance and the inductive reactance per unit length of conductor, carrying the complex current I.

The problem is to find the impedance drop in each of the pairs of conductors.

Referring to the figure, if S represents a very large but finite distance, the vector flux linking conductor a, due to all the currents, is

$$\phi_a = 2\left(I_a ln\frac{S}{r} + I_b ln\frac{S}{D} - I_b ln\frac{S}{D(1+k)} - I_a ln\frac{S}{D(2+k)}\right) + I_a\frac{\mu}{2}$$
$$= I_a\left(2ln\frac{D(2+k)}{r} + \frac{\mu}{2}\right) + 2I_b ln(1+k) \tag{134}$$

Similarly

$$\phi_b = 2\left(I_a ln\frac{S}{D} + I_b ln\frac{S}{r} - I_a ln\frac{S}{D(1+k)} - I_b ln\frac{S}{kD}\right) + \frac{I_b\mu}{2}$$
$$= I_b\left(2ln\frac{kD}{r} + \frac{\mu}{2}\right) + 2I_a ln(1+k) \tag{135}$$

However,

$$\phi_a = L_a I_a$$

and hence

$$Z_a I_a = RI_a + j\omega\phi_a$$

and, similarly,

$$Z_b I_b = RI_b + j\omega\phi_b.$$

Accordingly, since a and b are in parallel,

$$RI_a + j\omega\left(2I_a ln\frac{D(2+k)}{r} + 2I_b ln(1+k) + \frac{I_a\mu}{2}\right) = RI_b$$
$$+ j\omega\left(2I_b ln\frac{kD}{r} + 2I_a ln(1+k) + I_b\frac{\mu}{2}\right). \tag{136}$$

Substituting $I_b = I_0 - I_a$ and solving for I_a,

$$I_a = -I_0\left(\frac{ln\frac{2+k}{k}}{\frac{2R}{j\omega} + \mu + 2ln\frac{kD^2(2+k)}{r^2(1+k)^2}} - \frac{1}{2}\right) \text{ vector amp.} \tag{137}$$

and

$$I_b = I_0\left(\frac{ln\frac{2+k}{k}}{\frac{2R}{j\omega} + \mu + 2ln\frac{kD^2(2+k)}{r^2(1+k)^2}} + \frac{1}{2}\right) \text{ vector amp.} \tag{138}$$

The impedance drop in each of the conductors may be found by substituting these values of current in the above equation for ZI_a or ZI_b, since the drop is the same in each of the two conductors

That is,

$$ZI_a = I_0\left\{\left[R + j\omega\left(2ln\frac{D(2+k)}{r} + \frac{\mu}{2}\right)\right]\right.$$

$$\times\left[\frac{1}{2} - \frac{ln\frac{2+k}{k}}{\frac{2R}{j\omega} + \mu + 2ln\frac{kD^2(2+k)}{r^2(1+k)^2}}\right]$$

$$\left. + 2ln(2+k)\left[\frac{1}{2} + \frac{ln\frac{2+k}{k}}{\frac{2R}{j\omega} + \mu + 2ln\frac{kD^2(2+k)}{r^2(1+k)^2}}\right]\right\} \qquad (139)$$

This equation can be somewhat simplified. It will, however, serve the purpose about as well to leave the expression as it stands. Assuming I_0 to be known, the complex currents may be evaluated from Eq. (137) and (138), after which these values may readily be substituted in Eq. (139) to find the drop.

Inductance of Three-phase Lines. General Equations.—To illustrate the general method of calculating the drop in each conductor of a three-phase line, regardless of whether or not the currents are balanced, assume an untransposed three-phase line made up of three equal conductors, each of radius r, and having any spacings whatever, as in Fig. 28. Let S be a very large finite distance, and let the spacings, D_1, D_2 and D_3, as well as S and r, all be measured in the same units of length.

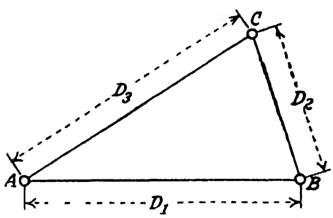


Fig. 28.—Triangular spacing for a three-phase line.

Then, in complex notation, the flux lines per centimeter, linking each of the three conductors, are the following:

$$\phi_a = 2\left(I_a ln\frac{S}{r} + I_b ln\frac{S}{D_1} + I_c ln\frac{S}{D_3}\right) + I_a\frac{\mu}{2} \qquad (140)$$

$$\phi_b = 2\left(I_a ln\frac{S}{D_1} + I_b ln\frac{S}{r} + I_c ln\frac{S}{D_2}\right) + I_b\frac{\mu}{2} \qquad (141)$$

$$\phi_c = 2\left(I_a ln\frac{S}{D_3} + I_b ln\frac{S}{D_2} + I_c\, ln\frac{S}{r}\right) + I_c\frac{\mu}{2}. \qquad (142)$$

The three complex currents are I_a, I_b and I_c, and their sum is zero. Thus,

$$I_a + I_b + I_c = 0.$$

If in Eq. (140), $-(I_a + I_b)$ be substituted for I_c, there results the relation

$$\phi_a = 2\left(I_a ln\frac{D_3}{r} + I_b ln\frac{D_3}{D_1}\right) + I_a\frac{\mu}{2} \tag{143}$$

since, as S approaches infinity, all the values of S cancel out.

The above equation may be written

$$\phi_a = 2\left(I_a ln\frac{1}{r} + I_b ln\frac{1}{D_1} - (I_a + I_b) ln\frac{1}{D_3}\right) + I_a\frac{\mu}{2}$$

or, since

$$-(I_a + I_b) = I_c$$

$$\phi_a = 2\left(I_a ln\frac{1}{r} + I_b ln\frac{1}{D_1} + I_c ln\frac{1}{D_3}\right) + I_a\frac{\mu}{2}. \tag{144}$$

Similarly,

$$\phi_b = 2\left(I_a ln\frac{1}{D_1} + I_b ln\frac{1}{r} + I_c ln\frac{1}{D_2}\right) + I_b\frac{\mu}{2}. \tag{145}$$

$$\phi_c = 2\left(I_a ln\frac{1}{D_3} + I_b ln\frac{1}{D_2} + I_c ln\frac{1}{r}\right) + I_c\frac{\mu}{2}. \tag{146}$$

The above are the complex linkages per centimeter of wire in absolute units of current. To change these to practical units per mile of wire and $\log_{10}$, the quantities in the brackets must be multiplied by the conversion 370.56×10^{-6}, while the last term of each equation is multiplied by the factor 160.94×10^{-6}. Thus, in henries per mile and amperes,

$$\phi_a = 741.13 \times 10^{-6}\left(I_a \log_{10}\frac{1}{r} + I_b \log_{10}\frac{1}{D_1} + I_c \log_{10}\frac{1}{D_3}\right) + 80.47 \times 10^{-6}\,\mu I_a \tag{147}$$

$$\phi_b = 741.13 \times 10^{-6}\left(I_a \log_{10}\frac{1}{D_1} + I_b \log_{10}\frac{1}{r} + I_c \log_{10}\frac{1}{D_2}\right) + 80.47 \times 10^{-6}\,\mu I_b \tag{148}$$

$$\phi_c = 741.13 \times 10^{-6}\left(I_a \log_{10}\frac{1}{D_3} + I_b \log_{10}\frac{1}{D_2} + I_c \log_{10}\frac{1}{r}\right) + 80.47 \times 10^{-6}\mu I_c. \tag{149}$$

Given the three complex potential differences at the supply end of the above three-phase line, together with the three expres-

sions for the complex currents, I_a, I_b and I_c, the drop in each wire can be calculated, and, by subtracting these from the supply voltages, the receiver voltages may be found.

Let the three potential differences to neutral at the supply end be

$$\left.\begin{aligned} E_a &= E(1 + j0) \\ E_b &= -\frac{E}{2}(1 + j\sqrt{3}) \\ E_c &= \frac{E}{2}(-1 + j\sqrt{3}) \end{aligned}\right\} \qquad (150)$$

and assume the three complex currents in the wires to be

$$\left.\begin{aligned} I_a &= I_{a_1}(\cos\theta - j\sin\theta) \\ I_b &= I_{b_1}\left[\cos\left(\theta + \frac{2\pi}{3}\right) - j\sin\left(\theta + \frac{2\pi}{3}\right)\right] \\ & I_{c_1}\left[\cos\left(\theta + \frac{4\pi}{3}\right) - j\sin\left(\theta + \frac{4\pi}{3}\right)\right] \end{aligned}\right\} \qquad (151)$$

where I_{a_1}, I_{b_1} and I_{c_1}, are the amplitudes of the corresponding complex currents I_a, I_b and I_c. The impedance drop in conductor a is

$$\begin{aligned} Z_a I_a &= (R + j\omega L_a)I_a \\ &= RI_a + j\omega\phi_a. \end{aligned} \qquad (152)$$

By substituting in Eq. (152) the value of I_a from Eq. (151), and the value of ϕ_a from Eq. (147), the drop may be calculated, since the currents I_a, I_b and I_c are all assumed to be known and defined by Eq. (151). The impedance drops in the remaining two conductors may be found in a similar manner.

Conductor Arrangements and Transpositions.—The conductors of three-phase transmission lines are supported on poles or towers in various arrangements, depending upon the type of supports used, the voltage of the line, the number of circuits per tower line, etc. In general, however, the three conductors are supported either at the three corners of a triangle, or they all lie in a single plane, usually a horizontal plane. When the triangular arrangement is used, and the sides of the triangle are all equal, the three loops of the line are balanced. That is, the inductive and condensive reactances of all loops are the same. This is the only arrangement of untransposed conductors for which such balance exists. While it is general practice to transpose the conductors of power lines in order to minimize possible inductive inter-

ference with adjacent lines, particularly with communication circuits, it is desirable to transpose three-phase lines having unsymmetrical conductor arrangements in order to secure like performance in the three phases.

Balanced phases are secured in three-phase lines by so arranging the conductors that each of the three conductors occupies each of the three possible positions throughout a total of one-third of the length of the line. Two complete transpositions of the line, dividing it into three equal sections, is the minimum number which will accomplish the desired result.

Equilateral Arrangement. Value of L. In this type of arrangement the conductors form the three edges of an equiangular prism, as illustrated in Fig. 29. The spacings are equal, and the flux through the loop formed by any two conductors is independent of the magnetizing force of the current in the third conductor. Thus for loop AB, since $D_1 = D_2 = D_3 = D$,

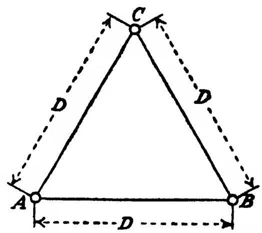


FIG. 29.—Equilateral spacing for a three-phase line.

$$\dot{\phi}_{ab} = 2ln\frac{D}{r}(I_a - I_b) \text{ vector linkages.} \tag{153}$$

From symmetry it is apparent that the partial self-inductances per unit length are equal for all conductors; that is, ${}_1L_a = {}_1L_b = {}_1L_c = L_1$. Also, the resultant vector linkages of a loop are equal to the vector differences of the linkages contributed by the individual conductors. Thus,

$$\phi_{ab} = L_1(I_a - I_b) \text{ vector linkages} \tag{154}$$

whence, from Eq. (153),

$$L_1 = 2ln\frac{D}{r} \text{ abhenries per centimeter of conductor.} \tag{155}$$

If we add the linkages per ampere within the conductor to the result expressed by Eq. (155), substitute $\log_{10}$ and write the result in terms of henries per mile of one conductor, the total inductance per mile of one conductor is found. It is

$$L = (741.13 \log_{10} \frac{D}{r} + 80.47\mu)10^{-6} \text{ henries per mile of one conductor.} \tag{156}$$

This equation is identical with Eq. (133). That is, the inductance of one conductor of a three-phase equilateral line is the same as that for a single-phase line having like conductors and spacings.

Unsymmetrically Arranged, Transposed, Three-phase Lines. Value of L.—Consider a three-phase line arranged as illustrated in Fig. 30, in which the sides D_1, D_2 and D_3 of the triangle may have any possible relative values. The values of D_1, D_2 and D_3 are large as compared with r, so that $(D - r) \div r$ is approximately equal to $D \div r$. Assume the line to be transposed so that the three phases are balanced. In short lines, so far as the power circuit alone is concerned, regardless of how many complete transpositions the line may actually have (so long as the number

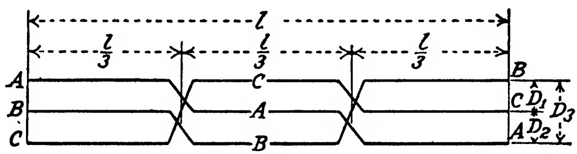


FIG. 30.—Unsymmetrically arranged three-phase line with transpositions.

is not zero), the line is approximately electrically equivalent to a like line having only one complete spiral, as illustrated in the figure. For very long lines this is not strictly true, since both the line voltages between conductors and the currents in them may differ considerably between the two ends of a line.

Since the line is transposed, each of the conductors in turn occupies each of the three possible positions in the circuit. The numerical value of the flux-linkages is the same for all conductors, and all conductors have the same coefficient of self-inductance. It will, therefore, be necessary to find the average vector flux linkages for one loop only. This is done by computing the linkages per centimeter of loop for each of the three conductor arrangements in the transposed line and averaging them. Thus (neglecting the linkages within the conductors), for the loop ab, the linkages in each of the positions are

$$_1\phi_{ab} = 2\left(I_a ln\frac{D_1}{r} - I_b ln\frac{D_1}{r} + I_c ln\frac{D_2}{D_3}\right)$$

$$_2\phi_{ab} = 2\left(I_a ln\frac{D_2}{r} - I_b ln\frac{D_2}{r} + I_c ln\frac{D_3}{D_1}\right)$$

$$_3\phi_{ab} = 2\left(I_a ln\frac{D_3}{r} - I_b ln\frac{D_3}{r} + I_c ln\frac{D_1}{D_2}\right)$$

and the average flux linkages, per centimeter of loop of the transposed line is one-third the sum of the above partial linkages; or

$$\phi_{ab} = \frac{2}{3}(I_a - I_b)ln\frac{D_1D_2D_3}{r^3}. \tag{157}$$

Since the line is transposed, and therefore electrically balanced, the partial inductances of the three like conductors are equal. Representing these as ${}_1L_a$, ${}_1L_b$ and ${}_1L_c$, we have

$${}_1L_a = {}_1L_b = {}_1L_c = L_1.$$

Furthermore, the partial inductance of the loop is such that

$${}_1L_a(I_a - I_b) = \phi_{ab} = \frac{2}{3}(I_a - I_b)ln\frac{D_1D_2D_3}{r^3}$$

or

$$L_1 = {}_1L_a = \frac{2}{3}ln\frac{D_1D_2D_3}{r^3} \text{ abhenries per centimeter.} \tag{158}$$

Adding to this the coefficient due to the linkages lying within the conductor, and reducing to practical units and $\log_{10}$,

$$L = \left(741.13 \log_{10}\frac{\sqrt[3]{D_1D_2D_3}}{r} + 80.47\mu\right) \times 10^{-6}$$

henries per mile of conductor. (159)

Equivalent Spacing.—If we assume an untransposed loop of an equilaterally arranged line of spacing D', such that its inductance per unit length of conductor is equal to that found in Eq. (159) for the unequally spaced, transposed line, the spacing D' may be called the *equivalent equilateral spacing* of the unequally spaced line. This is equivalent to equating the right-hand members of Eqs. (155) and (158). That is,

$$L_1 = 2ln\frac{D'}{r} = \frac{2}{3}ln\frac{D_1D_2D_3}{r^3}$$

and

$$D' = \sqrt[3]{D_1D_2D_3}. \tag{160}$$

Thus, the partial inductance L_1 per mile of conductor of any transposed, irregularly spaced line (or of an equilaterally spaced line whether transposed or not), may be calculated from Eq. (156) if the equivalent spacing given by Eq. (160) is used.

For the so-called flat spacing arrangement, the three conductors lie in a single plane, usually a horizontal or a vertical plane for high-voltage lines. Furthermore, the distance between conductors is usually such that $D_1 = D_2 = \frac{D_3}{2}$. For this case the equivalent separation is

$$\begin{aligned} D' &= D\sqrt[3]{2} \\ &= 1.26D. \end{aligned} \tag{161}$$

The Inductance of Untransposed, Double-circuit, Three-phase Lines.—Consider two untransposed, three-phase circuits as in Fig. 31, in which opposite conductors constitute the two conductors of a phase. The conductors of each pair are in parallel between generating and receiving station busses, and hence have the same impedance drop. Furthermore, the vector sum of the six currents, I_a, $I_{a'}$, I_b, $I_{b'}$, I_c, and $I_{c'}$ must equal zero. These are the basic conditions from which inductance of each wire may be calculated. If the arrangement is a symmetrical one, as in this instance, the currents are equal in pairs, whence

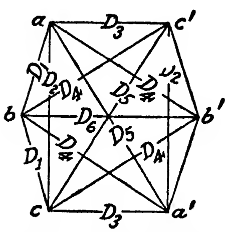


Fig. 31.—The double-circuit three-phase line.

$$\begin{aligned} I_a &= I_{a'} \\ I_b &= I_{b'} \\ I_c &= I_{c'}. \end{aligned}$$

In this case there are only three unknown currents instead of six, and the problem is accordingly considerably simplified.

It is apparent that in any unsymmetrical arrangement the inductance will have a different value for each conductor, while, with a symmetrical arrangement like that illustrated here, the numerical inductances of the outside wires a and c are equal for balanced currents; but the latter will not equal the inductance of b, except in the special case when $D_1 = D_3$.

The method of solving for L will be illustrated for the two-circuit, three-phase line of Fig. 31, in which half the current of a phase flows in each of the corresponding phase cables, as stated above. Let r be the radius of each of the equal cables, let S be a very large but finite distance, and let both r and S be measured in the same units as the distances D_1, D_2, etc. Assume the vector currents to be given by the equations

$$\left.\begin{aligned} I_a &= I_{a'} = \frac{I_0}{2}(1 + j0) \\ I_b &= I_{b'} = \frac{-I_0}{4}(1 + j\sqrt{3}) \\ I_c &= I_{c'} = \frac{I_0}{4}(-1 + j\sqrt{3}) \end{aligned}\right\} \tag{162}$$

The total vector flux linkages, per centimeter for the three conductors, are

$$\left.\begin{aligned} \phi_a &= 2\left(I_a ln\frac{S}{r} + I_b ln\frac{S}{D_1} + I_c ln\frac{S}{D_2}\right. \\ &\qquad \left. + I_{a'} ln\frac{S}{D_5} + I_{b'} ln\frac{S}{D_4} + I_{c'} ln\frac{S}{D_3}\right) + I_a\frac{\mu}{2} \\ \phi_b &= 2\left(I_a ln\frac{S}{D_1} + I_b ln\frac{S}{r} + I_c ln\frac{S}{D_1}\right. \\ &\qquad \left. + I_{a'} ln\frac{S}{D_4} + I_{b'} ln\frac{S}{D_1} + I_{c'} ln\frac{S}{D_4}\right) + I_b\frac{\mu}{2} \\ \phi_c &= 2\left(I_a ln\frac{S}{D_2} + I_b ln\frac{S}{D_1} + I_c ln\frac{S}{r}\right. \\ &\qquad \left. + I_{a'} ln\frac{S}{D_3} + I_{b'} ln\frac{S}{D_4} + I_{c'} ln\frac{S}{D_5}\right) + I_c\frac{\mu}{2} \end{aligned}\right\} \tag{163}$$

Substituting in these equations the values of I_a, $I_{a'}$, I_b, $I_{b'}$, I_c and $I_{c'}$, in terms of I_0, from Eq. (162), we get

$$\phi_a = I_0(1 + 0)\left(\frac{1}{2}ln\frac{D_1D_2D_3D_4}{r^2D_5^2} + \frac{\mu}{4} + j\frac{\sqrt{3}}{2}ln\frac{D_1D_4}{D_2D_3}\right) \tag{164}$$

$$\phi_b = -I_0\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(ln\frac{D_1D_4}{rD_6} + \frac{\mu}{4}\right) \tag{165}$$

$$\phi_c = I_0\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2}ln\frac{D_1D_2D_3D_4}{r^2D_5^2} + \frac{\mu}{4} - j\frac{\sqrt{3}}{2}ln\frac{D_1D_4}{D_2D_3}\right). \tag{166}$$

Since, in general, $LI = \phi$, it is apparent that in absolute units the inductances are obtained by dividing the vector flux linking each conductor by the vector current in the conductor. Performing this operation and reducing to practical units yields the inductances given below:

$$L_a = 10^{-5}\left(18.53 \log_{10}\frac{D_1D_2D_3D_4}{r^2D_5^2} + 40.2\mu + j32.09 \log_{10}\frac{D_1D_4}{D_2D_3}\right) \text{ henries per mile} \tag{167}$$

$$L_b = 10^{-5}\left(37.06 \log_{10}\frac{D_1D_4}{rD_5} + 40.2\mu\right) \text{ henries per mile} \tag{168}$$

$$L_c = 10^{-5}\left(18.53 \log_{10}\frac{D_1D_2D_3D_4}{r^2D_5^2} + 40.2\mu - j32.09 \log_{10}\frac{D_1D_4}{D_2D_3}\right) \text{ henries per mile.} \tag{169}$$

Inductance of Transposed, Double-circuit, Three-phase Line. It will be of interest to consider the effect on the inductance of a transposed, three-phase line of the close proximity, of a second similar transposed line, operated in parallel with the first.

Let the lines be spaced as in Fig. 31 and assume each of the lines to have sufficient transpositions to balance the phases. It is assumed further that the paralleled pairs consist of the conductors a and a', b and b' and c and c', respectively, and that the transpositions are so made that the conductors of a pair such as a and a', etc., are always opposite each other in the figure. The inductance per centimeter of conductor will then be the same for all conductors.

There are six currents, each presumably contributing something to the linkages encircling a given conductor, or linking a given loop. Because of the symmetrical construction, however, and since the two lines are operated in parallel, the vector currents are equal in pairs, whence

$$
\begin{aligned}
I_a &= I_{a'} \\
I_b &= I_{b'} \\
I_c &= I_{c'}
\end{aligned}
$$

so that, in reality, only three currents remain to be considered.

Writing the partial flux-linkages per centimeter for the loop ab in each of its three positions, we find that

$$
\left.
\begin{aligned}
{}_1\phi_{ab} &= 2\Big(I_a ln\frac{D_1}{r} - I_b ln\frac{D_1}{r} - I_c ln\frac{D_2}{D_1} - I_a ln\frac{D_5}{D_4} \\
&\qquad\qquad - I_b ln\frac{D_4}{D_6} + I_c ln\frac{D_4}{D_3}\Big) \\
{}_2\phi_{ab} &= 2\Big(I_a ln\frac{D_1}{r} - I_b ln\frac{D_1}{r} + I_c ln\frac{D_2}{D_1} + I_a ln\frac{D_4}{D_6} \\
&\qquad\qquad + I_b ln\frac{D_5}{D_4} + I_c ln\frac{D_3}{D_4}\Big) \\
{}_3\phi_{ab} &= 2\Big(I_a ln\frac{D_2}{r} - I_b ln\frac{D_2}{r} + I_c ln\frac{D_1}{D_1} - I_a ln\frac{D_5}{D_3} \\
&\qquad\qquad + I_b ln\frac{D_5}{D_3} + I_c ln\frac{D_4}{D_4}\Big)
\end{aligned}
\right\} \qquad (170)
$$

Adding the above partial linkages and averaging by dividing by 3,

$$
\phi_{ab} = \frac{2}{3}\left[(I_a - I_b) ln\frac{D_1^2 D_4^2 D_2 D_3}{r^3 D_5^2 D_6}\right]. \qquad (171)
$$

Since, for a balanced line,

$$\phi_{ab} = L_1(I_a - I_b)$$

$$L_1 = 2ln\frac{1}{r}\sqrt[3]{\frac{D_4^2D_1^2D_2D_3}{D_5^2D_6}} \text{ abhenries per centimeter of wire.} \quad (172)$$

Adding the coefficient due to the linkages within the conductor and converting to practical units,

$$L = 10^{-6}\left(741.13\log_{10}\frac{1}{r}\sqrt[3]{\frac{D_1^2D_4^2D_2D_3}{D_5^2D_6}} + 80.45\mu\right) \text{ henries per mile of one wire.} \quad (173)$$

Example.—The 250,000-cir. mil conductors of a double-circuit, 110-kv. three-phase line are spaced much as in Fig. 31. The vertical separation is 10 ft., the horizontal offset of the middle conductor is 5 ft., and the horizontal spacing D is 20 ft. The lines are transposed as described above. Required to find, (*a*) the reactance at 60 cycles per mile of conductor, assuming that one of the circuits is removed; (*b*) the reactance at 60 cycles per mile of conductor, assuming the presence of both lines with the spacings given; (*c*) the per cent change in reactance based on the reactance of a single line.

Solution.—Using the spacings given, the following values are found:

$$D_1 = \sqrt{10^2 + 5^2} = 11.18 \text{ ft.} \qquad D_4 = \sqrt{10^2 + 25^2} = 26.93 \text{ ft.}$$
$$D_2 = \qquad = 20 \text{ ft.} \qquad D_5 = \sqrt{20^2 + 20^2} = 28.25 \text{ ft.}$$
$$D_3 \qquad = 20 \text{ ft.} \qquad D_6 = \qquad = 30.0 \text{ ft.}$$
$$r = \frac{0.576}{2 \times 12} = 0.0240 \text{ ft.}$$

For the single line,

$$\frac{D'}{r} = \frac{\sqrt[3]{11.18 \times 11.18 \times 20}}{0.0240} = 565.8$$
$$\log_{10} 565.8 = 2.753$$
$$x = \omega L = 377(741.13 \times 2.753 + 80.45)10^{-6}$$
$$= 0.7995 \text{ ohm per mile.}$$

For the double-circuit line,

$$\frac{D'}{r} = \frac{1}{r}\sqrt[3]{\frac{11.18^2 \times 26.93^2 \times 20 \times 20}{28.28^2 \times 30}}$$
$$= 476.3$$
$$\log_{10} 476.3 = 2.6778$$
$$x = \omega L = 377(741.13 \times 2.6778 + 80.45)10^{-6}$$
$$= 0.778 \text{ ohm per mile.}$$

Due to the presence of a second, similar circuit, the reactance of each circuit is decreased by the percentage,

$$\frac{100(0.799 - 0.778)}{0.799} = 2.63 \text{ per cent.}$$

Influence of Stranding and Spiraling on the Value of L.—Conductors of the larger sizes are made up of varying numbers of smaller, round wires twisted about a central conductor or group of conductors to form a stranded cable, as illustrated in Fig. 32, for a seven-strand cable. Owing to the somewhat larger space occupied by such a conductor, as compared with a solid, circular conductor of equal area, its inductance is somewhat smaller than for the equivalent solid rod. (The equivalent solid rod is a solid, round conductor having the same conducting area as the stranded cable.)

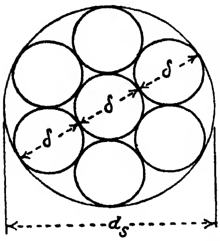


FIG. 32.—Section of a seven-strand cable.

The exact value for a given cable depends upon its size, the number of strands and the pitch of the spiral used. Spiraling tends to increase the inductance of the cable, since it produces a solenoidal effect. Since the pitch of the spiral is large, however, this effect is quite negligible in transmission-line cables.

Equations[1] have been developed for calculating the inductances of transmission-line cables for cables of standard stranding, and for the various commercial sizes. There is little to be gained, however, by attempting too great a refinement in the calculation of inductance. For variations from the assumed values, in the spacings and in the actual lengths of cables, together with the influence of the tower in somewhat increasing the inductance where steel towers are used, may leave a larger margin of uncertainty than that introduced by the use of the approximate equation.

It is customary, in dealing with stranded cables for aerial power transmission work, to calculate their inductances by the use of Eq. (156), in which, however, it is preferable that the value of r be taken as the radius of the equivalent solid rod. This substitution yields values of inductance which are somewhat too high. The error is therefore on the safe side.

For example, a seven-strand cable is made up of seven round wires of diameter δ each, the seven being twisted together to form a cable of outside diameter $d_s = 3\delta$. Neglecting the effect of spiraling, the cross-sectional area of the cable in circular mils is

$$\text{cir. mil} = 7\delta^2 = 7\left(\frac{d_s}{3}\right)^2.$$

[1] DWIGHT, H. B., "Transmission Line Formulas."

If d be the diameter of the equivalent solid rod, it follows that

$$7\left(\frac{d_s}{3}\right)^2 = d^2$$

or

$$d_s = 3\sqrt{\frac{d^2}{7}}$$

and

$$d_s = 1.1338d.$$

The value of r to substitute in Eq. (133) is

$$r = \frac{d_s}{2 \times 1.1338}.$$

Table 6 gives the values of the overall diameters in terms of circular mil areas, and the corresponding ratios of $d_s \div d$ for cables of standard stranding up to cables of 127 strands.

TABLE 6

Number of strands	d_s	$d_s \div d$
1	$\sqrt{\text{cir. mil}}$	1
7	$3\sqrt{\frac{\text{cir. mil}}{7}}$	$\frac{3}{\sqrt{7}} = 1.1338$
19	$5\sqrt{\frac{\text{cir. mil}}{19}}$	$\frac{5}{\sqrt{19}} = 1.1471$
37	$7\sqrt{\frac{\text{cir. mil}}{37}}$	$\frac{7}{\sqrt{37}} = 1.1508$
61	$9\sqrt{\frac{\text{cir. mil}}{61}}$	$\frac{9}{\sqrt{61}} = 1.1523$
91	$11\sqrt{\frac{\text{cir. mil}}{91}}$	$\frac{11}{\sqrt{91}} = 1.1531$
127	$13\sqrt{\frac{\text{cir. mil}}{127}}$	$\frac{13}{\sqrt{127}} = 1.1536$

PROBLEMS

1. The two round conductors of a parallel-sided loop are each 0.60 in. in diameter. The distance between centers of conductors is 12 ft. The current in the conductors is 225 amp. In a sectional view taken normal to the plane of the loop, draw 5 lines of equal magnetic potentials, so spacing the lines that the magnetic potential drop between each 2 adjacent lines shall be one-eighth of the total magnetomotive force of the loop. Let the

straight line joining the two conductors be one of the lines, as well as the axis of symmetry, of the completed figure.

2. For the circuit of Problem 1, in the sectional view, draw magnetic lines of force so spaced that each 2 adjacent lines shall include between them one-tenth of the total flux through the loop. Let the straight line drawn normal to the plane of the loop and midway between conductors, be one of the lines, as well as the axis of symmetry of the figure. How much flux passes between adjacent lines per mile of circuit when the current is 225 amp.?

3. An untransposed, split-conductor, single-phase, 60-cycle line, 2,000 ft. long, is built of four 00 copper conductors, each having a resistance of 0.46 ohm per mile. The conductors lie in a horizontal plane, and reading from left to right, their order is a, b, a', b'. The conductors ab form the outgoing pair, while $a'b'$ are the corresponding return conductors. The distances of separation ab and $a'b'$ are each 2 ft., while ba' is 4 ft. When the impressed voltage is 2,300 volts, and the load current is 220 amp., what is the potential difference $E_{ab'}$ at the receiver end of the line?

4. An untransposed, three-phase, 60-cycle line has three 0000 conductors of diameters 0.533 in. spaced 6 ft. apart in a horizontal plane. The resistance of the conductors at working temperatures is 0.28 ohm per mile. The currents in the conductors are unbalanced and equal to

$$\left.\begin{aligned} I_a &= 156 - j\,90 \\ I_b &= -111 - j\,108 \\ I_c &= -\,45 + j\,198 \end{aligned}\right\} \text{vector amp.}$$

The supply voltages to neutral are balanced, and have the following complex values:

$$\left.\begin{aligned} E_a &= 3{,}800(1 + j0) \\ E_b &= -1{,}900(1 + j\sqrt{3}) \\ E_c &= 1{,}900(-1 + j\sqrt{3}) \end{aligned}\right\} \text{vector volts.}$$

Find the complex expressions for the line-to-line voltages E_{ab}, E_{bc} and E_{ca} at the receiver end.

5. What is the 60-cycle, inductive reactance per mile of a conductor for the transposed, double-circuit line in Problem 5, Chap. IV?

CHAPTER IV

THE DIELECTRIC CIRCUIT AND CAPACITANCE

Any space in which a charge of electricity experiences a force, may be called a dielectric field. Dielectric fields are represented by dielectric *lines of force* in much the same way as magnetic fields are represented by magnetic lines of force. At any point in a dielectric field the direction and sense of the dielectric lines of force indicate the direction and sense of the resultant force on unit positive quantity of electricity at that point. The magnitude of the force is indicated by the number of lines crossing per unit of area taken normal to their path, that is, by the dielectric flux density.

Concepts, Definitions and Units.—Unless otherwise stated, the units used in the definitions and discussions which follow are the *electrostatic units*.

Measurements of dielectric field intensities are based on the concept of unit point charge of electricity. The charge is assumed to be collected at a point, and to be of such amount that, in air and when separated from an equal charge of like sign, it will experience a force of 1 dyne. Such a charge is called a *unit point charge.*

The *electrical potential of a point* is the work done on a unit positive point charge in bringing it from the edge of the dielectric field, where the force on the charge is zero, up to the point in question.

The *field intensity* of a dielectric field is the force in dynes experienced by a unit positive point charge in the field. It is the electrical *potential gradient* at the point, measured along a line of force. This follows from the definition of potential given above. When measured in statvolts per centimeter the gradient is designated by the symbol K; when practical electromagnetic units are used it is measured in volts per centimeter, for which the symbol is G. G and K are related by the equation

$$G = \frac{v^2K}{10^9} \tag{174}$$

where v is the velocity of propagation of the electric field in centimeters per second, and is equal approximately to the velocity of light.

That is,

$$v = 3 \times 10^{10} \text{ cm. per second.} \tag{175}$$

Since the direction of a dielectric line of force represents the direction of the resultant dielectric field at every point, the electrical *potential difference* between any two points in the field is the integral of the field intensity taken along a line of force. The field intensity is

$$K = \frac{de}{dl \cdot \cos \theta} \tag{176}$$

from which

$$e = \int_{(L)} K \cdot dl \cdot \cos \theta \tag{177}$$

where e = the potential difference between the two points defined by the limits of the integration.

dl = an elementary length of path.

θ = the angle between the direction of the path and the direction of a line of force at every point.

The unit of dielectric flux is a *dielectric line of force.* The total flux crossing a given area is represented by the symbol ψ, while the *flux density* is

$$D = \frac{\psi}{A} = kK \tag{178}$$

where A is the area in square centimeters of path of the dielectric flux, taken normal to its direction. The unit of density is 1 line per square centimeter.

In a medium of air, a field intensity of 1 statvolt per centimeter sets up a flux density of 1 dielectric line per square centimeter. This is equivalent to saying that for the abstat system of units the dielectrical conductivity of air per cubic centimeter is unity. Materials other than air have various conductivities, and, in them, a field intensity of unity will accordingly produce various dielectric flux densities, depending upon the nature of the material The ratio

$$k = \frac{D}{K} \tag{179}$$

of flux density to field intensity, in a dielectric field, is called the *permittivity* of the material in which the field is established.

The *law of the dielectric circuit,* analogous to Ohm's law for the electric and magnetic circuits, is

$$E = \frac{\psi S}{4\pi}$$

or

$$\psi = \frac{4\pi E}{S} \tag{180}$$

where E is the total potential difference required to set up the dielectric flux ψ over the length l of the dielectric circuit. The constant $S = \frac{4\pi l}{kA}$ is called the *elastance* of the dielectric circuit, analogous to the reluctance of a magnetic circuit. In practical electromagnetic units, the elastance is

$$S = \frac{4\pi v^2 l}{10^9 kA} \text{ darafs} \tag{181}$$

whence, using volts of potential difference and c.g.s. units of length and area, the law of the dielectric circuit, in practical electromagnetic units, becomes

$$E = \frac{v^2 \psi l}{10^9 kA} \text{ volts.} \tag{182}$$

When a difference of electrical potential is applied to a dielectric circuit, thereby producing a dielectric field, variations in potential difference are accompanied by corresponding changes in the dielectric flux. The varying flux is accompanied by a flow of electricity, that is, by an electric current, which flows in such a direction as to oppose the change in potential which causes it. The collapse of each unit of dielectric flux may be thought of as releasing a definite quantity of electricity. The current set up is proportional to the negative rate of change of flux and hence also to the negative rate of change of the potential difference. In abstat units, these relations are expressed by the equation,

$$i = -\frac{Cde}{dt} = -\frac{d\psi}{dt} \tag{183}$$

where C is the proportionality factor between the time rate of change of potential difference and the current. From Eq. (183) it follows that

$$CE = \frac{\psi}{4\pi} = \frac{E}{S}$$

and

$$C = \frac{\psi}{4\pi E} = \frac{1}{S} \tag{184}$$

For any electrical circuit the constant C as defined by Eq. (184) is called the *capacitance* of the circuit. In stat units it is the number of dielectric flux lines set up per statvolt of potential difference impressed, called the *statfarad*.

The practical unit of capacitance is the *farad*. This unit is related to the stat unit as shown in Eq. (185).

$$\text{Farads} = \text{statfarads} \times \frac{10^9}{v^2} \tag{185}$$

By multiplying Eq. (183) through by $e \cdot dt$ and integrating, the stored energy of the dielectric field, when the potential difference is E, is found to be

$$E = \frac{E^2C}{2} \text{ joules} \tag{186}$$

where C is in farads and E is in volts.

Potential Gradient and Capacitance.—In dealing with the circuits of transmission systems, it is frequently necessary to compute both the potential gradients about the conductors or within the insulating materials which isolate them, and the capacitances between the various conductors or between one conductor and neutral. The potential gradient must be kept below the value at which the dielectric medium breaks down, and the capacitance must be known in order that the operating characteristics of the circuit may be predetermined. In the simplest cases these calculations are quite readily made or at least may be closely approximated. The general method of making them is much the same in all cases, as will be apparent from the illustrations which follow.

The Parallel-plate Condenser.—A condenser consisting of two equal parallel plates separated by a homogeneous dielectric

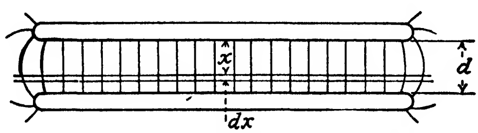


Fig. 33.—The parallel-plate condenser.

material is perhaps the simplest form of a capacitance. Such a condenser is illustrated in Fig. 33.

Let d = the distance between plates in centimeters.
A = the number of square centimeters area per plate.
E = the potential difference between the equipotential plates.
ψ = the total number of dielectric lines passing between plates.

Let it be assumed that, compared with the area of the plates, the distance of separation is small, so that the distortion of the field around the edges of the plates is negligible, and the entire field may be considered uniform. The elastance of an elementary solid, with lateral faces parallel to the plates of the condenser, of width dx and of area equal to that of one of the plates is

$$\begin{aligned} dS &= \frac{4\pi v^2 dx}{10^9 kA} \\ &= 4\pi\lambda\frac{dx}{A} \end{aligned} \tag{187}$$

where k = the permittivity of the dielectric material.

$\lambda = \dfrac{v^2}{10^9 k}$ (used to simplify the notation).

The drop of potential between parallel faces of the elementary solid is

$$\begin{aligned} de &= \frac{\psi dS}{4\pi} \\ &= \lambda\psi\frac{dx}{A}. \end{aligned} \tag{188}$$

The potential gradient in the dielectric is

$$\begin{aligned} G &= \frac{de}{dx} \\ &= \frac{\lambda\psi}{A} \text{ volts per centimeter.} \end{aligned} \tag{189}$$

The gradient is seen to be independent of x and therefore constant throughout the dielectric.

The total flux in the dielectric is obtained from Eq. (188) by equating the integral of the right-hand member, taken between the limits of 0 and d of the variable, to the potential difference between plates, whence

$$E = \frac{\lambda\psi d}{A} \text{ volts}$$

and

$$\psi = \frac{AE}{\lambda d} \text{ lines.} \tag{190}$$

By Eq. (184) the capacitance of the condenser is

$$C = \frac{\psi}{4\pi E}$$

$$= \frac{A}{4\pi\lambda d} \text{ farad} \tag{191}$$

$$= \frac{8.842kA \times 10^{-8}}{d} \text{ mf.} \tag{192}$$

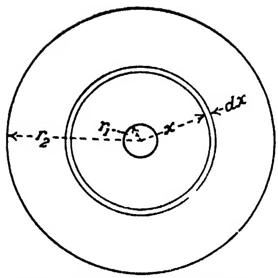


FIG. 34.—The lead-covered cable.

Concentric Cylinders.—The single-conductor, lead-covered cable illustrates this case. It consists of a single conductor of circular cross-section covered with a certain thickness of homogeneous, insulating material, the whole, in turn, being enclosed in a lead sheath, as in Fig. 34. Let the sheath of the cable be grounded, and let the potential difference of the conductor to neutral be E_0. It is assumed that the drop of potential in the conductor is negligible, and hence both the conductor and the sheath are equipotential surfaces.

Let r_1 = radius of conductor in centimeters.
r_2 = inside radius of the sheath.
ψ = the total number of dielectric lines per centimeter of conductor length, passing between core and sheath.

The elastance of any elementary, concentric cylinder of the dielectric, of unit length, of radius x and of thickness of wall dx is

$$dS = \frac{4\pi v^2 dx}{10^9 \cdot 2\pi kx}$$

$$= \frac{2\lambda dx}{x}.$$

The potential drop across the walls of the elementary cylinder is

$$de = \frac{\psi dS}{4\pi}$$

$$= \frac{2\lambda\psi dx}{4\pi x} = \frac{\lambda\psi dx}{2\pi x}$$

whence the total potential difference between the core and sheath is

$$E_0 = \frac{\lambda\psi}{2\pi}\int_{r_1}^{r_2}\frac{dx}{x}$$

$$= \frac{\lambda\psi}{2\pi}\psi ln\frac{r_2}{r_1} \text{ volts.} \tag{193}$$

Accordingly, the flux passing out from the core, per unit length of conductor, is

$$\psi = \frac{2\pi E_0}{\lambda ln\frac{r_2}{r_1}} \tag{194}$$

and the corresponding capacitance is

$$C_0 = \frac{\psi}{4\pi E_0}$$

$$= \frac{1}{2\lambda ln\frac{r_2}{r_1}} \text{ farads per centimeter} \tag{195}$$

$$= \frac{0.03883k}{\log_{10}\frac{r_2}{r_1}} \text{ mf. per mile.} \tag{196}$$

The potential gradient within the dielectric at a distance of x cm. from the center of the core is

$$G = \frac{de}{dx}$$

$$= \frac{\lambda\psi}{2\pi x} \text{ volts per centimeter.} \tag{197}$$

Substituting the value of ψ from Eq. (194) in Eq. (197) transforms the latter to

$$G = \frac{E_0}{x ln\frac{r_2}{r_1}} \text{ volts per centimeter.}$$

$$= \frac{0.4343E_0}{x\log_{10}\frac{r_2}{r_1}} \text{ volts per centimeter.} \tag{198}$$

Eq. (198) shows that, for this case, the gradient is maximum at the surface of the conductor and varies inversely as the distance from the center of the core.

The Dielectric Field about a Long, Straight Cylinder.—At any point distant r cm. from a long, straight cylinder suspended in a medium of constant permittivity k, the field intensity is given by the equation,

$$K = \frac{2Q}{kr}$$

$$= \frac{\psi}{2\pi kr} \text{ abstat units per centimeter} \tag{199}$$

where $\psi = 4\pi Q$, the number of dielectric lines leaving the conductor per centimeter of its length.

This conclusion results from a proof very similar to that given in the preceding chapter for the magnetic field intensity near a long straight wire.

Equation (199) is reduced to practical, electromagnetic units by introducing the conversion factor $\lambda = \frac{v^2}{10^9 k}$, whence, by Eq. (174),

$$G = \frac{v^2 K}{10^9}$$

$$= \frac{\lambda\psi}{2\pi r} \text{ volts per centimeter.} \tag{200}$$

Equation (200) shows that all points distant r cm. from the center of the cylinder have the same value of potential gradient. At all points about the conductor the gradient is radially directed. Since dielectric lines of force are everywhere drawn in the direction of the resultant

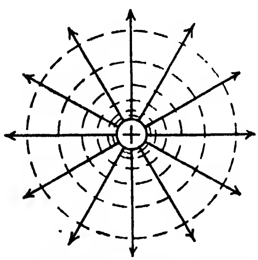


FIG. 35.—The dielectric field about an isolated straight wire.

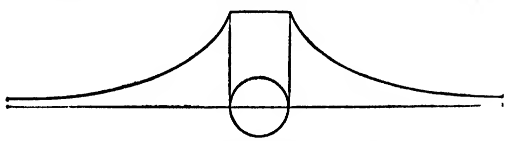


FIG. 36.—The potential gradient about a straight round wire.

gradient, they, too, are radial lines. Normal to these are the lines of equal potentials which, by Equation (199), are circles concentric with the conductor axis. In Fig. 35 these lines are shown, the equipotential circles as broken, and the lines of force as full lines. It should be noted that the lines of constant potential are also magnetic lines of force, while the dielectric lines of force are

the lines of constant magnetic potentials (compare Fig. 35, with Fig. 17).

Figure 36 shows how the potential gradient varies in the region on and outside of the isolated cylindrical conductor.

The Dielectric Field about Two Equal, Parallel, Cylindrical Conductors.—Let the equal, parallel, cylindrical conductors A and B of Fig. 37 be suspended in a medium of constant permittivity k. Assume that the distance of separation D of the conductors is great as compared with the radius of the conductor. Then the charges on the conductors, due to their potential difference, will be approximately uniformly distributed over the conductor surfaces. Let the equal charges on A and B, per unit length of conductor, be $+Q$ and $-Q$ units, respectively. These give rise to the corresponding, outwardly directed, equal dielectric fluxes $+\psi$ and $-\psi$ lines per centimeter. Let P be a point which moves along a line of the resultant dielectric field due to the potentials of A and B, and let the fixed point F, lying on the normal bisector of AB, be a point on the locus of P. The dielectric fluxes per centimeter length of conductor passing *outwardly* through the loop PF, are

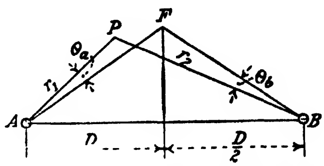


Fig. 37.—The dielectric lines of force between two equal, parallel, cylindrical conductors of a return loop are circles.

$$\left.\begin{aligned} \psi_a &= \frac{\psi\theta_a}{2\pi} \\ \psi_b &= -\frac{\psi\theta_b}{2\pi} \end{aligned}\right\} \tag{201}$$

Since P is assumed to move along a line of force of the resultant dielectric field, however, the potential gradient normal to its path must everywhere be zero, and the total outgoing flux through the loop PF is therefore likewise zero. Therefore,

$$\psi_a + \psi_b = 0$$

and

$$\theta_a = \theta_b. \tag{202}$$

This condition defines a circle passing through the points A, P, F and B. Thus the dielectric lines of force of the resultant field are circles. They are also the circles of constant, magnetic potentials, as already shown in the previous chapter (see p. 43).

The Lines of Constant Potential about Two Equal, Parallel, Cylindrical Conductors.—The constant-potential lines for this case are also circles and are identical with the circular magnetic lines of force already discussed in the previous chapter. The proof is exactly similar to that already given for the magnetic lines of force. It will be given, however, for the sake of completing the parallelism between the two cases.

Let A and B of Fig. 38 be two equal, parallel, cylindrical conductors, and let their surfaces be equipotential surfaces subjected to a potential difference of $E = 2E_0$ volts. The conductors are assumed to be widely separated so that the charges on their surfaces are uniformly distributed, and, so far as the region beyond the conductor surfaces are concerned, they are therefore electrically equivalent to equal charges uniformly distributed along the length of conductor filaments coincident with the axis of the conductors. Again, as before, let the outwardly going equal, uniformly distributed, radial fluxes per centimeter length of the conductors be $+\psi$ and $-\psi$. Let the point P move along a line of constant electrical potential, and let F be the point on the line AB where the locus of P crosses it, distant m and n centimeters from A and B, respectively.

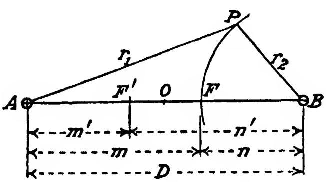


FIG. 38.—Lines of constant electrical potentials about two equal, parallel round wires of a return loop.

From the definition of potential difference and Eq. (200), the potential differences between F and P due to A and B each acting alone, are

$$E_a = \frac{\lambda\psi}{2\pi}\int_m^{r_1}\frac{dr}{r}$$

$$\left.\begin{aligned} E_a &= \frac{\lambda\psi}{2\pi}ln\frac{r_1}{m} \\ \text{and, similarly,} \\ E_b &= -\frac{\lambda\psi}{2\pi}ln\frac{r_2}{n} \end{aligned}\right\} \qquad (203)$$

Since, by assumption, P moves along a line of constant potential

$$E_a + E_b = 0$$

whence

$$\frac{r_1}{m} = \frac{r_2}{n}$$

and

$$\frac{r_1}{r_2} = \frac{m}{n} = \text{a constant.}$$

By the theorem of inverse points of a circle, this condition defines a circle passing through the points P and F. The equations of these circles are given in the previous chapter.

Figure 39 shows the two families of circles discussed in this and the previous article. The full lines represent the dielectric lines of force (also lines of constant, magnetic potentials), while the dotted lines are the lines of constant, electrical potential, and also the magnetic lines of force (Fig. 24).

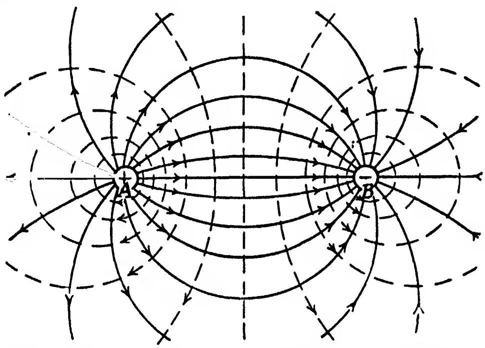


FIG. 39.—The dielectric lines of force between the two parallel, cylindrical conductors of a return loop.

Calculation of Capacitance. General Method.—The general method of solving for the capacitance between two conductors, or between one conductor and a plane of zero potential, has already been illustrated in the solution of the problem for the concentric-core cable. The procedure was: First, to solve for the flux emanating from unit length of the conductor for any assumed potential difference; and secondly, to compute the capacitance by dividing the flux by the impressed potential difference. The same general procedure will be followed in finding the capacity for single-phase and three-phase lines.

The Capacitance of Two Parallel, Round Wires. Approximate Equation for Single-phase Line.—Let the two equal, parallel, cylindrical conductors A and B of Fig. 38 be suspended in a medium of constant permittivity k; let r be the radius of each of the cylinders, and assume that the separation D is large as compared with r. (This condition always prevails in high-

tension, aerial-transmission work.) As before, assume that the potential difference $E = 2E_0$ is impressed across the conductors, and is of such value that the resulting, uniformly distributed charge produced per unit length of the conductors is $+Q$ and $-Q$ for A and B, respectively. The corresponding outwardly radiating dielectric fluxes per unit length of conductor will then be $+\psi$ and $-\psi$ lines.

At the point P, the component potential gradients due to the separate potentials of the two conductors, are, by Eq. (200),

$$\left.\begin{aligned} G_a &= \frac{\lambda\psi}{2\pi r_1} \text{ directed outward along } r_1 \\ G_b &= -\frac{\lambda\psi}{2\pi r_2} \text{ directed inward along } r_2 \end{aligned}\right\} \qquad (204)$$

The resultant gradient, made up of the vector sum of these two values, is at every point in the direction of the resultant line of force through P. The potential difference between any two points in a dielectric field is the line integral of the resultant potential gradient between the two points. Thus, in any plane normal to the conductors the potential drop between conductors is

$$E = \int_{(L)} G_0 \cos\theta \cdot dl$$

the integral to be taken from A to B over any path whatever,

where G_0 = the numerical value of the resultant gradient at P.
θ = the angle between the direction of the gradient and the path.
dl = the elementary length of path.

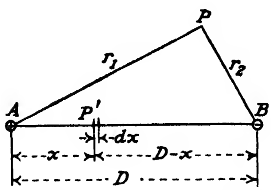


FIG. 40.—The capacitance of round, parallel wires.

This integral is most readily evaluated along the straight-line path from A to B, for along it the resultant gradient is at every point equal to the arithmetic sum of its two components, and its direction and sense are along the line AB from A towards B.

Thus, in Fig. 40, if P be moved to the position P', the component gradients become

$$G_a = +\frac{\lambda\psi}{2\pi x}$$

$$G_b = -\frac{\lambda\psi}{2\pi(D - x)}$$

and

$$\begin{aligned} G_0 &= G_a - G_b \\ &= \frac{\lambda\psi}{2\pi}\left(\frac{1}{x} + \frac{1}{D - x}\right). \end{aligned} \tag{205}$$

Therefore,

$$\begin{aligned} E &= \frac{\lambda\psi}{2\pi}\int_r^{D-r}\left[\frac{1}{x} + \frac{1}{D - x}\right]dx \\ &= \frac{\lambda\psi}{\pi} ln \frac{D - r}{r} \text{ volts} \end{aligned} \tag{206}$$

and

$$\psi = \frac{\pi E}{\lambda ln \dfrac{D - r}{r}}. \tag{207}$$

The capacitance between wires is

$$\begin{aligned} C &= \frac{\psi}{4\pi E} \\ &= \frac{1}{4\lambda ln \dfrac{D - r}{r}} \text{ farads per centimeter.} \end{aligned} \tag{208}$$

It is apparent that the potential of the point on AB midway between conductors is zero. Accordingly, if in deriving Eq. (206), the potential difference to neutral had been used in the left-hand member, and on the right the integration had been carried from r to $\frac{D}{2}$, the capacitance of one wire to neutral would result. It is just twice the value given in Eq. (208). Hence

$$C_0 = \frac{1}{2\lambda ln \dfrac{D - r}{r}} \text{ farads per centimeter (one wire to neutral).} \tag{209}$$

Since D is large compared with r, the substitution of D for $(D - r)$ is permissible. Making this substitution, writing in terms of $\log_{10}$, evaluating the constant, and giving k a value of unity for the permittivity of air, Eqs. (208) and (209) are converted to

Capacitance between two conductors,

$$C = \frac{3.68}{10^3 \log_{10} \frac{D}{r}} \text{ mf. per 1,000 ft.} \tag{210}$$

$$= \frac{0.01942}{\log_{10} \frac{D}{r}} \text{ mf. per mile} \tag{211}$$

Capacitance of one conductor to neutral

$$C_0 = \frac{7.36}{10^3 \log_{10} \frac{D}{r}} \text{ mf. per 1,000 ft.} \tag{212}$$

$$= \frac{3.883}{100 \log_{10} \frac{D}{r}} \text{ mf. per mile.} \tag{213}$$

While the above equations are not theoretically exact, the errors introduced by their use are entirely negligible for all practical transmission spacings. To illustrate, when the spacing of wires between centers is only five times the conductor diameter, these equations yield results 0.36 per cent too low. The error diminishes rapidly as the spacing increases.

The Exact Value of Capacitance between Two Parallel Cylinders.—When the parallel conductors are close together, more exact equations than those given above may be desired. Exact expressions covering the case of two parallel, round conductors are developed below.

It has already been shown that if the potential drop in the conductors themselves be neglected, the equipotential lines linking a parallel-sided loop of two circular conductors are a family of circles to which the equipotential surfaces of the conductors themselves belong. Since, however, no restriction is now placed on the distance of separation between the two opposite sides of the loop, it is no longer permissible to assume the filaments, along which the charges are uniformly distributed, to be coincident with the conductor axes. In general, it is apparent that in a return loop the filaments, which are the electrical equivalents of the actual, non-uniformly distributed charges on the conductors, will be somewhat closer together than are the conductor axes, as shown in Fig. 41. Let the separation between filaments be represented by $2s$ while the separation of the conductors is the slightly greater distance D.

From Fig. 38 and Eq. (203) it is apparent that the difference of potential between any point P, and any other point F on the line AB, due to the filaments at A and B, is

$$E_{FP} = \frac{\lambda\psi}{2\pi}\left(ln\,\frac{r_1}{m'} - ln\,\frac{r_2}{n'}\right).$$

If the point F is halfway between A and B at O, then $m = n$, and the potential difference between O and P will be the electrical potential of P, since the potential of O is zero. Denoting the potential of P by E,

$$E = \frac{\lambda\psi}{2\pi}ln\frac{r_1}{r_2}. \qquad (214)$$

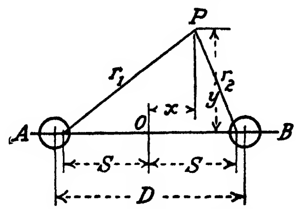


FIG. 41.—The capacitance between two parallel, round wires that are close together.

Using O as the origin of the cartesian coordinates, it is apparent that

$$\frac{r_1}{r_2} = \sqrt{\frac{(s + x)^2 + y^2}{(s - x)^2 + y^2}} \qquad (215)$$

which, substituted in (214), when simplified, yields

$$E = \frac{\lambda\psi}{4\pi}\,ln\,\frac{(s + x)^2 + y^2}{(s - x)^2 + y^2} \qquad (216)$$

or

$$\epsilon^{\frac{4\pi E}{\lambda\psi}} = \frac{(s + x)^2 + y^2}{(s - x)^2 + y^2}. \qquad (217)$$

Equation (217), when expanded and simplified, results in

$$y^2 + x^2 + 2sx\left[\frac{1 + \epsilon^{\frac{4\pi E}{\lambda\psi}}}{1 - \epsilon^{\frac{4\pi E}{\lambda\psi}}}\right]^2 + s^2 = 0 \qquad (218)$$

or

$$y^2 + \left[x + s\left(\frac{1 + \epsilon^{\frac{4\pi E}{\lambda\psi}}}{1 - \epsilon^{\frac{4\pi E}{\lambda\psi}}}\right)\right]^2 = s^2\left[\left(\frac{1 + \epsilon^{\frac{4\pi E}{\lambda\psi}}}{1 - \epsilon^{\frac{4\pi E}{\lambda\psi}}}\right)^2 - 1\right]. \qquad (219)$$

This is the equation of a family of equipotential circles, the individuals of which are determined by the values assigned to E. The center of the circle has the coordinates

$$y = 0$$

$$x = -s\left(\frac{1 + \epsilon^{\frac{4\pi E}{\lambda\psi}}}{1 - \epsilon^{\frac{4\pi E}{\lambda\psi}}}\right) \qquad (220)$$

and the radius is

$$\rho = s\sqrt{\left(\frac{1 + \epsilon^{\frac{4\pi E}{\lambda\psi}}}{1 - \epsilon^{\frac{4\pi E}{\lambda\psi}}}\right)^2 - 1}$$

$$= \frac{2s\epsilon^{\frac{4\pi E}{2\lambda\psi}}}{1 - \epsilon^{\frac{4\pi E}{\lambda\psi}}}. \qquad (221)$$

For $E = E_0$, the potential difference between the conductor A and the neutral plane through O, the equation yields the equipotential circle represented by a section normal to the circular conductor A. If the radius of the conductor is r, then, by Eq. (221),

$$r = \frac{2s\epsilon^{\frac{4\pi E_0}{2\lambda\psi}}}{1 - \epsilon^{\frac{4\pi E_0}{\lambda\psi}}} \qquad (222)$$

The center of the conductor is at

$$x = \frac{D}{2} = \frac{s\left(1 + \epsilon^{\frac{4\pi E_0}{\lambda\psi}}\right)}{1 - \epsilon^{\frac{4\pi E_0}{\lambda\psi}}} \qquad (223)$$

whence

$$s = \frac{D}{2}\frac{\left(1 - \epsilon^{\frac{4\pi E_0}{\lambda\psi}}\right)}{\left(1 + \epsilon^{\frac{4\pi E_0}{\lambda\psi}}\right)} \qquad (224)$$

Substituting Eq. (224) in Eq. (222),

$$r = \frac{D}{\epsilon^{\frac{2\pi E_0}{\lambda\psi}} + \epsilon^{-\frac{2\pi E_0}{\lambda\psi}}}$$

$$= \frac{D}{2\cosh\frac{2\pi E_0}{\lambda\psi}}. \qquad (225)$$

That is,

$$\frac{D}{2r} = \cosh\frac{2\pi E_0}{\lambda\psi}$$

or

$$\psi = \frac{2\pi E_0}{\psi\cosh^{-1}\frac{D}{2r}}. \qquad (226)$$

Since

$$\psi = 4\pi C_0 E_0$$

the capacitance of one conductor to neutral is

$$C_0 = \frac{1}{2\lambda \cosh^{-1}\frac{D}{2r}} \text{ farads per centimeter} \tag{227}$$

$$= \frac{8.948k}{100 \cosh^{-1}\frac{D}{2r}} \text{ mf. per mile.} \tag{228}$$

The capacitance between conductors is one half the value to neutral, or

$$C = \frac{1}{4\lambda \cosh^{-1}\frac{D}{2r}} \text{ farads per centimeter} \tag{229}$$

$$= \frac{4.474k}{100 \cosh^{-1}\frac{D}{2r}} \text{ mf. per mile.} \tag{230}$$

Since. $\cosh^{-1} u = ln(u + \sqrt{u^2 - 1})$, Eq. (227) may also be written[1]

$$C_0 = \frac{1}{2\lambda ln\left[\frac{D}{2r} + \sqrt{\left(\frac{D}{2r}\right)^2 - 1}\right]}.$$

The Capacitance of a Three-phase Line. Triangular Spacing.—Figure 42 represents a three-phase line having the three conductors unequally spaced. The plane of the paper is normal to the line and intersects the conductors in the points, A, B and C. The three unequal spacings measured in this plane are D_1, D_2 and D_3, all of which are assumed to be large compared with the radius r of the three equal conductors. Assume balanced, three-phase voltages to be impressed on the line, and let their vector values be

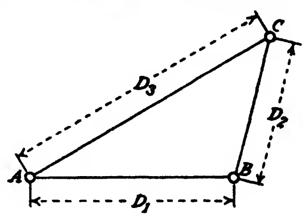


FIG. 42.—The capacitance of three-phase lines with triangular arrangement of conductors.

[1] See RUSSELL, ALEXANDER, Alternating Currents vol. 1, p. 102, Eq. (8).

$$\left.\begin{aligned} E_{ab} &= E(-1 + j0) \\ E_{bc} &= E\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) \\ E_{ca} &= E\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \end{aligned}\right\} \tag{231}$$

Let the dielectric vector fluxes per unit length of line, set up on the three conductors by the impressed voltages, be represented by ψ_a, ψ_b and ψ_c. Then equating the vector impressed voltage in each loop to the drop in the corresponding dielectric circuit (the drop in the conductors themselves is assumed to be zero), the following two independent equations are obtained:

$$\left.\begin{aligned} E_{ab} &= \frac{\lambda}{2\pi}\left[\psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} + \psi_c ln\frac{D_2}{D_3}\right] \\ E_{bc} &= \frac{\lambda}{2\pi}\left[\psi_a ln\frac{D_3}{D_1} + \psi_b ln\frac{D_2}{r} - \psi_c ln\frac{D_2}{r}\right] \end{aligned}\right\} \text{vector volts.} \tag{232}$$

The equation for E_{ca} could, of course, have been used in place of either of the two above.

If the effect of the ground be neglected and it be assumed that no other charges are present to influence the distribution of electrical charges on the line conductors, then the neutral is at zero potential, and the vector sum of the three dielectric fluxes is zero.

Accordingly,

$$\psi_a + \psi_b + \psi_c = 0 \text{ vector lines per centimeter.} \tag{233}$$

By solving the simultaneous Eqs. (232) and (233), and substituting the known, impressed voltages in Eq. (231), the vector fluxes ψ_a, ψ_b and ψ_c may be found for any given values of the spacings D_1, D_2 and D_3. The magnitude of each of the fluxes is then readily determined from the corresponding vector expression. The potential difference between any conductor and neutral (which is also the potential of the conductor under the above assumption), may be found from Eq. (231). Dividing the flux per unit length of each conductor by the potential of the conductor will yield the corresponding capacity to neutral. For an unsymmetrical, untransposed line such as is here being considered, the capacity to neutral has a separate value for each of the three conductors, and the line is electrically unbalanced. All impor-

tant transmission lines are transposed, however, as pointed out in a succeeding paragraph. Without transposition, electrical balance in a three-phase line is possible only when the three conductors are supported at the points of an equilateral triangle. This case is next considered.

The Capacitance of a Three-phase Line. Conductors Supported at the Points of an Equilateral Triangle.—This arrangement is shown in Fig. 43. Let the three-phase vector voltages be those of Eq. (231). The expressions for the fluxes per unit length of the conductors are derived from Eq. (232) by substituting $D_1 = D_2 = D_3 = D$.

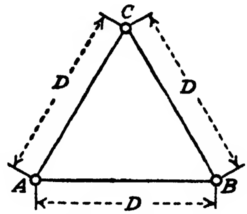


FIG. 43.—The capacitance of three-phase lines with equilateral arrangement of conductors.

The vector potential differences are

$$E_{ab} = \frac{\lambda}{2\pi}\left(\psi_a ln\frac{D}{r} - \psi_b ln\frac{D}{r} + \psi_c ln 1\right) \text{ vector volts} \qquad (234)$$

$$E_{bc} = \frac{\lambda}{2\pi}\left(\psi_a ln 1 + \psi_b ln\frac{D}{r} - \psi_c ln\frac{D}{r}\right) \text{ vector volts} \qquad (235)$$

and, for grounded neutral,

$$\psi_a + \psi_b + \psi_c = 0 \text{ vector lines.}$$

It is to be noted also that, for this case, due to the symmetry of arrangement, the numerical value of the flux leaving per unit length of conductor is the same for all conductors. That is, numerically, $\psi_a = \psi_b = \psi_c$.

Substituting

$$\psi_a = -\psi_b - \psi_c \text{ in Eq. (234),}$$

$$E_{ab} = \frac{\lambda}{2\pi}\left(2\psi_b ln\frac{D}{r} + \psi_c ln\frac{D}{r}\right). \qquad (236)$$

From Eq. (235),

$$E_{bc} = \frac{\lambda}{2\pi}\left(\psi_b ln\frac{D}{r} - \psi_c ln\frac{D}{r}\right). \qquad (237)$$

Adding Eqs. (236) and (237),

$$E_{bc} - E_{ab} = \frac{\lambda}{2\pi}\left(3\psi_b ln\frac{D}{r}\right). \qquad (238)$$

Substituting the vector voltages from Eq. (231) in Eq. (238), and solving,

$$\psi_b = \frac{\pi E(3 - j\sqrt{3})}{3\lambda ln\frac{D}{r}} \text{ vector lines} \tag{239}$$

and

$$\psi = \psi_b = \frac{2\pi\sqrt{3}E}{3\lambda ln\frac{D}{r}} \text{ lines (numerical).} \tag{240}$$

If E_0 be the numerical potential difference from one line conductor to neutral, and since $\sqrt{3}E_0 = E$, from Eq. (240),

$$\psi = \frac{2\pi E_0}{\lambda ln\frac{D}{r}}. \tag{241}$$

The capacitance of one conductor to neutral is

$$C_0 = \frac{\psi}{4\pi E_0} = \frac{1}{2\lambda ln\frac{D}{r}} \text{ farads per centimeter} \tag{242}$$

$$= \frac{7.36 \times 10^{-3}}{\log_{10}\frac{D}{r}} \text{ microfarads per 1,000 ft.} \tag{243}$$

$$= \frac{3.883}{100 \log_{10}\frac{D}{r}} \text{ microfarads per mile.} \tag{244}$$

A comparison of Eqs. (244) and (213) shows that the capacitance of one conductor to neutral for a symmetrical three-phase line as illustrated in Fig. 43, is identical with that for the single-phase line having like conductors and spacing D. Comparing Eq. (241) with Eq. (207) reveals the relation of the fluxes per unit of length of conductor for the two cases. Denoting the three-phase value of Eq. (241) by ψ_3, the single-phase value of Eq. (207) by ψ_1, writing $\sqrt{3}E_0$ for E, and $\frac{D}{r}$ for $\frac{D-r}{r}$ in Eq. (213), and dividing, yields

$$\psi_3 = \frac{2\psi_1}{\sqrt{3}}. \tag{245}$$

Unsymmetrically Arranged, Transposed, Three-phase Lines. Value of C.—It has already been pointed out, in the previous

chapter, how transmission lines are transposed in order to prevent inductive interference with adjacent communication circuits. In the case of three-phase lines having unsymmetrical arrangement of conductors, transpositions are also required, in order to secure electrical balance of the three phases. For properly transposed, short, three-phase lines having unsymmetrical arrangement of conductors, it was shown that the inductance of one conductor could be expressed in terms of an equivalent spacing D', equal to the geometric mean of the individual spacings (Eq. (160)). A similar relation holds for the value of C_0, the capacitance of one conductor to neutral. This relation may be shown as follows:

Since the line is assumed to be properly transposed, each of the three conductors occupies each of the three possible positions throughout a total of one-third the length of the line. Therefore, the average capacitance to neutral, per unit length of conductor, is the same for all conductors. That is,

$C_0 = C_a = C_b = C_c$ (the capacitance to neutral).

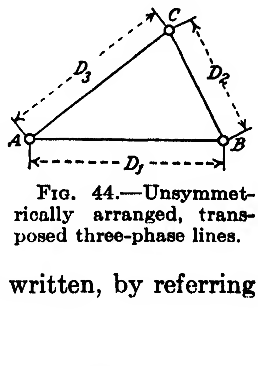


FIG. 44.—Unsymmetrically arranged, transposed three-phase lines.

Since each conductor occupies each of the three possible positions throughout a total of one-third the length of line, the vector voltage of phase AB, for example, may be written, by referring to Fig. 44, as

$$E_{ab} = \frac{\lambda}{6\pi}\left[(\psi_a - \psi_b)ln\frac{D_1}{r} - \psi_c ln\frac{D_3}{D_2} + (\psi_a - \psi_b)ln\frac{D_2}{r} - \psi_c ln\frac{D_1}{D_3} + (\psi_a - \psi_b)ln\frac{D_3}{r} - \psi_c ln\frac{D_2}{D_1}\right] \quad (246)$$

$$= \frac{\lambda}{6\pi}(\psi_a - \psi_b)ln\frac{D_1D_2D_3}{r^3}. \quad (247)$$

If E_a, E_b and E_c denote the vector voltages of the three conductors to neutral,

$$\left.\begin{aligned} \psi_a &= 4\pi C_0 E_a \\ \psi_b &= 4\pi C_0 E_b \\ \psi_c &= 4\pi C_0 E_c \end{aligned}\right\} \text{vector lines} \quad (248)$$

and

$$\begin{aligned} E_{ab} &= E_a - E_b \\ E_{bc} &= E_b - E_c \quad \text{vector volts.} \\ E_{ca} &= E_c - E_a \end{aligned} \tag{249}$$

Substituting the values of ψ_a and ψ_b from Eq. (248) and E_{ab} from Eq. (249) in Eq. (247),

$$E_a - E_b = \frac{2\lambda C_0}{3}(E_a - E_b)ln\frac{D_1D_2D_3}{r^3} \text{ vector volts,}$$

and

$$C_0 = \frac{1}{\frac{2\lambda}{3}ln\frac{D_1D_2D_3}{r^3}} \text{ farads per centimeter.} \tag{250}$$

Equivalent Spacing.—By Eq. (242) the capacitance to neutral for an equilaterally spaced, three-phase line is given as

$$C_0 = \frac{1}{2\lambda ln\frac{D}{r}}.$$

Consequently, for a given conductor size, the symmetrical line which has the same capacitance to neutral as the unsymmetrical, transposed line, and which may therefore be said to be equivalent to it, must have a spacing D' of a value such that

$$2\lambda ln\frac{D'}{r} = \frac{2\lambda}{3}ln\frac{D_1D_2D_3}{r^3}. \tag{251}$$

Thus,

$$\frac{D'}{r} = \sqrt[3]{\frac{D_1D_2D_3}{r^3}}$$

and

$$D' = \sqrt[3]{D_1D_2D_3}. \tag{252}$$

D' is called the equivalent spacing of the unsymmetrical line and is equal to the geometric mean of the three separate spacings. For the flat line, in which $D_1 = D_2 = \frac{D_3}{2} = D$

$$\begin{aligned} D' &= D\sqrt[3]{2} \\ &= 1.26D. \end{aligned} \tag{253}$$

These relations are identical with those already developed for the equivalent spacing used in computing the inductance of a transposed, unsymmetrical line, [Eq. (161)].

The Capacitance of Transmission Conductors, Including the Effect of Ground.—This problem will be solved for a three-phase

line in which the three conductors are in a horizontal plane, as in Fig. 45. The method, however, is no different for any other kind of conductor arrangement.

It is assumed that the earth is a zero-potential plane. Above earth, the line conductors A, B and C are supported at the height h, and, below it, are shown the equal conductors or images $A'B'C'$ at the distance $-h$ from the neutral plane. The images are assumed to carry charges, and therefore fluxes, of exactly the same amounts per unit length of conductor as the conductors themselves, but of opposite signs. This system of conductors and images will establish a plane of zero potential, coinciding with the surface of the earth, and the distribution of the dielectric flux about the conductors may be calculated by the method already employed for the conductors in which the influence of ground was neglected.

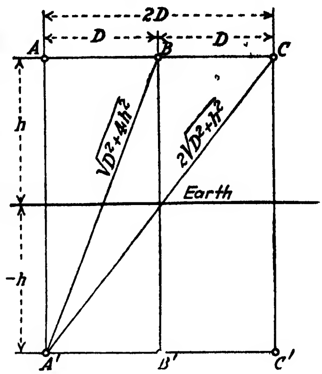


FIG. 45.—The capacitance of three-phase lines including the effect of earth.

Equating the potential differences between phases to the drops in the corresponding dielectric circuits,

$$\left.\begin{aligned} E_{ab} &= \frac{\lambda}{2\pi}\left[(\psi_a - \psi_b)ln\frac{D}{r} - \frac{1}{2}(\psi_a - \psi_b)ln\frac{D^2+4h^2}{4h^2}\right. \\ &\qquad\left. + \psi_c\left(\frac{1}{2}ln\frac{4(D^2+h^2)}{D^2+4h^2} - ln2\right)\right] \\ E_{bc} &= \frac{\lambda}{2\pi}\left[(\psi_b - \psi_c)ln\frac{D}{r} - \frac{1}{2}(\psi_b - \psi_c)ln\frac{D^2+4h^2}{4h^2}\right. \\ &\qquad\left. - \psi_a\left(\frac{1}{2}ln\frac{4(D^2+h^2)}{D^2+4h^2} - ln2\right)\right] \\ E_{ca} &= \frac{\lambda}{2\pi}\left[(\psi_c - \psi_a)ln\frac{2D}{r} - \frac{1}{2}(\psi_c - \psi_a)ln\frac{D^2+h^2}{h^2}\right]. \end{aligned}\right\} \tag{254}$$

Again,

$$\psi_a + \psi_b + \psi_c = 0 \tag{255}$$

$$\left.\begin{aligned} E_{ab} &= E(-1 + j0) \\ E_{bc} &= E\left(\frac{1}{2} - j\frac{\sqrt{3}}{2}\right) \\ E_{ca} &= E\left(\frac{1}{2} + j\frac{\sqrt{3}}{2}\right) \end{aligned}\right\} \text{vector volts.} \tag{256}$$

By substituting values of voltages from Eq. (256), dielectric flux from Eq. (255) and known values of spacings D and h in two of the three Eqs. (254), and solving, one may find the vector fluxes ψ_a, ψ_b and ψ_c. The corresponding numerical values may thus also be found. Having the latter, the capacitance to neutral follows from the relation

$$C_0 = \frac{\psi}{4\pi E_0} \text{ (numeric).} \tag{257}$$

Double-circuit, Three-phase Lines. Value of C.—The general method of procedure to be followed in finding the capacitance of double-circuit lines is identical with that already outlined in the previous section for unsymmetrically arranged, three-phase lines.

The conductors of double-circuit lines may be arranged in a number of ways. A common arrangement is that illustrated in Fig. 31. Here each of the circuits occupies one side of the tower. The three cables of a given circuit are arranged vertically one above the other, with the middle conductor slightly offset outwardly. This arrangement of circuits will be used in the following computations illustrating the method of solving for the capacitance.

The additional assumptions made are: The two three-phase lines are operated in parallel between generating and receiving stations. The vector potentials of the conductors a and a', b and b' and c and c' are E_a, E_b and E_c with respect to the neutral, whose potential is assumed to be zero. Each of the circuits has several complete transpositions, but the corresponding conductors a and a', b and b' and c and c' are always opposite each other in the hexagonal figure formed by the six conductors on the towers.

Since the lines are transposed, each conductor occupies each of the three possible positions in rotation; the phases are properly balanced, and the capacitance to neutral is the same for all conductors.

Let the vector fluxes per centimeter of cable from a, b and c, be ψ_a, ψ_b and ψ_c, respectively. The corresponding fluxes from a', b' and c' are likewise ψ_a, ψ_b and ψ_c, since $E_a = E_{a'}$, $E_b = E_{b'}$ and $E_c = E_{c'}$.

The potential difference for phase ab of the transposed line may now be written for each of the three positions. The average value for the transposed line is

$$E_{ab} = \frac{\lambda}{6\pi}\left(\psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} - \psi_c ln\frac{D_2}{D_1} - \psi_a ln\frac{D_5}{D_4} - \psi_b ln\frac{D_4}{D_6} + \psi_c ln\frac{D_4}{D_3}\right.$$
$$+ \psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} + \psi_c ln\frac{D_2}{D_1} + \psi_b ln\frac{D_4}{D_6} + \psi_b ln\frac{D_5}{D_4} - \psi_c ln\frac{D_4}{D_3}$$
$$\left.+ \psi_a ln\frac{D_2}{r} - \psi_b ln\frac{D_2}{r} + \psi_c ln\frac{D_1}{D_1} - \psi_a ln\frac{D_5}{D_3} + \psi_b ln\frac{D_5}{D_3} + \psi_c ln\frac{D_4}{D_4}\right)$$

$$= \frac{\lambda}{6\pi}\left[ln\frac{D_1^2D_4^2D_2D_3}{r^3D_5^2D_6}(\psi_a - \psi_b) - \psi_c ln\frac{D_2D_3}{D_1D_4}\cdot\frac{D_1D_4}{D_2D_3}\cdot\frac{D_1D_4}{D_1D_4}\right]. \quad (258)$$

The last term in the above equation is evidently zero. Since the circuit is transposed, the capacitance to neutral is the same for all conductors, and

$$\frac{1}{4\pi C_0}(\psi_a - \psi_b) = E_a - E_b = E_{ab}$$

the vector potential difference between the conductors a and b. Hence, from Eq. (258),

$$1 = \frac{2\lambda C_0}{3} ln\frac{D_1^2D_4^2D_2D_3}{r^3D_5^2D_6}$$

and

$$C_0 = \frac{1}{\frac{2\lambda}{3} ln\frac{D_1^2D_4^2D_2D_3}{r^3D_5^2D_6}} \text{ farads per centimeter.} \quad (259)$$

Using $\log_{10}$ and expressing the capacitance in microfarads per mile, we get

$$C_0 = \frac{0.03883}{\log_{10}\sqrt[3]{\frac{D_1^2D_4^2D_2D_3}{r^3D_5^2D_6}}} \text{ mf. per mile.} \quad (260)$$

If we represent the equivalent spacing by D', giving D' a value such that the capacitance obtained from the equation

$$C_0 = \frac{0.03883}{\log_{10}\frac{D'}{r}}$$

is the same as that found from Eq. (260), the equivalent spacing is

$$D' = \sqrt[3]{\frac{D_1^2D_4^2D_2D_3}{D_5^2D_6}}. \quad (261)$$

The effect of the presence of two parallel circuits in close proximity is to increase the effective capacitance of each. This may be illustrated by means of an example.

Example.—In the arrangement of Fig. 31, let the vertical distance between adjacent conductors be 13 ft., the offset of the middle conductor from the vertical plane through a and c, be 3 ft., and let D_3, the horizontal spacing, be 22 ft. These are suitable spacings for a 132-kv. circuit. Let the conductor be 350,000-cir. mil, stranded cable having a diameter of 0.682 in. Hence, its radius is 0.0284 ft. The various spacings in the figure are as follows:

$$\begin{aligned} D_1 &= 13.3 \text{ ft.} & D_4 &= 28\ 2 \text{ ft.} \\ D_2 &= 26.0 \text{ ft.} & D_5 &= 34.0 \text{ ft.} \\ D_3 &= 22.0 \text{ ft.} & D_6 &= 28.0 \text{ ft.} \end{aligned}$$

By Eq. (261),

$$D' = \sqrt[3]{\frac{(13.3)^2 \times (28.2)^2 \times 26 \times 22}{(34)^2 \times 28}}$$
$$= \sqrt[3]{2485.9} = 13.55 \text{ ft.}$$

and

$$C_0 = \frac{0.03883}{\log_{10} \dfrac{13.55}{0.0284}} = \frac{0.03883}{2.6786}$$
$$= 0.0145 \text{ mf., per mile.}$$

For a transposed single-circuit line having the same spacings as one of the lines of the figure, the equivalent spacing is

$$D' = \sqrt[3]{(13.3)^2 \times 26} = 16.63 \text{ ft.}$$

and

$$C_0 = 0.0140 \text{ mf. per mile.}$$

Percentage increase in single-circuit capacitance, due to the proximity of the second similar circuit, is

$$\frac{500}{140} = 3.57 \text{ per cent.}$$

PROBLEMS

1. The two round conductors of a parallel-sided loop are each 0.60 in. in diameter. They are separated 12 ft. between centers. A potential difference of 100 kv. is impressed across the loop. In a sectional view normal to the plane of the loop, draw the equipotential lines for each 5 kv. of potential difference.

2. For the circuit of Problem 1, in the sectional view, draw 5 dielectric lines of force, so chosen that each 2 adjacent lines shall include between them one-eighth of the flux per centimeter of conductor. Let the straight line joining the two conductors be one of the lines as well as the axis of symmetry of the completed figure.

3. Two equations are given for the capacity of two round parallel wires, namely,

$$(a)\ C_0 = \frac{3.883}{100 \log_{10} \dfrac{D - r}{r}} \text{ mf. per mile}$$

$$(b)\ C_0 = \frac{8.948}{100 \cosh^{-1} \dfrac{D}{2r}} \text{ mf. per mile.}$$

Calculate the capacity of two parallel conductors each 1 in. in diameter, by each of the above equations, for various separations between centers from slightly more than 1 in. up to 36 in. Between these limits calculate the percentage of error made when using the approximate equation and plot error percentage *vs.* separation in inches.

4. A three-phase, 60-cycle, 110-kv. line is built of 300,000-cir. mil cables whose diameters are 0.631 in. The conductors are supported at the three points of a triangle whose sides are 12, 12, and 20 ft. The line is transposed to equalize the phases. What is the charging current per phase, per mile of the line?

5. A double-circuit, 150-kv., 60-cycle, three-phase line has six conductors, each of diameter 0.773 in., supported vertically on each side of the tower line in a manner similar to that shown in Fig. 31. The vertical separation between adjacent conductors is 11 ft. 6 in., the middle conductor is offset from the vertical plane, and away from the tower, by 4 ft. 6 in., and the separation between the planes of the two circuits is 14 ft. 6 in. The mean distance to earth of lowest conductor is 30 ft. Beginning at the upper left-hand corner of the hexagonal figure representing the conductor arrangement, and reading around the figure counter-clockwise, the conductors lie in the order a, b, c, c', b', a'. The conductors a and a', b and b' and c and c' are connected to the busses A, B and C, respectively.

If the lines are transposed to balance phases, calculate (*a*) the capacity to neutral of each line, neglecting the effect of ground; (*b*) the capacity to neutral of each line including the effect of ground. Assume the neutral plane at the earth's surface.

CHAPTER V

CORONA

Description.—Our interest here in the phenomenon of corona concerns its appearance on the parallel conductors of a transmission-line circuit, suspended from insulators in air, a medium of unit permittivity. Under conditions of low-potential difference impressed on the conductors, air is practically a perfect insulator. If the impressed potential difference is alternating, the dielectric flux is alternating, giving rise to a *displacement current* in the dielectric circuit and the usual *charging current* in the conductors. The energy flow in the circuit is purely reactive, since no energy is lost in the dielectric.

As the voltage is continually raised, the potential gradient about the conductors is correspondingly increased until, at the critical gradient of about 30 kv. per centimeter, the air in the immediate vicinity of the conductor becomes conducting. If viewed in the dark, a violet glow becomes visible, a hissing sound is heard, the conductor tends to vibrate, and, if conditions are favorable, the presence of ozone may be detected by its characteristic odor. The phenomenon, the evidences of which have just been recited, is called *corona.* The minimum voltage at which the glow is observed is called the *critical visual corona voltage.* If the conductors are rough or dirty, the potential gradient will not be uniform for all parts of the conductor, but will be greatest at the roughened surfaces where the curvature is greatest. Corona will appear first at these points, and will continue there in intensified form, giving the conductor the appearance of lumpiness. A wattmeter placed in the electrical circuit will indicate a loss of power, showing that energy is being dissipated in the leakage circuit. The lost energy appears in various forms such as heat, light, chemical energy and mechanical energy of sound and vibration generally. A measure of the charging current under corona conditions indicates an apparent increase in the capacitance of the line. The explanation usually given has been that the corona increases the effective diameter of the

conductor to the outer boundary of the corona envelope. An increase in the capacitance of the conductor would naturally follow. Later experiments[1] make it appear that no actual increase in capacitance occurs, but that the apparent increase is due to the presence of harmonic currents introduced by the corona cycle.

If the voltage be raised to still higher values, the luminous glow will reach out farther and farther from the conductors until, finally, the insulation of the air is broken down completely and an arc passes between the conductors. This voltage, which for widely spaced conductors is considerably higher than the visual corona voltage, is called the *sparkover voltage*. If the voltage is held at this value, the sparks or arcs may be intermittent, corona reappearing between successive discharges. For small, closely spaced conductors, sparkover may appear without previous formation of corona.

Theory of Corona Formation.—The formation of corona may be accounted for on the basis of the electron theory somewhat as follows: Under ordinary conditions, even when free from electrical stress, air contains a certain number of free electrons; that is, it is slightly ionized. When a potential difference is gradually applied between conductors, a potential gradient is established in the space about the conductors and between them. In response to this force the electrons acquire a uniformly accelerated motion, and in very short distances attain high velocities. The velocity acquired in a given time depends upon the mass of the electron, the size of its charge, and the potential gradient of the field causing the acceleration.

The moving electrons collide with one another, and with the larger and more slowly moving, neutral molecules. The *average distance* which the electrons travel before making a collision, called the *mean free path*, depends upon the number of molecules per unit volume of the air, and thus upon the temperature and barometric pressure. When the potential gradient reaches a value of about 30 kv. per centimeter, assuming standard conditions of temperature and pressure, the electrons acquire sufficient velocity in moving a distance equal to their mean free path, to dislodge one or more electrons from a neutral molecule when

[1] Gardner, Murray F., "Corona Investigation on an Artificial Line," *Proc.*, A. I. E. E., p. 183, August, 1925.

Peek, F. W., Jr., "Voltage and Current Harmonies Caused by Corona," *Trans.*, A. I. E. E., p. 1155, 1921.

colliding with it. This results in additional free electrons, and molecules deficient in electrons, called ions. Both of these in turn are accelerated, resulting in increasing numbers of new collisions, free electrons and ions. The numbers of these present in a given volume thus rapidly accumulate, until saturation is reached, the insulating properties of the air are destroyed, the air becomes conducting, and corona forms or a spark passes between the conductors.

It has been found experimentally that the constant gradient of approximately 30 kv. per centimeter is the gradient at which the cumulative effect of ionization will cause corona to form. This, however, is not the gradient at the surface of the conductor, but at a distance from the center of a conductor whose radius[1] is r of $x = (r + 0.301\sqrt{r})$. The gradient at the surface of the conductor, designated as G_v, the visual corona gradient, is somewhat higher, and is given by the equation

$$G_v = G_0\left(1 + \frac{0.301}{\sqrt{r}}\right).$$

The observed results are explained by the fact that energy must be transferred from the electric circuit to the surrounding air before corona can form. The amount of energy required is the sum of all the energies possessed by the moving ions and electrons in the vicinity of the conductor when saturation is reached, and the air becomes conducting. If the gradient of the surface of the conductor were but 30 kv. per centimeter, the gradients at greater distances from the center would be less, and therefore not sufficient to cause the cumulative ionization required.

Experimental Investigation.[2] Factors Influencing Corona.—As a result of extensive experiments, and particularly as a result of the very complete investigation carried out under the supervision of F. W. Peek, Jr., at the General Electric Company

[1] Peek, F. W., Jr., "Law of Corona," *Proc.*, A. I. E. E., June, 1911.

[2] The present body of knowledge pertaining to the subject of corona is the result of many experiments performed by various investigators and reported in the electrical journals. These experiments began many years ago and are still being reported from year to year. Among those who have made important contributions to this knowledge are: Mershon, Ryan, Steinmetz, Peek, Whitehead and Harding. For results of their experiments, see *Trans.*, A. I. E. E., 1908 to 1924; also, Peek, F. W., Jr., "Dielectric Phenomena in High Voltage Engineering."

about 1910, the factors which influence the formation of corona and the laws which govern it, are now fairly well established. Among the conclusions reached, the following, applying to the parallel conductors of a transmission line, are important to the engineers:

1. *Conductor, Diameter and Spacing.*—The stress at any point in a dielectric is measured by its corresponding potential gradient, G.

From Eq. (205), by substituting r for x, we get

$$_{\max}G = \frac{\lambda\psi}{2\pi}\left(\frac{1}{r} + \frac{1}{D - r}\right)$$

and by slightly rearranging Eq. (207) one may write

$$\psi = \frac{2\pi E_0}{\lambda ln\frac{D}{r}}$$

Eliminating λ from the equation for maximum gradient and simplifying yields the potential gradient at the surface of one of two parallel wires spaced far apart, as in transmission-line practice. It is, approximately,

$$_{max}G = \frac{de}{dx} = \frac{E_n}{rln\frac{D}{r}}$$

where r = radius of the conductor in centimeters.
D = spacing between conductors in centimeters.
E_n = potential difference from conductor to neutral.

Since the gradient varies inversely as the distance from the center of the conductor, the above is the maximum gradient. If the voltage to neutral E_n, be raised to the critical value E_v, at which visual corona appears, the corresponding gradient G_v is then

$$G_v = \frac{E_v}{rln\frac{D}{r}} \qquad (262)$$

$$= \frac{0.4343E_v}{r\log_{10}\frac{D}{r}}.$$

It is apparent from the above that both the conductor diameter and the spacing are factors influencing the surface gradient, and therefore the voltage at which visual corona starts.

Experiment shows that at sea level under standard conditions of temperature and barometric pressure, namely 25° C. and 76 cm. of mercury barometer, respectively, visual corona appears on parallel, round conductors at varying surface gradients. At a distance from the center of the conductor of

$$x = r + 0.301\sqrt{r} \text{ cm.} \tag{263}$$

the gradient is, however, constant and equal to approximately 30 kv. per centimeter. For sine waves of e.m.f. this is the maximum value. The corresponding gradient in terms of effective volts is 21.1 kv. per centimeter or 53.6 kv. per inch. This constant gradient G_0, is less than the surface gradient G_v. The two gradients are related as shown in Eq. (264). That is,

$$G_v = G_0\left(1 + \frac{0.301}{\sqrt{r}}\right). \tag{264}$$

It is apparent from Eq. (264) that, to start visual corona, small conductors require a higher surface gradient than large ones.

By substituting the value of G_v from Eq. (262) in Eq. (264), writing for G_0 its equivalent, and solving for E_v, there results:

$$E_v = 21.1\left(1 + \frac{0.301}{\sqrt{r}}\right)r\ln\frac{D}{r} \text{ kv. effective to neutral.} \tag{265}$$

This is the equation for the voltage to neutral required to start visual corona in fair weather under standard conditions of temperature and pressure.

2. *Density of Air.*—As already intimated, the air density is a factor affecting the voltage at which corona begins. This factor, denoted δ, is unity at 760-mm. pressure and 25° C. temperature. For other temperatures or barometric pressures, it is the fraction

$$\delta = \frac{3.92b}{273 + t} \tag{266}$$

where b = barometer reading in centimeters of mercury
t = temperature in degrees Centigrade.

The factor is directly proportional to the barometric pressure and inversely proportional to the absolute temperature.

Obviously also, it decreases with increase in altitude, other conditions remaining unchanged.

3. *Roughness of Conductor Surface.*—On conductors with rough surfaces, corona starts at voltages lower than for conductors with smooth surfaces. The reason for this is apparent. For a rough spot has the effect of increasing the curvature of the surface at that point, and thus of increasing the local, potential gradient. A factor m, called the *roughness factor*, is therefore introduced to give proper weight to the influence of the condition of conductor surface upon the corona voltage. Thus, the corona starting voltage becomes proportional also to m.

Deposits on the conductor surface, such as moisture, snow or sleet, produce a roughness effect and lower the corona voltage.

Corona Voltage and Altitude Roughness Factors Included.—Equation (265) gives the visual corona voltage to neutral for standard conditions of temperature and pressure and for a smooth conductor. For other conditions the correction factors δ and m are introduced, and Eq. (265) becomes

$$E_v = 21.1 m_v \delta r \left(1 + \frac{0.301}{\sqrt{\delta r}}\right) ln \frac{D}{r} \text{ kv. effective to neutral.} \quad (267)$$

The corresponding disruptive critical voltage is

$$E_0 = 21.1 m_0 \delta r ln \frac{D}{r} \text{ kv. effective to neutral.} \quad (268)$$

In Eqs. (267) and (268) c.g.s. units and natural logarithms are used. When logarithms to the base 10, degrees Fahrenheit and inches are used, these equations become

$$E_v = 123 m_v \delta r \left(1 + \frac{0.189}{\sqrt{\delta r}}\right) \log_{10} \frac{D}{r} \text{ kv. effective to neutral} \quad (269)$$

$$E_0 = 123 m_0 \delta r \log_{10} \frac{D}{r} \text{ kv. effective to neutral} \quad (270)$$

where

δ = air density factor

$= \dfrac{17.9b}{459 + t}$

= 1 at 77° F. and 29.9 in., barometer

t = temperature in degrees Fahrenheit

b = barometer reading in inches of mercury

r = radius of conductor in inches

D = interaxial spacing in inches

m_v = irregularity factor for E_v
m_0 = irregularity factor for E_0

$$G_0 = \frac{123}{2.302} = 53.6 \text{ kv. per inch.}$$

The irregularity factor has the approximate values given below:

m_0 = 1 for polished wires
= 0.98 to 0.93 for roughened or weathered wires
= 0.87 to 0.83 for seven-strand cable
= 0.85 to 0.80 for 19-, 37- and 61-strand, concentric, lay cables.

A value of $m_0 = 0.68$ for a piece of new, 49-strand, 500,000-cir. mil rope lay cable, was reported by Wilkins.[1] Three years later a test of the line showed $m_0 = 0.72$. While these low values are no doubt due to the use of rope lay cable, they clearly illustrate the improvement to be effected in the value of m_0 with use. This is due to the disappearance of the roughened surfaces and irregularities by oxidation. They also argue powerfully against the use of rope lay cables for high-voltage lines. For large, concentric lay cables from which the rough points left by the manufacturing and handling processes have been weathered away by some years of use, a value of $m_0 = 0.85$ is probably a safe value to use. Since the accuracy of the results to be expected from the loss equation, are largely dependent upon the accuracy with which m_0 is known, the importance of properly determining this factor is apparent.

Calculation of Maximum Potential Gradients.—The maximum potential gradient is at the surface of the conductor, and is proportional to the dielectric flux density. The latter, in turn, is equal to the dielectric flux per unit length of conductor divided by the circumference of the conductor; or

$$G = \frac{\lambda\psi}{2\pi r}$$

where

$$\lambda = \frac{v^2}{10^9 k}.$$

Thus, to compare the maximum gradients produced at the surfaces of conductors of a given diameter, but having various spacing arrangements, it is only necessary to compare their dielectric

[1] Wilkins, Roy, "Corona Loss Tests on 202-mile, 60-cycle, 220-kv., Pitt-Vacca Transmission Line," *Proc.*, A. I. E. E., December, 1924.

fluxes. It will be assumed in all cases that the spacing is large and that the error made in substituting D for $D - r$ is negligible.

1. *Single-phase Line.*—From Eq. (207), the dielectric flux, per unit length of conductor for a single-phase line, is readily seen to be

$$\psi = \frac{2.7288 E_n}{\lambda \log_{10} \frac{D}{r}} \tag{271}$$

and the corresponding maximum gradient is

$$_{\max} G = \frac{0.4343 E_n}{r \log_{10} \frac{D}{r}}. \tag{272}$$

2. *Three-phase Line, General.*—Consider a three-phase line having loops of widths D_1, D_2 and D_3, whose corresponding vector voltages are E_{ab}, E_{bc} and E_{ca}, producing, respectively, per unit length of line, the dielectric vector fluxes ψ_a, ψ_b and ψ_c. The voltage of a loop is

$$4\pi E = \int_r^{D-r} \psi \cdot dS.$$

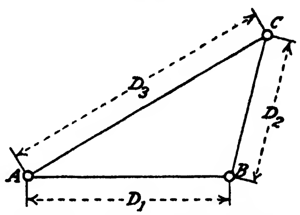


FIG. 46.—Potential gradients on three-phase lines.

Writing these integrals for two of the loops, and replacing $D - r$ by D,

$$E_{ab} = \frac{\lambda}{2\pi}\left[\psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} - \psi_c ln\frac{D_3}{D_2}\right] \text{ vector volts} \tag{273}$$

$$E_{bc} = \frac{\lambda}{2\pi}\left[\psi_a ln\frac{D_3}{D_1} + \psi_b ln\frac{D_2}{r} - \psi_c ln\frac{D_2}{r}\right] \text{ vector volts} \tag{274}$$

$$E_{ca} = \frac{\lambda}{2\pi}\left[-\psi_a ln\frac{D_3}{r} + \psi_b ln\frac{D_1}{D_2} + \psi_c ln\frac{D_3}{r}\right] \text{ vector volts.} \tag{275}$$

Also,

$$0 = \psi_a + \psi_b + \psi_c \tag{276}$$

since the vector sum of all the dielectric fluxes is zero. The vector voltages may be written,

$$E_{ab} = E(1 + j0)$$

$$E_{bc} = E\left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$E_{ca} = E\left(-\frac{1}{2} - \frac{\sqrt{3}}{2}\right).$$

If the spacings D_1, D_2 and D_3 and the radius of the conductor are known, the values of ψ_a, ψ_b and ψ_c for the corresponding three-phase line may be found.

3. *Three-phase Line, Equilateral Spacing.*—For this case $D_1 = D_2 = D_3 = D$. Substituting these values in Eqs. (273) and (274), and solving for ψ_a, ψ_b and ψ_c from Eqs. (273), (274), and (276)

$$\psi_b = \frac{4\pi E(-3 + j\sqrt{3})}{2\lambda \cdot 6ln\frac{D}{r}} \text{ vector lines} \tag{277}$$

where E is the numerical value of the line voltage

and

$$\psi_b = \frac{2.7288E_n}{\lambda \log_{10}\frac{D}{r}} \text{ lines absolute,} \tag{278}$$

whence

$$G_b = \frac{0.4343E_n}{r \log_{10}\frac{D}{r}} \text{ kv. per inch.} \tag{279}$$

Equations (278) and (279) are the same as the corresponding equations for the single-phase line. By symmetry, the absolute values of ψ_a and ψ_c are each equal to ψ_b. Comparing Eqs. (278) and (279) with Eqs. (271) and (272) shows that, for conductors of like diameters and spacings, the maximum gradient for the three-phase case bears the ratio of $2 \div \sqrt{3}$ to the maximum, single-phase gradient for equal line voltages. They are equal for equal voltages to neutral.

4. *Three-phase Line, Flat Spacing.*—Let the conductors lie in a single plane, and let $D_1 = D_2 = \frac{D_3}{2} = D$. Substituting these values in Eqs. (273) and (274), and solving as before, the flux per unit length of middle conductor is

$$\psi_b = -\frac{4\pi E(3 - j\sqrt{3})}{2\lambda \cdot 2\left(3ln\frac{D}{r} - ln2\right)} \text{ vector lines} \tag{280}$$

and

$$\psi_b = \frac{2\pi E}{\lambda\left(\sqrt{3}ln\frac{D}{r} - \frac{1}{\sqrt{3}}ln2\right)} \text{ lines absolute} \tag{281}$$

whence

$$G_b = \frac{E}{r\left(\sqrt{3}\,ln\frac{D}{r} - \frac{1}{\sqrt{3}}ln2\right)} \text{ kv. per inch} \tag{282}$$

$$= \frac{0.4343E_n}{r\left(\log_{10}\frac{D}{r} - 0.333\log_{10}2\right)} \text{ kv. per inch.} \tag{283}$$

In a similar manner, ψ_a may be calculated, after which ψ_c may conveniently be found by substituting the value of ψ_b from Eq. (281) in Eq. (276), after substituting for ψ_a in the latter. The scalar values of ψ_a and ψ_c are equal, from symmetry. The value of ψ_c is

$$\psi_c = +\frac{2\pi E}{\lambda}\left[\frac{3 - j\sqrt{3}}{12ln\frac{D}{r} - 4ln2} - \frac{1 + j\sqrt{3}}{4\left(ln\frac{D}{r} + ln2\right)}\right] \text{vector lines.} \tag{284}$$

The absolute values of ψ_a and ψ_c are obtained by substitution in Eq. (284), simplifying to get the resultant vector and subsequently evaluating. Table 7 gives the maximum gradients for the middle and outside conductors in percentage of the maximum gradient for the equilateral three-phase line and for given values of D and r.

TABLE 7.—PERCENTAGE RATIOS $\frac{G_F}{G_\Delta}$ FOR THREE-PHASE LINES AND FOR FLAT SPACING $\left(D_1 = D_2 = \frac{D_3}{2} = D\right)$

	Per cent ratio $G_F \div G_\Delta$ for three-phase line, flat spacing	
$\log_{10}\frac{D}{r}$	Outside conductors	Middle conductors
1.00	86 7	111 2
1.25	88 4	108.7
1.50	89.8	107.2
1 75	91 0	106.1
2.00	91.9	105.3
2.25	92.6	104.7
2.50	93.2	104.2
2.75	93.5	103.8
3.00	94.0	103.4
3.25	94.0	103.2

G_F = maximum gradient for flat spacing.
G_Δ = maximum gradient for equilateral triangle type of spacing

Corona Loss.[1]—Energy loss due to corona begins at the *disruptive critical voltage* of Eq. (270), a value somewhat below the visual voltage of Eq. (269). Peek found, as a result of a long series of tests on an experimental transmission line, that the loss is proportional to the square of the voltage in excess of the disruptive, critical value, and is influenced by several other factors. For single-phase lines and three-phase lines having equilateral spacings, the fair-weather loss is

$$P = \frac{390}{\delta}(f + 25)\sqrt{\frac{r}{D}}(E_n - E_0)^2$$
$$\times 10^{-5} \text{ kw. per mile of one conductor.} \qquad (285)$$

where f is the frequency employed in cycles per second and E_n and E_o are the impressed and critical, disruptive kilovolts to neutral, respectively. All other symbols in this equation are in the inch system of units as previously defined.

To approximate the loss under storm conditions, E_0 is taken as 0.80 of its corresponding fair-weather value.

When the spacings employed on three-phase lines are not equal, as for example when flat, horizontal spacings or triangular spacings with unequal sides are used, Eq. (285) gives results which are in error. For the flat spacing, corona starts on the middle conductor at a voltage approximately 4 per cent below the critical value for the equilateral arrangement having the same spacing D, while on the outer conductors it starts at a voltage about 6 per cent higher (Table 7). The reason for the differences is apparent, since the gradient at the middle conductor must necessarily be higher than that for the outside ones.

The gradient is the determining factor, and, since in unequally spaced lines it will not be the same for all conductors, it is desirable to express the loss equation in terms of gradients rather than voltages, to take care of loss calculations for such cases. This is done by substituting in Eq. (285) the corresponding expression for voltage, or

$$E_n = 2.302Gr \log_{10} \frac{D}{r} \text{ from Eqs. (204) and (207)} \qquad (286)$$

and

$$E_0 = 2.302G_0m_0\delta r \log_{10} \frac{D}{r} \text{ from Eq. (270).} \qquad (287)$$

[1] PEEK, F. W., JR., "Law of the Corona," *Trans.*, A. I. E. E., "Dielectric Phenomena in High-voltage Engineering," p. 204 and following, 1920.

Equation (285) then becomes

$$P = \frac{2070}{\delta}(f + 25)\sqrt{\frac{r}{D}}\left(\log_{10}\frac{D}{r}\right)^2 r^2(G - m_0\delta G_0)^2 \times 10^{-5}$$
kw. per mile of one conductor. (288)

This equation, like Eq. (285), employs the inch units of length. Equation (288) may be used to calculate the loss on each conductor separately as soon as the gradients are known. For flat spacings the factors of Table 7 may be used to find the gradients.

Corona Loss Calculation.—Let it be required to estimate the corona loss per mile of a transmission line for which the following data apply:

Cable diameter = 0.681 in. (concentric lay cable)
Number of strands = 37
Spacing, flat $D_1 = D_2 = \frac{D_3}{2} = D = 16$ ft.

Frequency = 60 cycles per second
Roughness factor $m_0 = 0.83$
Altitude = 1,000 ft.
Fair-weather value of $\delta = 0.95$
Disruptive, critical gradient $G_0 = 53.6$ kv. per inch
Applied voltage = 97 kv. to neutral

$$E = 2.302 G_0 m_0 \delta r \log_{10}\frac{D}{r}$$
$$= 0.302 \times 53.6 \times 0.83 \times 0.95 \times 0.340 \times 2.752$$
= 91,000 volts to neutral for equilateral spacing of 16 ft.

$$G = \frac{0.4343 E_n}{r \log_{10}\frac{D}{r}},$$ from Eq. (262)

$$= \frac{0.97 \times 0.4343}{0.340 \times 2.752}$$ = 45 kv. per inch for three-phase, equilateral spacing.

From Table 7,

$G_a = G_c = 0.935 \times 45 = 42 +$ kv. per inch for outside conductors.
$G_b = 1.038 \times 45 = 46.6$ kv. per inch for middle conductor.

By Eq. (288),

$$P = \frac{2{,}070}{\delta}(f + 25)\sqrt{\frac{r}{D}}\left(\log_{10}\frac{D}{r}\right)^2 r^2(G - m_0\delta G_0)^2 \times 10^{-5}$$ kw. per mile of one conductor.

$$= \frac{2{,}070 \times 85}{0.95}\sqrt{\frac{0.340}{192}}(2.75)^2(0.340)^2(G - 0.83 \times 0.95 \times 53.6)^2 \times 10^{-5}$$

$$= 0.06829(G - 42.3)^2$$

From Table 7, for $\log_{10}\frac{D}{r} = 2.75$, as before,

$G_a = 42$ kv. per inch for two outside conductors.
$G_b = 46.6$ kv. per inch for mid-conductor.

There is no corona on the outside conductors since $G_a = G_c <$ $(42.3 = m_0G_0\delta)$.
The loss per mile on the mid-conductor is

$$P = 0.06829(48.6 - 42.3)^2$$
$$= 1.26 \text{ kw. per mile.}$$

Corona as a Factor in Line Design.—A study of the economics of transmission-line design shows that, generally speaking, the most economical line for a given project is the one having minimum, annual, fixed charges. Such a line will fulfill the requirements of Kelvin's Law and will operate at or near corona voltage. What the exact value of the voltage should be, that is, whether it should be a little above or a little below the disruptive, critical value for average, fair-weather conditions, is a question which is difficult to determine. Consideration must be given to the fact that seasonal temperature changes may be sufficient to cause a variation of critical, disruptive voltage of perhaps 20 per cent between winter and summer. Variations in altitude for different sections of the line and the prevalence or absence of storms are other factors which should be given proper weight. From the purely economic standpoint alone, it would seem that the correct operating voltage should be slightly under the critical value for fairweather and summer temperatures. A choice of this voltage would eliminate corona under all fair-weather, normal, operating conditions and still would leave incidental advantages to be gained under normal voltages.

Lines which are built to operate near the critical voltage not only are the most economical, but they have additional protection against the destructive effects of high-voltage, high-frequency surges occasioned by lightning, switching and other transient disturbances. It has been pointed out by Peek, Whitehead and others, that surges, due to suddenly applied over-voltage resulting in corona formation, are greatly attenuated in traveling over a comparatively short stretch of line. The energy of the surge is dissipated in the corona leakance, and the excess voltage is thereby reduced to a safe value. By normally operating

close to the critical voltage, the maximum advantage is gained from this protective feature.

PROBLEMS

1. A 60-cycle, three-phase line is built of 250,000-cir. mil, concentric lay cable, for which the roughness factor is 0.85. The conductors are spaced 12 ft. apart at the vertices of an equilateral triangle. Assuming δ to vary with altitude, as in Table 24, at what voltage will corona begin to form on this line at 2,000 ft. altitude? At sea level?

2. For the line in problem 1, at what voltage would the corona loss be 5 kw. per mile at 2,000 ft. altitude?

3. Assuming an equilateral arrangement of conductors and a line free from corona loss, what is the maximum allowable voltage for a 0000 solid conductor line, whose spacing of conductors is 11 ft. between centers? $m_0 = 0.93$. $\delta = 0.96$.

4. The three equal cables of a 165-kv., three-phase line lie in a horizontal plane, and are separated by 20 ft. between centers. The irregularity factor is 0.83 and the value of δ is 0.94. What is the minimum conductor diameter that may be used on this line if corona loss is to be avoided?

5. If we assume (*a*) that for single-circuit lines with flat spacing, the separation of conductors in inches, demanded by good practice, is given by the equation:

$$D' = 0.002E_n \text{ (Eq. 528)}$$

and (*b*) that near sea level the voltage E_n to neutral in k.v. is 240 times the conductor radius in inches, what is then the minimum permissible separation of the conductors of a 165-kv. line having no corona? The conductors are arranged in a plane and the elevation of the line is sea level.

CHAPTER VI

INDUCTIVE INTERFERENCE

When power circuits and communication lines are in close proximity, and particularly when they parallel one another for considerable distances, potential differences are induced between the wires of the communication circuits, due to the varying magnetic and dielectric fields set up by the power circuits. These potential differences establish currents in the wires, instruments, relays and other devices that compose the essential elements of the communication system. Since the amount of current required to produce an audible sound in a telephone receiver is very small indeed, even feeble currents may be troublesome, especially if their frequencies lie within the range of voice frequencies. In telephone systems the induced currents cause a humming noise in the receiver, thereby annoying the user and reducing the intelligibility of conversation, while in telegraph circuits the disturbance may cause chattering of relays, reduced speed of transmission and impaired clearness of signaling. Disturbances in telegraph circuits are due largely to the fundamental frequency, while the harmonics are the source of the disturbances most affecting telephonic communication. The situation thus created by the proximity of power and communication circuits constitutes the problem of *inductive interference.* Since power circuits cannot be divested of their dielectric and magnetic fields and communication circuits cannot be operated with complete satisfaction when indiscriminately immersed in them, a satisfactory solution of the problem can only be had as a result of the closest cooperation between the power and communication interests. The problem has received a good deal of attention at the hands of engineers and considerable literature on the subject is available. Perhaps the most extensive single source is to be found in the report of the California Railroad Commission entitled "Inductive Interference."

Electromagnetically-induced Voltages. 1. *Single-phase Line.* In Chap. III is discussed in some detail how and to what

extent alternating currents in one conductor produce varying magnetic fields that reach out and link other adjacent conductors, thereby inducing potential differences in them. These potential differences, even though very small as compared with those impressed upon power circuits, may nevertheless produce a disturbance in a nearby telephone circuit owing to the sensitivity of the instruments used.

To illustrate how electromagnetically-induced potential differences are set up, assume a single-phase power circuit ab, and a nearby, parallel telephone circuit mn, m being the near conductor and n the more remote one. Let the power circuit carry a sine-wave current whose value is I absolute units. If the current in a be taken as positive, that in b is negative, and their sum is zero; that is,

$$I_a = I_b = 0.$$

Representing the distances from a to m, a to n, etc., by the appropriate letters am, etc., the flux linking the telephone circuit per centimeter of loop is

$$\begin{aligned} \phi_{mn} &= 2I\left(ln\frac{an}{am} - ln\frac{bn}{bm}\right) \\ &= 2Iln\frac{an \cdot bm}{am \cdot bn} \text{ linkages.} \end{aligned} \tag{289}$$

Neglecting the effect of the telephonic currents on the power circuit (they are extremely small), the coefficient of mutual induction per centimeter of loop is

$$M = \frac{\phi_{mn}}{I} = 2ln\frac{an \cdot bm}{am \cdot bn}$$

and the induced voltage per centimeter of loop is

$$\begin{aligned} E_{mn} &= j\omega MI = j\omega\phi_{mn} \\ &= 2j\omega Iln\frac{an \cdot bm}{am \cdot bn} \text{ abvolts per centimeter.} \end{aligned} \tag{290}$$

Only that part of the flux from each conductor which threads the loop of the telephone circuit induces any resultant potential difference, for all flux lines which link both conductors of a circuit induce therein equal voltages that are oppositely directed around the loop. The amount of flux through the loop depends upon the current and upon the difference of the logarithms of the

distances between the disturbing wires and the telephone loop, as in Eq. (289). The induced voltage is proportional to the flux (and hence to the current), and to the frequency of the inducing current. As the distance between the power wires is reduced, the fraction whose logarithm is involved in Eq. (290) approaches zero, as does also the induced potential difference.

2. *Three-phase Line.*—If the disturbing circuit is a three-phase power line, the magnetic fields due to the currents in the three phases all induce potential differences in the loop of the communication circuit. The resultant potential difference induced is the vector sum of the individual potential differences. Thus if I_a, I_b and I_c are the known vector sine-wave currents in the three line conductors a, b and c of the power circuit, and m and n are the nearer and the farther wires of the communication circuit respectively, the induced vector potential difference in the communication circuit loop is

$$\begin{aligned} E_{mn} &= j\omega\phi_{mn} \\ &= j\omega\left[2I_a ln\frac{an}{am} + 2I_b ln\frac{bn}{bm} + 2I_c ln\frac{cn}{cm}\right] \text{ vector abvolts per centimeter.} \quad (291) \end{aligned}$$

When harmonics are present, the induced potential difference due to each is separately considered. If, per unit of induced potential difference in the given circuit, all harmonics may be considered as being approximately equal in their disturbing effects, the effect due to all harmonics may be taken as approximately proportional to the effective value of the total induced potential difference, that is, to the square root of the sum of the squares of the potential differences due to the individual harmonics.

If the three-phase currents are balanced, their vector sum is zero. The three vector currents may then be represented as

$$\begin{aligned} I_a &= I_0(1 + j0) \\ I_b &= -\frac{I_0}{2}(1 + j\sqrt{3}) \\ I_c &= \frac{I_0}{2}(-1 + j\sqrt{3}). \end{aligned}$$

Substituting these currents in Eq. (291), the induced potential difference in mn is

$$E_{mn} = j\omega\left(2I_0 ln\frac{an}{am} - I_0 ln\frac{bn}{bm} - I_0 ln\frac{cn}{cm} - jI_0\sqrt{3}ln\frac{bn}{bm} + jI_0\sqrt{3}ln\frac{cn}{cm}\right)$$

$$= I_0\omega\left(jln\frac{\overline{an}^2 \cdot bm \cdot cm}{\overline{am}^2 \cdot bn \cdot cn} - \sqrt{3}ln\frac{cn \cdot bm}{cm \cdot bn}\right) \text{ vector abvolts per centimeter of loop} \quad (292)$$

or, in amount, the potential difference is

$$E_{mn} = 370.56 \times 10^{-6} I_0\omega \times \sqrt{\left(\log_{10}\frac{\overline{an}^2 \cdot bm \cdot cm}{\overline{am}^2 \cdot bn \cdot cn}\right)^2 + 3\left(\log_{10}\frac{cn \cdot bm}{cm \cdot bn}\right)^2} \text{ abvolts per mile of loop.} \quad (293)$$

Residual Currents in Unbalanced Three-phase Lines.—The currents in a three-phase line are said to be balanced when their vector sum is zero. Conversely, in an unbalanced line the vector sum of the three currents yields a resultant current which is not zero. This resultant current is called the *residual current.* A convenient method[1] of calculating the inductive effects of such currents is to deal with their equivalent, balanced, three-phase and single-phase components as discussed below.

The three line currents of an unbalanced three-phase line may be resolved into (*a*) three equal line currents differing in phase by 120° (balanced currents); (*b*) three equal, residual currents, each of which is one-third of the residual current, as defined above, flowing in a loop of which one side is the neutral conductor (or ground return) and the other consists of the three line wires in multiple; and (*c*) a single-phase current flowing out in one line conductor and returning in another. A vector diagram showing these relations is given in Fig. 47. The inductive effects of unbalanced currents may be calculated by dealing with these components in place of the unbalanced currents themselves, using the methods already described.

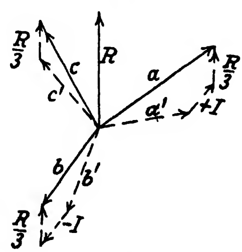


FIG. 47.—Residual currents in unbalanced three-phase lines.

[1] "Inductive Interference between Elctric Power and Communication Circuits," California Railroad Commission, p. 120.

In the diagram a, b and c are the unbalanced three-phase currents, equivalent to the three balanced currents a', b' and c', plus the single-phase residual R, plus the single-phase current I circulating in the leads A and B.

The effect of the residual currents is, in general, of far greater importance than that of the balanced, three-phase currents, because the ratio of the distances from the disturbing to the disturbed conductors, shown in Eq. (293), is usually greater for the residuals.

Electrostatically-induced Voltages.—Electrostatically-induced potentials are set up in conductors occupying any space in which there is a dielectric field. In the space about a power circuit, each power conductor sets up a varying dielectric field which is at all times proportional to the relative potential of the conductor. In such a field the two conductors of a communication line paralleling a power line will, in general, be subjected to fields of unequal intensities; the electrical potentials of the conductors will be unequal, and thus there will be a difference of potential between them. There will also be a difference of potential between the wires and ground, which, in general, is many times as large as that between the two wires of a circuit.

The electrostatically-induced potential differences may be calculated by the methods of Chap. IV. Let a, b and c be the three conductors of an untransposed, three-phase power circuit and let m and n be the two conductors of a communication circuit paralleling the power line, m being the near conductor as before. Let r be the radius of each of the three equal power cables and represent the distances from each power cable to the wires of the communication circuit by the appropriate letters as am, bm, cn, etc. Representing the vector flux, per centimeter of power conductor for the three conductors, by ψ_a, ψ_b and ψ_c in practical units, and neglecting the capacitance of the power line to ground, the vector potential differences between the three phases of the power line are (Eq. (232))

$$\left.\begin{aligned} E_{ab} &= \frac{\lambda}{2\pi}\left(\psi_a ln\frac{D_1}{r} - \psi_b ln\frac{D_1}{r} + \psi_c ln\frac{D_2}{D_3}\right) \\ E_{bc} &= \frac{\lambda}{2\pi}\left(\psi_a ln\frac{D_3}{D_1} + \psi_b ln\frac{D_2}{r} - \psi_c ln\frac{D_2}{r}\right) \\ E_{ca} &= \frac{\lambda}{2\pi}\left(\psi_a ln\frac{r}{D_3} + \psi_b ln\frac{D_1}{D_2} + \psi_c ln\frac{D_3}{r}\right) \end{aligned}\right\} \text{vector volts} \qquad (294)$$

Where λ is the conversion factor $10^{-9}v^2$ and v is the velocity of light in centimeters per second, or, approximately, $v = 3 \times 10^{10}$ cm. per second.

The vector line potential differences E_{ab}, E_{bc} and E_{ca} are known. Their magnitudes are measurable and the vectors form an equilateral triangle. In the group of three equations (Eq. (294)) there are two independent equations only, containing three unknowns. It is necessary to have one additional, independent equation in order to solve for the three fluxes.

Balanced Three-phase Voltages.—If the three-phase line is balanced, the vector sum of the three potential differences between line conductors and neutral is zero, as is also the vector sum of the dielectric fluxes. Thus, for a balanced line,

$$\psi_a + \psi_b + \psi_c = 0$$

is the third required, independent equation. Once the three fluxes are known, the potential difference induced between the conductors m and n by electrostatic induction may be found. It is

$$E_{mn} = \frac{\lambda}{2\pi}\left(\psi_a ln\frac{an}{am} + \psi_b ln\frac{bn}{bm} + \psi_c ln\frac{cn}{cm}\right) \text{ vector volts.} \qquad (295)$$

Since the circuit is balanced, the three equal fluxes are 120 time degrees apart and so may be represented as

$$\psi_a = \psi_0(1 + j0)$$

$$\psi_b = -\frac{\psi_0}{2}(1 + j\sqrt{3})$$

$$\psi_c = \frac{\psi_0}{2}(-1 + j\sqrt{3}).$$

When these values are substituted in Eq. (295), a solution of the resulting equation for the amount of the potential difference yields

$$E_{mn} = \frac{2.302\lambda\psi_0}{4\pi}\sqrt{\left(\log_{10}\frac{\overline{an}^2 \cdot bm \cdot cm}{\overline{am}^2 \cdot bn \cdot cn}\right)^2 + 3\left(\log_{10}\frac{cn \cdot bm}{cm \cdot bn}\right)^2} \text{ volts} \qquad (296)$$

This is the potential difference impressed across the capacitance of the communication line at its mid-point. The disturbing current produced by it is the charging current of this condenser. When the condenser is charging, current flows from the two ends of one wire toward its middle, and away from the middle of the second wire toward its ends. When the condenser discharges, the reverse action takes place.

Unbalanced, Three-phase Voltages.—The voltages of a three-phase line are unbalanced when the vector sums of the three voltages to neutral is not zero. The line-to-line voltages supplied by the generators, however, are three equal voltages displaced by 120 time degrees from each other. Thus, in Fig. 48, the three-line-to-line voltages of an unbalanced line are represented by the vectors AB, BC and CA. The three unbalanced voltages to neutral or ground potential are OA, OB and OC. If the voltages were balanced the three potential differences to neutral would be $O'A$, $O'B$ and $O'C$, and the neutral O would be at O'. In the vector diagram however, it is seen that

$$\begin{aligned} OO' + O'A &= OA \\ OO' + O'B &= OB \\ OO' + O'C &= OC \end{aligned}$$

and thus the three unbalanced voltages are equivalent to the three balanced voltages plus three times the potential difference between ground and the neutral of the balanced system. The potential difference OO' is the *residual voltage* of each conductor, while the *residual voltage of the system* is three times this value, and is in phase with OO'.

The potential differences induced in the communication circuit may be estimated either directly or by splitting the three line potential differences to neutral into their residual and balanced components. If the vector potential differences OA, OB and OC are known, the residual voltage may be found.

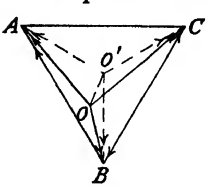


FIG. 48.—Residual voltages in unbalanced three-phase lines.

Considering the three line conductors of Fig. 48 and their images (not shown), and using the method of images of Chap. IV, the known vector potential differences to neutral in terms of dielectric fluxes are

$$\left.\begin{aligned} 4\pi E_{Oa} &= \frac{1}{2}\left[2\lambda\left(\psi_a ln\left(\frac{aa'}{r}\right)^2 + \psi_b ln\left(\frac{ba'}{ba}\right)^2 + \psi_c ln\left(\frac{ca'}{ca}\right)^2\right)\right] \\ &= 2\lambda\left(\psi_a ln\frac{aa'}{r} + \psi_b ln\frac{ba'}{ba} + \psi_c ln\frac{ca'}{ca}\right) \\ 4\pi E_{Ob} &= 2\lambda\left(\psi_a ln\frac{ab'}{ab} + \psi_b ln\frac{bb'}{r} + \psi_c ln\frac{cb'}{cb}\right) \\ 4\pi E_{Oc} &= 2\lambda\left(\psi_a ln\frac{ac'}{ac} + \psi_b ln\frac{bc'}{bc} + \psi_c ln\frac{cc'}{r}\right) \end{aligned}\right\} \quad (297)$$

where aa', ba' and ca, etc., are the distances between the conductors designated by the letters A and A', B and A', C and A, etc., respectively.

Since the line is unbalanced, the vector sum of the fluxes is not zero. The separate fluxes may, however, be found by solving the above three equations simultaneously. Once the fluxes are known, the induced potential difference between the two wires m and n of the communication circuit may be set down by evaluating the expression

$$E = \frac{\lambda\psi}{2\pi}\int_{x_1}^{x_2}\frac{dx}{x}$$

for each power conductor and its image. The fluxes ψ_a, ψ_b and ψ_c are taken as positive, and are respectively equal to the negatives of the fluxes ψ_a', ψ_b' and ψ_c'. The values x_1 and x_2 are the distances from the power conductors to the near and far wires of the communication circuit, respectively. Thus, the induced voltage in circuit mn is

$$E_{mn} = \frac{\lambda}{2\pi}\left(\psi_a ln\frac{an \cdot a'm}{am \cdot a'n} + \psi_b ln\frac{bn \cdot b'm}{bm \cdot b'n} + \psi_c ln\frac{cn \cdot c'm}{cm \cdot c'n}\right) \text{ vector volts.} \quad (298)$$

Instead of proceeding as above one may deal with the residual and balanced components of line voltage and their induced potential differences separately. The induced potential difference due to the balanced components may be calculated from Eq. (295), while that due to the residual is

$$\text{Residual } E_{mn} = \frac{\lambda}{2\pi}\psi_R ln\frac{Rn \cdot R'm}{Rm \cdot R'n} \text{ vector volts} \quad (299)$$

where

$$\psi_R = \text{the complex residual flux per centimeter}$$
$$= \psi_a + \psi_b + \psi_c.$$

R is the point at the center of mass of the conductor configuration triangle, and R' is a point midway between R and its image.

Causes of Current and Voltage Unbalance.—Two general types of networks are in use; namely, (a) insulated systems in which no ground connections are made between neutral and ground at any point, and (b) grounded-neutral systems in which the generator neutrals as well as the neutrals at the load centers are grounded.

In isolated systems perhaps the principal sources of trouble are unbalanced capacitances and leakances between the different phases and ground. The former may largely be corrected by suitable transpositions, while the latter is chiefly a question of careful construction and maintenance. The unbalanced capacitance is less for lines having flat horizontal spacings than for those with vertical or triangular arrangements of conductors.

In grounded-neutral systems unequal load impedances from line to neutral will cause unbalanced components of current to flow in the neutral or ground return. Third harmonics and their odd multiples are set up by the cyclic variation of the core flux in transformers, the percentage of the harmonic being dependent, to a considerable extent, upon the degree of saturation of the core. In a grounded-neutral system using star-to-star transformer connections, these harmonics will appear as unbalanced currents flowing in the neutral connection, the currents due to the three phases being additive in the neutral return. The use of delta connection on the secondary side provides a short-circuit path for these harmonics and greatly reduces them. When a resistance or a reactance is inserted in the connection between neutral and ground, currents flowing to ground will cause voltage unbalance.

Harmonics.—The current and voltage waves of power circuits are always more or less distorted. The positive and negative loops of succeeding cycles, however, are alike in shape. If the negative loop be rotated about the zero axis and then moved over to match the positive loop, the two loops are found to be identical in shape. Owing to symmetry of construction, all waves generated by rotating, electrical machinery have this type of symmetry. Composite waves of this type are in fact made up of a number of sine waves of various frequencies, consisting of fundamental and odd integral multiples of the fundamental frequency. Harmonics may be introduced by generators or large synchronous motors on account of pulsating, air-gap reluctance in slotted armatures, by transformers due to the cyclic variation of core reluctances, by certain types of load, such as electric arcs and rectifiers, and by the formation of corona at the peaks of the voltage waves.

From the standpoint of interference, the harmonics, even though of relatively small magnitudes, are of greater importance than the fundamental. The disturbing effect set up in a telephone

circuit depends upon the amount of the induced voltage and upon the frequency of the induced current. Since the voltage itself is proportional to the frequency, within the approximate range between say 200 and 800 cycles, the disturbing effect is proportional to the square of the frequency. Frequencies which fall within the range of the average voice frequency produce far greater disturbances per unit of induced voltage than do those lying without or near one end of the voice-frequency band.

The voice-frequency band approximately covers the range from 200 cycles per second to 2,000 cycles per second. Near its middle, or in the neighborhood of 800 cycles per second, lies what is usually taken as the average voice frequency. At approximately this frequency, the telephone receiver is most sensitive. The standard fundamental frequency of 50 or 60 cycles, therefore, is ordinarily of little importance in producing disturbances in telephone lines, while in telegraph lines this frequency may be the source of interference troubles.

PROBLEMS

1. The three equal conductors of a 60-kv., three-phase, 60-cycle power line are each 0.533 in. in diameter. The conductors are supported at the vertices of an equilateral triangle whose base is parallel to the earth's surface. Beginning with the uppermost conductor and reading counter-clockwise, the conductors are a, c, b, the conductor a lying on the center line of the supporting structure. The conductors are 6 ft. apart between centers, and the plane cb is 40 ft. above ground. Fifty feet to the right of the center line of the power circuit is the center line of a horizontal telephone loop mn, running parallel to the power circuit. The telephone wire m is 30 ft. above ground, and the width of the loop is 2 ft. The balanced voltages to neutral of the power circuit are

$$\left.\begin{aligned} E_{0a} &= 34{,}600(1 + j0) \\ E_{0b} &= -17{,}300(1 + j\sqrt{3}) \\ E_{0c} &= 17{,}300(-1 + j\sqrt{3}) \end{aligned}\right\} \text{ vector volts}$$

and the balanced currents are each equal to 175 amp. and lag the above voltages by 30 time degrees, respectively. Calculate the value of (a) the electromagnetically-induced voltage, and (b) the electrostatically-induced voltage in the telephone loop per mile of untransposed parallel.

2. A 110-kv., 60-cycle, three-phase power line is built of three conductors, each of 0.631-in. diameter. They are strung 11 ft. apart in a horizontal plane, 40 ft. above ground, the middle conductor lying on the center line of the support. The center line of a horizontal telephone loop is 60 ft. to the right of the center line of the power line, and its conductors m and n are 2 ft. apart. Reading from left to right in the diagram representing the arrangement of lines described above, the power conductors are a, b and c,

and the telephone conductors are m and n. The line-to-line voltage E_{ab} of the power line is

$$E_{ab} = 110{,}000 + j0$$

and the voltages to neutral are

$$\begin{aligned} E_{0a} &= 34{,}900 + j55{,}000 \\ E_{0b} &= 75{,}100 + j55{,}000 \\ E_{0c} &= 20{,}100 - j40{,}300. \end{aligned}$$

Assuming the earth's potential as zero, what is the electrostatically-induced voltage E_{mn}?

CHAPTER VII

SHORT-LINE CALCULATIONS

The structure of a transmission line is such that its electrical properties per unit length of line are practically constant. These properties are represented by the symbols r, g, L and C, and are called the line constants. The magnitudes of these constants depend only upon the conductor size, conductor material, and conductor arrangements in space.

Because the so-called constants of a given transmission line do not vary, the transfer of energy over the line follows laws which may readily be expressed in simple mathematical form when the current and voltage are sinusoidal. When automatic regulators are used to maintain constant voltage at either one or both ends of the line and at all loads, the performance calculations are still further simplified.

The exact equations of the transmission line involving the correct assumption of uniformly distributed, line capacitance, while simple in form, involve considerable calculation. In order to simplify the task of making transmission-line calculations, where such simplification is permissible without sacrificing too much in accuracy, two expedients have been resorted to.

1. The use of the first one or two terms only of the converging infinite series which are the equivalents of the hyperbolic functions used in the exact equations. This simplifies the operations considerably, but the error increases as the number of terms used in each series diminishes.

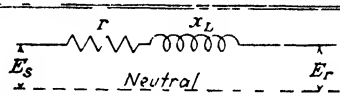
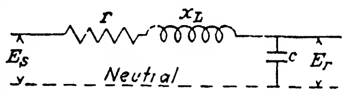
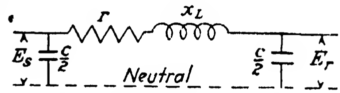
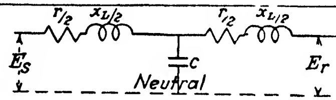
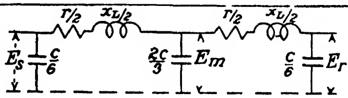
2. The substitution of hypothetical circuits having lumped capacitance or capacitances (or no capacitance at all), which perform very nearly like the actual transmission line. Under this classification various kinds of substitute circuits are possible.

The method described under (1) above will be discussed in the chapter dealing with the exact equations of the line. Under (2) a number of different approximations of various degrees of complexity are used, depending principally upon the length of the line under consideration, and the degree of accuracy desired.

These substitute circuits differ from each other chiefly in the methods employed to approximate the effect of the line capacitance. In lines under 30 miles in length the capacitance of the line is often entirely neglected in making line calculations.

A list of the substitute circuits used in line calculations is given in Table 8.

TABLE 8.—TYPES OF SUBSTITUTE CIRCUITS FOR APPROXIMATE CALCULATIONS

METHOD	CIRCUIT
Impedance	
Load-end Condenser	
Nominal π	
Nominal T	
Dr. Steinmetz' Three Condenser Method	
H. B. Dwights' K Formulas	*See Transmission Line Formulas by H. B. Dwight*

In addition to the above, graphical and semigraphical methods are available.

Simple Impedance Circuits.—In the simplest approximate circuit the line is assumed to have no capacitance. The circuit then reduces to a series circuit containing only resistance and inductive reactance. This approximately equivalent line will have a greater line drop, and, usually, a somewhat larger generator current and poorer power factor then the actual line. For 60-

cycle lines under 30 miles long delivering fair sized loads, however, the error is usually negligible. Even for 60-cycle lines up to 50 miles the error will probably usually not exceed 0.6 per cent. Thus, for lines up to perhaps 25 or 30 miles, there is little practical objection to the use of this approximately equivalent circuit.

In Fig. 49 is represented one leg of a three-phase line, over which one-third of the power delivered to the line is assumed to be transmitted. It is further assumed that the receiver-end conditions are known, that is, the receiver load, voltage, current and power factor. The problem is to find the corresponding quantities at the generator or input end, the line regulation, losses, etc. Quantities carrying the subscripts r refer to the receiver end, while those carrying the subscript s refer to the generator or supply end. For quantities which are the same at both ends the

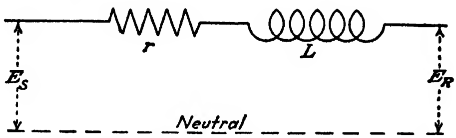


FIG. 49.—Diagram of simple impedance circuit.

subscripts are omitted. The supply-end voltage is assumed to be controlled by varying the generator excitation to whatever extent is necessary to keep the receiver voltage constant in value. The receiver voltage is used as the vector of reference. Since the conductance of the leakage path and the condensance of the line are both negligibly small, the supply and receiver currents are one and the same. Thus, assuming that the receiver current may be either leading or lagging, the line current is

$$I = I_1 \pm jI_2 \text{ vector amp.}$$

and the receiver power factor angle is

$$\theta_r = \pm \tan^{-1}\frac{I_2}{I_1}.$$

The line impedance is

$$Z = r + jx \text{ vector ohms}$$

where

r = the total resistance of one conductor

and

x = the total inductive reactance of one conductor.

The impedance drop in the line is

$$
\begin{aligned}
ZI &= (r + jx)(I_1 \pm jI_2) \\
&= rI_1 \mp xI_2 + j(xI_1 \pm rI_2) \qquad (300) \\
&= I_1(r + jx) + I_2(\mp x \pm jr) \text{ vector volts.} \qquad (301)
\end{aligned}
$$

The supply-end voltage is the receiver voltage plus the impedance drop, or

$$
\begin{aligned}
E_s &= E_r + ZI \\
&= E_r + rI_1 \mp xI_2 + j(xI_1 \pm rI_2) \qquad (302) \\
&= {}_sE_1 + j{}_sE_2 \text{ vector volts}
\end{aligned}
$$

where ${}_sE_1$ and ${}_sE_2$ are the real and the quadrature components of the supply voltage, respectively. The angle of the supply voltage referred to the receiver voltage is

$$
{}_e\theta_s = \tan^{-1} \frac{{}_sE_2}{{}_sE_1} \qquad (303)
$$

and the supply power factor angle is

$$
\begin{aligned}
\theta_s &= {}_e\theta_s - (\pm\theta_r) \\
&= {}_e\theta_s \mp \theta_r. \qquad (304)
\end{aligned}
$$

In terms of absolute values, the supply voltage is

$$
\begin{aligned}
E_s &= \sqrt{{}_sE_1^2 + {}_sE_2^2} \\
&= \sqrt{(E_r + rI_1 \mp xI_2)^2 + (xI_1 \pm rI_2)^2} \text{ volts.} \qquad (305)
\end{aligned}
$$

Note that, wherever the double signs appear, the upper one is the condition for leading and the lower one for lagging current.

The percentage of line regulation found by substituting full-load current values in Eq. (305) is

$$
\text{per cent regulation} = \frac{100(E_s - E_r)}{E_r}. \qquad (306)
$$

The supply output is

$$
Kw_s = E_sI \cos \theta_s \qquad (307)
$$

the receiver input is

$$
Kw_r = E_rI \cos \theta_r \qquad (308)
$$

and the line loss is

$$
\left.
\begin{aligned}
\text{Loss} &= E_sI \cos \theta_s - E_sI \cos \theta_r \\
\text{or} \quad \text{Loss} &= \frac{I^2r}{1{,}000} \text{ kw.}
\end{aligned}
\right\} \qquad (309)
$$

Vector Diagram for Impedance Circuit.—The vector diagram for this case is conveniently built up by adding the impedance drop of the line, as given by Eq. (301), to the receiver voltage to obtain the supply voltage. Thus, in Fig. 50, assuming a lagging current having components I_1 and $-jI_2$, the four component drops are laid off as indicated in the solid-line diagram. The first term of the right-hand member of Eq. (301) gives the drop due to the in-phase component of the current, I_1, resulting in the right triangle of which ZI_1 is the hypotenuse, while the second term yields the drop due to the quadrature component of the line current, as shown in the right triangle of which ZI_2 is the hypotenuse. The vector sum of ZI_1 and ZI_2 is the total impedance drop ZI. It is to be noted that the two right triangles above

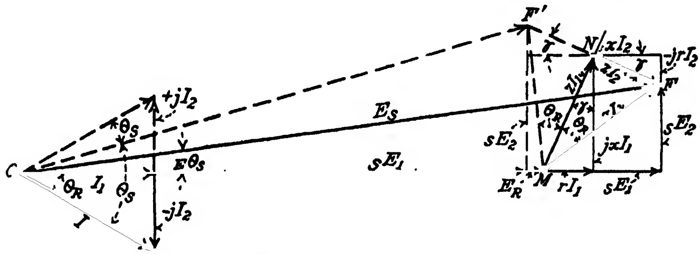


FIG. 50.—Vector diagram for the simple impedance circuit.

referred to are similar since their acute angles are each equal to the angle γ. That is,

$$\gamma = \tan^{-1}\frac{rI_1}{xI_1} = \tan^{-1}\frac{rI_2}{xI_2} = \tan^{-1}\frac{r}{x}.$$

The line ZI_2 is therefore perpendicular to ZI_1. Furthermore, in the triangle MNF, angle $NMF = \theta_r$, the receiver power factor angle, for

$$\angle NMF = \tan^{-1}\frac{ZI_2}{ZI_1}$$
$$= \theta_r.$$

The diagram may readily be extended to represent any power factor by simply including a series of lines to represent different values of θ_r. The effect, upon the receiver voltage, of introducing leading or lagging components of current, is clearly shown in the diagram. For given values of supply voltage, increasing lagging quadrature currents depress the receiver voltage, while increasing leading quadrature currents cause it to rise. This

changing effect is due to the angle at which the line impedance drop subtracts vectorially from the generator voltage to produce the receiver voltage.

Perrine-Baum Diagram.—This diagram is useful as an aid in visualizing the performance of a short line, and serves at the same time as a means of solving short-line problems. Graphical solutions taken from carefully drawn diagrams will be of sufficient accuracy for practical purposes. It is always desirable, however, to make at least one purely mathematical check calculation, when a graphical method is used.

The Perrine-Baum diagram follows at once by somewhat generalizing Fig. 50. The diagram is illustrated in Fig. 51. To draw the diagram, the procedure is as follows: Choose a suitable

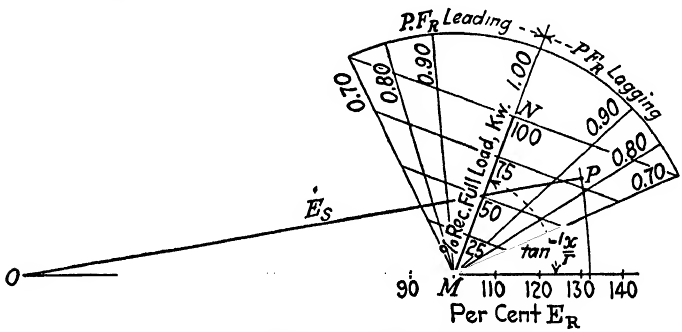


FIG. 51.—The Perrine-Baum diagram.

scale, and lay off OM along the X-axis equal to the receiver voltage E_r. Draw MN, making an angle $\theta = \tan^{-1}\frac{x}{r}$ with OM (x and r are the reactance and resistance of one line conductor, respectively). The length of MN to the voltage scale is the product of the full-load value of I_1 and the line impedance; that is,

$$MN = I_1\sqrt{r^2 + x^2} = I_1 z.$$

MN thus represents the impedance drop in the line at full load and unity power factor. By dividing MN into segments proportional to the loads, the impedance drop at unity power factor for any load may be read directly. Draw radial power factor lines to represent various power factors, both leading and lagging, such as MF, MF' etc., of Fig. 50. Lay off a scale of voltages along OM in percentage of the receiver voltage; that is, $OM = E_r = 100$ per cent, etc. This will serve as a measure of the supply voltage

when its vector is swung down to parallelism with OM. The line regulation may then readily be computed.

To illustrate the use of the diagram, suppose the line is loaded to 80 per cent of full-load kilowatts at a power factor of 85 per cent, current lagging. The end of the E_s vector is then found at P, and the supply voltage is seen to be 132 per cent of the receiver voltage.

Mershon Diagram.—Referring to Fig. 49, if the current be used as the axis of reference, the impedance drop in the line is then

$$ZI = I(r + jx) \text{ vector volts}$$

where I is the load current as before. The supply voltage is the sum of the receiver voltage and the impedance drop or

$$E_s = E_r + rI + jxI \text{ vector volts.} \tag{310}$$

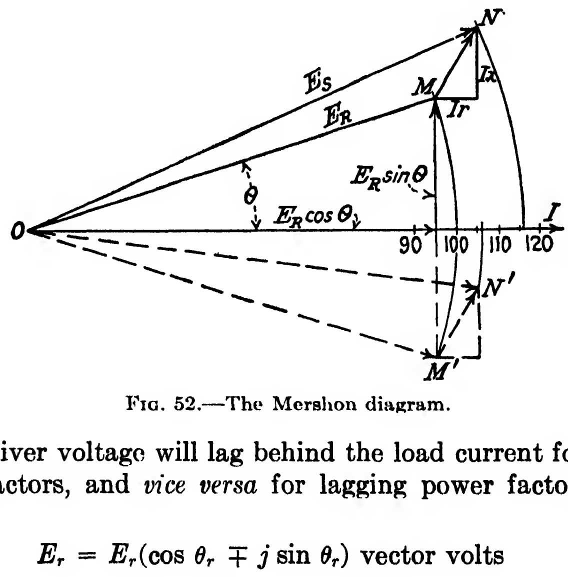


FIG. 52.—The Mershon diagram.

The receiver voltage will lag behind the load current for leading power factors, and *vice versa* for lagging power factors; or, in general,

$$E_r = E_r(\cos\theta_r \mp j\sin\theta_r) \text{ vector volts} \tag{311}$$

where the upper sign refers to leading currents, and $\cos\theta_r$ is the power factor of the receiver load. Combining Eqs. (310) and (311), the generator voltage is

$$E_s = E_r\cos\theta_r + Ir \mp j(E_r\sin\theta_r + Ix) \text{ vector volts.} \tag{312}$$

The vector diagram, of voltages corresponding to Eq. (312), is shown in Fig. (52). The solid-line diagram is for lagging current while the dotted-line diagram represents leading current.

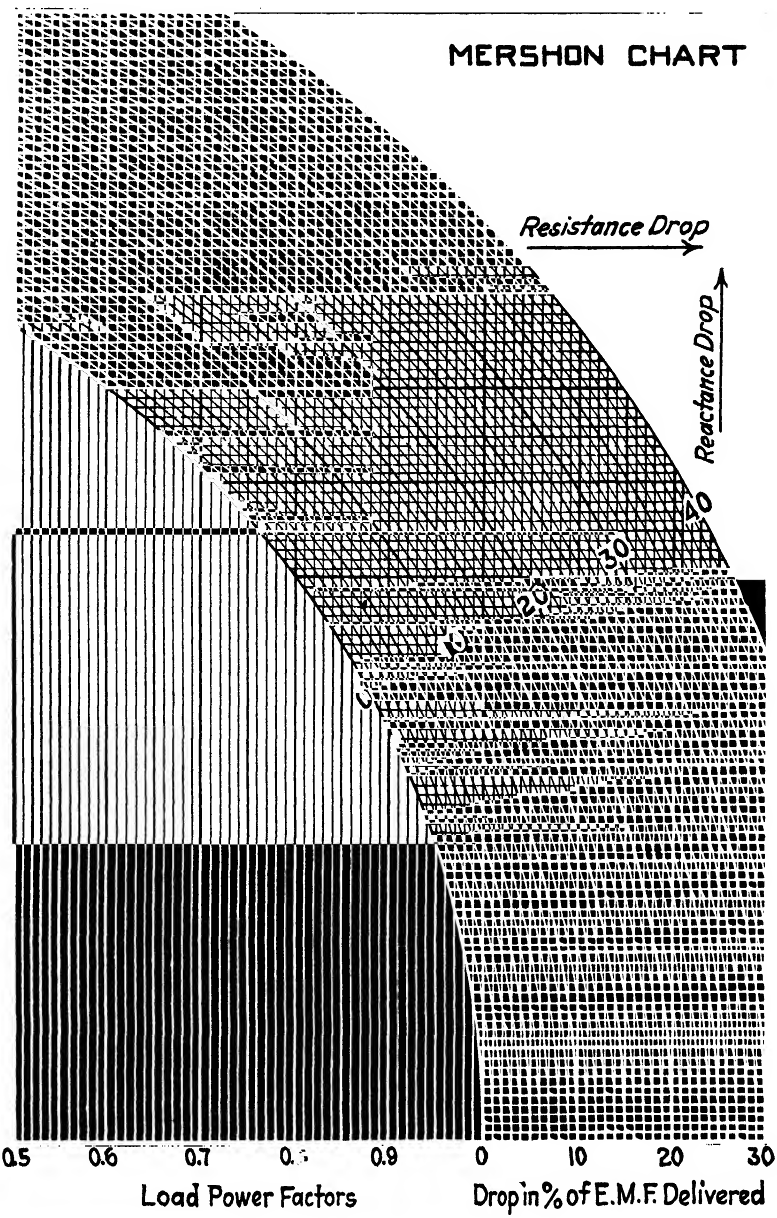


Fig. 53.—The Mershon chart.

It is apparent that if the receiver voltage remains constant in size, the point M will trace the arc of a circle as the power factor angle changes. Whatever the position of E_r, however, the impedance-drop line MN always has the same slope, and therefore adds vectorially to E_r at varying angles, to produce the supply voltage E_s. By expressing all components of voltage in percentage of E_r, the latter being 100 per cent, laying the percentage scale off on the X-axis as indicated, and drawing in a series of circles at 1 or 2 per cent intervals, the basis for the Mershon chart becomes apparent. This chart simply provides a ready means of adding the per cent impedance drop vector MN to E_r at any angle within the scope of the chart, and, by following down the arc of the circle passing through N, provides a means of reading off the scale on the X-axis the corresponding line drop in per cent of the receiver voltage. The Mershon chart is reproduced in Fig. 53.

Effect of Line Capacitance.—While the effect of line capacitance is negligible in 60-cycle lines up to 20 or 30 miles in length, it is a very important factor in the operation of very long lines, and, in general, must at least be approximated in the calculations for all lines except the short ones mentioned above. Since the various approximate solutions differ largely in the method used to approximate the capacitance effect, it is important to have a clear understanding of what happens in the circuit.

Capacitance, like resistance and inductance, is uniformly distributed along the line. Each unit length of line has a certain definite value of C, depending upon the size of conductors and their interaxial spacings. Each unit therefore requires a reactive component of leading current to charge the capacitance associated therewith. This current is a constant amount per unit length of line per volt of potential difference, and leads the potential difference between line conductors at the particular point considered by 90°. Since the vector-line voltage varies from point to point along the line, both in size and in angular position, however, it is apparent that the quadrature component, of current per line element, also varies in size and phase from point to point. Furthermore, since the line current is the sum of the load current and the aggregate of all quadrature currents required to charge the elements of the line lying between the chosen point and the receiver, the current in the line varies from point to point, and, likewise, the drop in the series impedance of the line per unit of length is a variable. Leading current flowing in the inductive

reactance of the line causes the voltage to rise towards the end of the line. The line capacitance thus has the effect of decreasing the line drop for a given load current (Figs. 50 and 52).

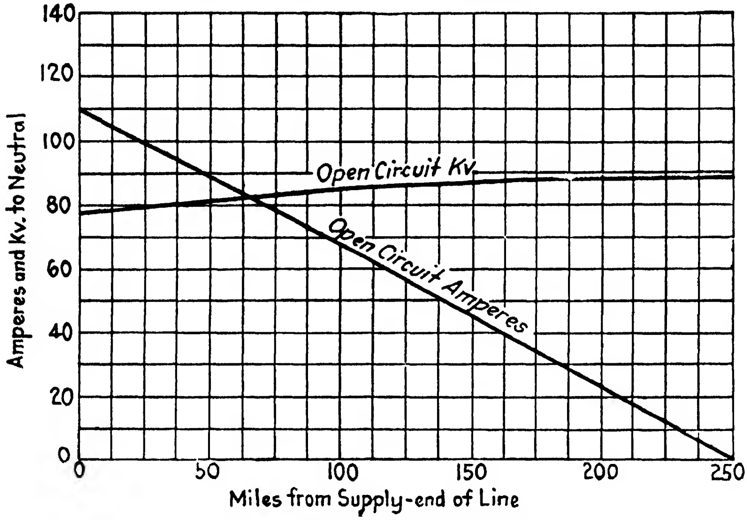


FIG. 54.—Open-circuit current and voltage of a 250-mile transmission line.

Charging Current.—Consider a transmission line which is open at the receiver end and is subjected to a potential difference at the supply end, of a value such that the receiver-end voltage is

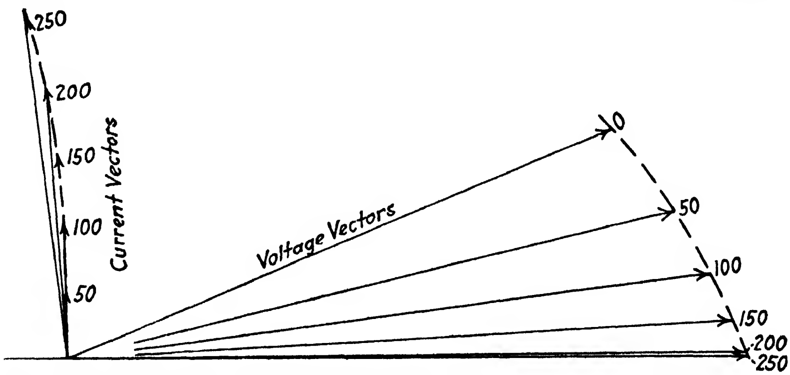


FIG. 55.—Open-circuit current and voltage vectors of a 250-mile transmission line.

normal. The current in such a line is zero at the receiver and increases towards the supply end, where it has its maximum value. The latter value is called the *charging current* of the line. The

voltage is maximum at the receiver end and decreases towards the supply end. In a long line the former may be considerably higher than the latter. For example, in the 200-mile line of Chap. XVII, with normal supply voltage impressed, the no-load receiver-end voltage rises to a value of 22.9 per cent above the supply-end voltage. The no-load charging current for the same line is 29.1 per cent of the full-load supply current or 88.1 amp., for normal receiver voltage. The variations in the size, of no-load current and voltage vectors along a 250-mile line for normal receiver-end voltage, are illustrated in the curves of Fig. 54. In Fig. 55 the corresponding polar diagrams are drawn.

Load-end Condenser Method.—The roughest approximation in which the line capacitance is considered, assumes a circuit like the second one of Table 8. Here the capacitance of one line conductor is assumed to be placed across the line in parallel with the load at the receiver end. The charging current is thus the product of the receiver voltage to neutral and the admittance of the capacity to neutral, and is assumed to be constant throughout the line, and leading the receiver voltage by 90°. The line current is the vector sum of the load current and the line-charging current, and the line drop is the product of this current and the series impedance of the line. The method of calculation, therefore, differs only slightly from that already given for the simple impedance circuit, and will not be developed in detail. The use of this method results in a generator voltage which is too low by about the same amount as it is too high when the impedance method is used. An average of the values obtained by the two methods is more nearly correct than either value. The impedance method is uncompensated for the rise in voltage towards the receiver end, due to the charging current, while the load-end condenser method overcompensates for this effect.

Nominal π Line.—This is the third circuit of Table 8. Here only one-half the line capacitance is assumed to be put in parallel with the load, the remaining one-half being put from line to neutral at the generator end. The method of calculation is identical with that of the single-load-end condenser, except that

(*a*) Only one-half of the line-charging current of the former is added to the load current in the latter.

(*b*) The supply current is the vector sum of the line current and the charging current of the supply-end condenser. The results secured by this method are considerably more accurate than those resulting from the use of the load-end condenser circuit.

Nominal *T* Line.—An inspection of Figs. 54 and 55 suggests the use of the approximation employed in this circuit. From these figures it is observed that the open-circuit current increases at a fairly uniform rate along the line. No great error will result in a short line if it be assumed as a constant amount per mile of line. Similarly, the drop in voltage along the line is fairly uniform, and the value of the potential difference at the middle of the line is nearly equal to the average value. If a line having only resistance and inductive reactance be assumed, and if a lumped capacitance, equal to the capacitance of one line conductor to neutral, be placed across the line at its mid-point, the charging current flowing in it will be approximately equal to the actual charging current, and the drop in the impedance of one-half the line due to this current, will be approximately equal to the drop in the actual line. This type of approximate circuit is called the "nominal *T*" line.

The circuit for this line is given below in Fig. 56.

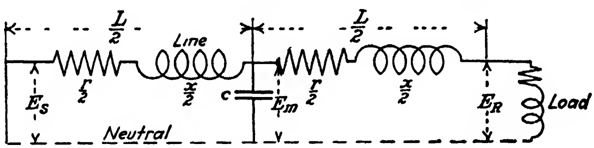


FIG 56.—The nominal *T* line.

Representing the load current by

$$I_r = I_1 - jI_2 \text{ vector amp.}$$

the potential difference across the lumped capacitance is

$$\begin{aligned} E_m &= E_r + \frac{1}{2}(I_1 - jI_2)(r + jx) \\ &= E_r + \frac{1}{2}(rI_1 + xI_2) + j\frac{1}{2}(xI_1 - rI_2) \qquad (313) \\ &= {}_mE_1 + j{}_mE_2 \text{ vector volts} \end{aligned}$$

where r and x are the resistance and reactance of one line conductor and ${}_mE_1$ and ${}_mE_2$ are the real and the quadrature components of the potential difference E_m, respectively.

The size of the vector E_m is

$$E_m = \sqrt{{}_mE_1^2 + {}_mE_2^2}. \qquad (314)$$

The current flowing in the line capacitance is

$$I_c = jb_cE_m \text{ vector amp.}$$

where $b_c = 2\pi fC$, the susceptance of one line conductor to neutral. Thus, from Eq. (313),

$$I_c = jb_c\left[E_r + \frac{1}{2}(rI_1 + xI_2) + j\frac{1}{2}(xI_1 - rI_2)\right]$$

$$= b_c\left\{\frac{1}{2}(rI_2 - xI_1) + j\left[E_r + \frac{1}{2}(rI_1 + xI_2)\right]\right\} \text{vector amp.} \quad (315)$$

and

$$I_s = I_r + I_c \text{ vector amp.}$$

$$= I_1 - jI_2 + I_c$$

$$= I_1 + b_c\left[\frac{1}{2}(rI_2 - xI_1)\right] + j\left\{b_c\left[E_r + \frac{1}{2}(rI_1 + xI_2)\right] - I_2\right\} \quad (316)$$

$$= {}_sI_1 \pm j{}_sI_2 \text{ vector amp.} \quad (317)$$

where ${}_sI_1$ and ${}_sI_2$ represent the real and quadrature components, respectively, of the supply current, as found from Eq. (316).

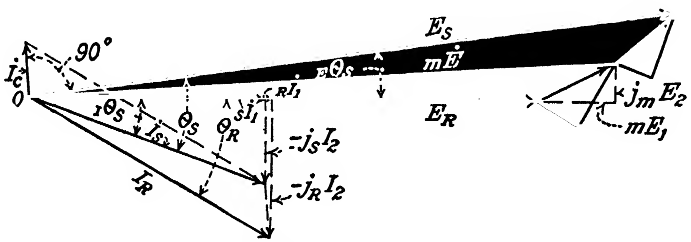


FIG 57 —Vector diagram for the nominal T line.

It is to be noted that the sign of ${}_sI_2$ of Eq. (317) may be either positive or negative, depending upon the relative magnitudes of $-I_2$ and the remainder of the j-component. Thus the second term may be either $+j{}_sI_2$ or $-j{}_sI_2$. The double sign here does not mean that it will have two values.

The angle of the supply current is

$$_1\theta_s = \tan^{-1}\frac{{}_sI_2}{{}_sI_1}. \quad (318)$$

The supply voltage is

$$E_s = E_m + \frac{1}{2}ZI_s$$

$$= E_m + \frac{1}{2}(r + jx)({}_sI_1 \pm j{}_sI_2)$$

$$= {}_sE_1 \pm j{}_sE_2 \text{ vector volts} \quad (319)$$

and its length is $E_s = \sqrt{{}_sE_1^2 + {}_sE_2^2}$ volts. (320)

The angle of the supply-voltage vector is

$$_e\theta_s = \pm \tan^{-1} \frac{_sE_2}{_sE_1} \tag{321}$$

and the supply power factor becomes

$$\cos \theta_s = \cos (_e\theta_s \pm {_i\theta_s}). \tag{322}$$

The line loss may now be computed. It is

$$\Sigma I^2r = E_sI_s \cos \theta_s - E_rI_r \cos \theta_r \tag{323}$$

$$= \frac{r}{2}(I_s^2 + I_r^2) \text{ watts.}$$

The vector diagram for the nominal T line is shown in Fig. 57.

The vector diagram of Fig. 50 may also be modified to compensate approximately for the rise in voltage due to the charging current of the nominal T line. The receiver voltage is assumed constant. The charging current is calculated, and is assumed to lead the receiver voltage by 90° at all loads. (This assumption is not strictly true for any load, but is very nearly so.) Referred to the receiver voltage, the charging current is jI_c, and the potential drop in one-half of the line due to this current is

$$j\frac{ZI}{2} = \frac{1}{2}(-xI_c + jrI_c) \text{ vector volts.}$$

This is approximately equal to the true rise in voltage in the entire line due to the actual charging current. The vector

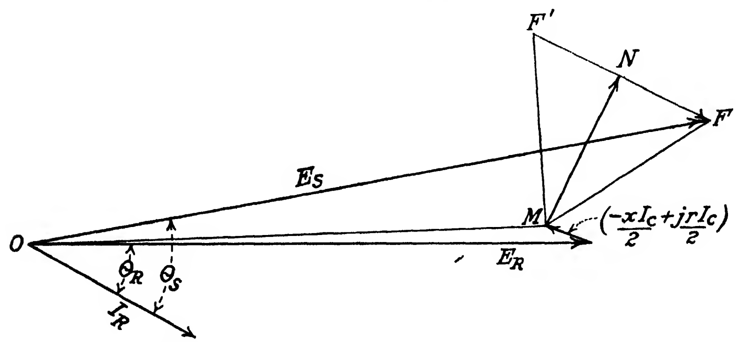


FIG. 58.—Diagram of voltages including the effect of charging current.

diagram of Fig. 50 may therefore be modified to include the effect of the charging current by adding the above value to the receiver voltage of Fig. 50, to locate the point M in Fig. 58. This

addition assumes that the charging current leads E_r by 90°, which is of course, not strictly true. The method of constructing the remainder of the diagram is the same as for Fig. 50.

Dr. Steinmetz' Split-condenser Method.—This method assumes the line capacitance to be divided into three parts, placed as shown in Fig. 59, $\frac{1}{6}C$ at each end of the line and $\frac{2}{3}C$ at the middle of the line, where C is the total capacitance of one conductor to neutral. The calculations are carried out step fashion in much the same way as for the nominal T method. Using the same notation as before, except that the currents in the two halves of the line are given the subscripts a and b in conformity with the notation in the figure, the charging current of load-end condenser is

$$I_c = j\frac{2\pi fCE_r}{6} = j_rI_c \text{ vector amp.}$$

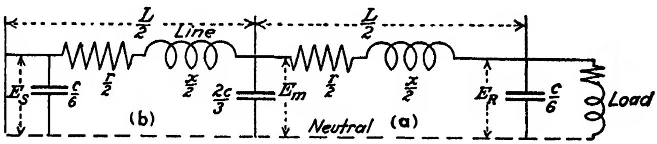


FIG. 59.—The Steinmetz split-condenser circuit.

For lagging power factors the load current is

$$I_r = I_1 - jI_2$$

whence

$$\begin{aligned} I_a &= I_1 - jI_2 + j_rI_c \\ &= I_1 - j(I_2 - {}_rI_c) \text{ vector amp.} \end{aligned}$$

$$\begin{aligned} E_m &= E_r + \frac{ZI_c}{2} \\ &= E_r + \frac{1}{2}(r + jx)[I_1 - j(I_2 - {}_rI_c)] \\ &= E_r + \frac{1}{2}[rI_1 + x(I_2 - {}_rI_c)] + \frac{1}{2}[(xI_1 - r(I_2 - {}_rI_c)] \qquad (324) \\ &= {}_mE_1 + j_mE_2 \text{ vector volts} \end{aligned}$$

and, in amount,

$$E_m = \sqrt{{}_mE_1^2 + {}_mE_2^2} \text{ volts absolute.} \qquad (325)$$

The charging current of the middle condenser is

$$
\begin{aligned}
{}_mI_c = jb_mE_m &= j\frac{4\pi fC}{3}({}_mE_1 + j_mE_2) \\
&= \frac{4\pi fC}{3}(-{}_mE_2 + j_mE_1) \qquad (326) \\
&= -{}_{mc}I_1 + j_{mc}I_2 \text{ vector amp.}
\end{aligned}
$$

Then

$$
\begin{aligned}
I_b &= I_a + {}_mI_c \\
&= I_1 - {}_{mc}I_1 - j(I_2 - {}_{rc}I_1 - {}_{mc}I_2) \qquad (327) \\
&= {}_bI_1 \pm j_bI_2 \text{ vector amp.} \qquad (328)
\end{aligned}
$$

The sign of I_b may be either positive or negative, depending upon the relative values of quantities within the parenthesis, hence the double sign in Eq. (328). The potential drop in the impedance of the remaining one-half of the line is

$$
\begin{aligned}
\frac{ZI_b}{2} &= \frac{1}{2}(r + jx)({}_bI_1 - j_bI_2) \\
&= \frac{1}{2}(r_bI_1 + x_bI_2) + j\frac{1}{2}(x_bI_1 - r_bI_2) \text{ vector volts} \qquad (329)
\end{aligned}
$$

and

$$
\begin{aligned}
E_s &= E_m + \frac{ZI_b}{2} \\
&= E_r + \frac{r}{2}(I_1 + {}_bI_1) + \frac{x}{2}(I_2 - {}_rI_c + {}_bI_2) \\
&\qquad + j\left[\frac{x}{2}(I_1 + {}_bI_1) - \frac{r}{2}(I_2 - {}_rI_c + {}_bI_2)\right] \qquad (330) \\
&= {}_sE_1 + j_sE_2 \text{ vector volts.} \qquad (331)
\end{aligned}
$$

Also

$$E_s = \sqrt{{}_sE_1^2 + {}_sE_2^2} \text{ volts absolute} \qquad (332)$$

and

$${}_e\theta_s = \tan^{-1}\frac{{}_sE_2}{{}_sE_1}. \qquad (333)$$

The charging current of the generator-end condenser is

$$
\begin{aligned}
{}_sI_c &= jb_sE_s \\
&= j\frac{2\pi fC}{3}({}_sE_1 + j_sE_2) \\
&= \frac{2\pi fC}{3}(-{}_sE_2 + j_sE_1) \\
&= -{}_{sc}I_1 + j_{sc}I_2 \text{ vector amp.} \qquad (334)
\end{aligned}
$$

and the generator current is

$$
\begin{aligned}
I_c &= I_b + {}_sI_c \\
&= {}_bI_1 - j_bI_2 - {}_{sc}I_1 + j_{sc}I_2 \\
&= {}_bI_1 - {}_{sc}I_1 + j({}_{sc}I_2 - {}_bI_2) && (335)\\
&= {}_sI_1 - j_sI_2 \text{ vector amp.} && (336)
\end{aligned}
$$

or

$$I_s = \sqrt{{}_sI_1^2 + {}_sI_2^2} \text{ volts absolute.} \qquad (337)$$

The angle of I_s is

$$\pm {}_i\theta_s = \tan^{-1} \frac{{}_sI_2}{{}_sI_1} \qquad (338)$$

and the generator power factor is

$$\cos \theta_s = \cos ({}_e\theta_s \pm {}_i\theta_s). \qquad (339)$$

In Eq. (339) the sign to be chosen for the second term is determined by the sign of ${}_i\theta_s$. The generator output and the line loss readily follow from the above relations.

This approximate method has the advantage of being sufficiently accurate in preliminary calculations for lines of any length yet built, but it is rather long and cumbersome.

Impedance of Line Transformers.—In the previous discussions of line calculations the impedance of raising and lowering transformers has been omitted. The transformer resistances and reactances, however, are usually not negligible. They may be combined with the line constants to give approximately correct results by simply adding the equivalent values, of resistance and reactance of transformers per line to neutral, to the corresponding constants already found for the line alone. This method of including the impedance offered by the transformers neglects the effect of the exciting current of the transformers, and is therefore not strictly correct. A method in which the admittance of the transformer exciting circuit is also taken into account will be found in a later chapter.

The constants of transformers are obtainable from the manufacturer, and are usually expressed as a percentage of the transformer voltage. They may be calculated in terms of either the high-side or the low-side voltage. For transformers of the types and sizes used in transmission line work, the resistance drop will probably be in the neighborhood of 0.6 per cent, and the reactance drop about 7.5 or 8 per cent.

Consider a three-phase line whose kilowatt power output per leg is

$$P_r = E_r I_r \cos \theta_r \times 10^{-3} \tag{340}$$

where

E_r = receiver-end line voltage to neutral

and

I_r = receiver-end line current.

Then, if

d_x = transformer reactance drop in per cent of the high-side voltage

d_r = transformer resistance drop in percentage of the high-side voltage

x = equivalent reactance per leg of lowering transformers

r = equivalent resistance per leg of lowering transformers

we have,

$$\frac{d_x}{100} = \frac{xI_r}{E_r}$$

or

$$x = \frac{E_r d_x}{100 I_r}. \tag{341}$$

Similarly

$$r = \frac{E_r d_r}{100 I_r}. \tag{342}$$

Substituting the value of I_r from Eq. (340) into Eqs. (341) and (342),

$$x = \frac{10^{-5} E_r^2 \cos \theta_r \cdot d_x}{P_r} \text{ ohms} \tag{343}$$

$$r = \frac{10^{-5} E_r^2 \cos \theta_r \cdot d_r}{P_r} \text{ ohms} \tag{344}$$

The constants for the raising transformers at the supply end are found in a similar manner. The sum of the lowering and raising transformer reactances is then added to the line reactance to give the equivalent reactance for the entire line including transformers. The equivalent line resistance is obtained in an exactly similar manner. The solution of a line problem may then be carried forward in the manner already indicated for the various approximate methods.

PROBLEMS

1. A 35-mile, 60-cycle, three-phase line has three 0000 solid copper conductors of 0.533-in. diameter, supported at the vertices of an equilateral

triangle that is 6 ft. on a side. The resistance of each conductor is 0.26 ohm per mile. Consider the capacitance of the line as concentrated at the middle. If the receiver voltage is held constant at 50 kv., what is the supply voltage: (*a*) at no load? (*b*) at full load of 16,000 kw. and 85 per cent lagging power factor? (*c*) What is the line loss at full load?

2. Recalculate (*a*), (*b*) and (*c*) of Problem 1 neglecting charging current.

3. A three-phase, 60-cycle line, 105 miles long, has the following constants per mile:

$$r = 0.130 \text{ ohm}$$
$$x = 0.805 \text{ ohm}$$
$$b = 5.12 \times 10^{-6} \text{ mho.}$$

The full-load receiver input is 60,000 kw. at 85 per cent lagging power factor. The receiver voltage is held constant at 154 kv. For full-load receiver input, calculate by Dr. Steinmetz' split-condenser method the following supply-end quantities: (*a*) voltage, (*b*) current, (*c*) power factor. What is the per cent line loss at full load? Using the receiver voltage as the axis of reference, draw vector diagrams showing the various component voltages and currents that combine to make the supply voltage and the supply current.

4. Recalculate Problem 3, assuming the total line capacitance to be concentrated in the middle of the line. Assuming the split-condenser method to give correct results, calculate the per cent error in each of the quantities called for in Problem 3.

CHAPTER VIII

LONG-LINE EQUATIONS. EXACT SOLUTION

LONG TRANSMISSION LINES

The Electric Circuit.—When dealing with the electric circuits of three-phase transmission lines it is most convenient to consider only one leg of the circuit, that is, one line conductor whose potential above ground is the potential E_n to neutral. The circuit under consideration may then be pictured as in Fig. 60. It consists of an infinite number of elemental circuits in series, parallel grouping. Since the line conductors are of uniform section and, on the average, are spaced throughout the line at a constant distance from the other conductors of the line, each elemental

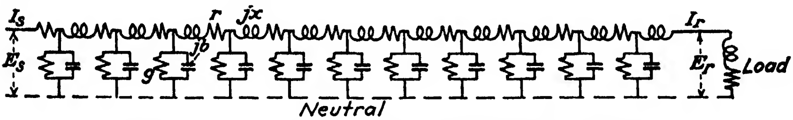


FIG. 60.—Transmission line with distributed constants.

length of the series circuit has a fixed resistance, and a fixed inductive reactance, constituting the series impedance of an element. Associated with the same element is a shunt circuit having a resistance and condensance in parallel, constituting the leakage conductance and susceptance of the element respectively, the two together forming the admittance of the leakage path. Between any two adjacent elements there is a drop of potential in the series circuit, due to the line current flowing in the impedance of the element, and so the potential to neutral at the beginning of the first element differs from that at the beginning of the second element, both in magnitude and in phase position. Likewise, the current in the first element differs from that in the second element by an amount equal to the current which flows in the admittance of the shunt circuit between them. In general, the current in the line therefore also varies from point to point both in size and in phase position.

One exception to the above statements is to be noted. This occurs when the voltage impressed between line conductors, and the current flowing in them, are of such values per unit length of line that the energy, stored in the magnetic field of the line during one-half cycle, is exactly equal to the energy released from the dielectric field during the same half cycle. Under these conditions, if the power factor of the receiver circuit is unity, no reactive power flows in any part of the line, the current in all parts of the line is the same, and the line drop is pure rI drop, in phase with the current at all points.

If one were to draw a diagram showing the vector values of voltages and currents along the line for given values of receiver load current and voltage, using a line conductor itself as one axis of a three-dimensional diagram, the result would be a picture somewhat as shown in Fig. 61. The surfaces formed by

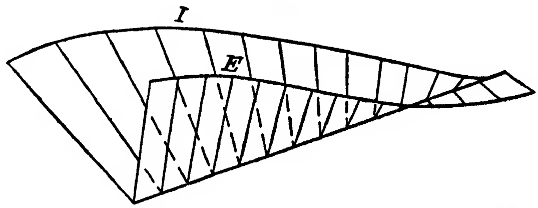


Fig. 61.—Perspective drawing of current and voltage vectors along a transmission line.

the current and voltage vectors are warped surfaces; the surface elements at any point in the line are the current and voltage vectors, and the angle between them is the power factor angle at the point in question.

The difference between the vector impedance drops of adjacent elements, or between the vector currents in adjacent elements of a line, is small. In short-line calculations the approximations made usually consist in assuming that the current flowing is the same, and that the impedance drop is constant, for all elements. In medium-length lines the line is sometimes divided into two equal sections with the above quantities assumed constant for each section but having different values for the two sections, as illustrated by the discussion of the previous chapter. In long lines, however, these differences existing cannot be neglected if accurate results are desired.

Approximate Methods.—While several of the approximate methods of Chap. VII yield results sufficiently accurate for

lines up to any length yet built in this country, and for the highest frequencies now used, yet the only ultimate check on the accuracy of these methods is the exact mathematical solution, and it therefore seems well worth while to always make this check on important lines. Furthermore, the exact method is not as difficult to apply as many engineers seem to think, and it is believed it will come into more general use as it is better and more generally understood.

The Fundamental Differential Equations.—In the discussion which follows one leg only of the transmission circuit will be considered. All voltages will be referred to neutral, and the length of line will be considered positive in the direction from receiver end to generator end.

At any point p, a distance l from the receiver end of the line, let

I = vector current in the line conductor
E = vector potential difference from conductor to neutral

and let the constants per mile of one line conductor be as follows:

r_1 = resistance of line conductor.
x_1 = inductive reactance of line conductor.
$z_1 = \sqrt{r_1^2 + x_1^2}$ = numerical impedance of series circuit.
$Z_1 = r_1 + jx_1$ = complex impedance of one conductor.
g_1 = conductance of the shunt leakage path per conductor.
b_1 = susceptance of the shunt leakage path per conductor.
$y_1 = \sqrt{g_1^2 + b_1^2}$ = numerical admittance of the shunt leakage path per conductor.
$Y_1 = g_1 + jb_1$ = complex admittance of one conductor to neutral.

The changes in voltage and current occurring from point to point in the line are readily illustrated. For example, let the vector voltage from line to neutral at a point l miles from the receiver, referred to the receiver voltage as zero vector, be

$$E = E_1 + jE_2 \text{ vector volts.}$$

If the vector current at the same point is

$$I = I_1 - jI_2 \text{ vector amp.}$$

the drop dE, due to the current flowing in an elemental length dl of the line, is

$$\begin{aligned} dE &= (I_1 - jI_2)(r_1 + jx_1)dl \\ &= (I_1r_1 + I_2x_1)dl + j(I_1x_1 - I_2r_1)dl \text{ vector volts} \qquad (345) \end{aligned}$$

and the potential difference from line to neutral at $l + dl$ miles from the receiver is

$$E + dE = E_1 + (I_1r_1 + I_2x_1)dl + j[E_2 + (I_1x_1 + I_2r_1)dl] \quad \text{vector volts.} \qquad (346)$$

The current in the leakage path of the element dl is

$$\begin{aligned} dI &= (E_1 + jE_2)(g_1 + jb_1)dl \\ &= (E_1g_1 - E_2b_1)dl + j(E_1b_1 + E_2g_1)dl \ \text{vector amp.} \end{aligned} \qquad (347)$$

The positive sign is used with dI because the charging current increases in the positive direction of l.

The current in the line, at a distance $l + dl$ miles from the receiver, is

$$I + dI = I_1 + (E_1g_1 - E_2b_1)dl - j[I_2 - (E_1b_1 + E_2g_1)dl].$$

The vector diagram showing these voltage and current increments of an element of line is shown in Fig. 62.

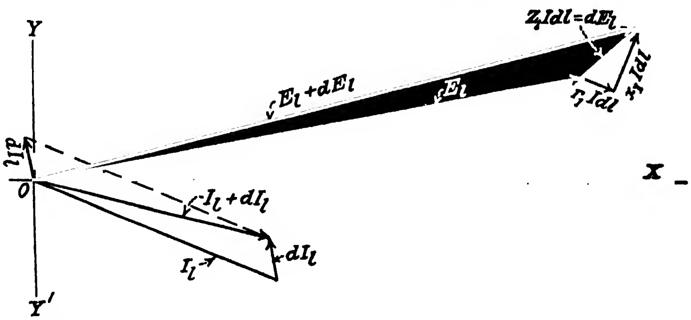


FIG. 62.—Voltage and current diagrams for element dl of line.

Using the more general form of notation, the increments, of vector-voltage rise in the element of the series circuit, is

$$dE = IZ_1dl$$

or

$$\frac{dE}{dl} = IZ_1 \qquad (348)$$

and the current, leaking to neutral from length dl of line, is

$$dI = EY_1dl$$

or

$$\frac{dI}{dl} = EY_1. \qquad (349)$$

Equations (348) and (349) are the fundamental differential equations of the circuit under consideration. Their solution will give the desired values of I and E at any point in the line.

Solution of Equations.—Differentiating Eq. (348) with respect to l,

$$\frac{d^2E}{dl^2} = Z_1\frac{dI}{dl}. \tag{350}$$

Differentiating Eq. (349) with respect to l,

$$\frac{d^2I}{dl^2} = Y_1\frac{dE}{dl}. \tag{351}$$

Substituting Eq. (349) in Eq. (350),

$$\frac{d^2E}{dl^2} = Y_1Z_1E. \tag{352}$$

Substituting Eq. (348) in Eq. (351),

$$\frac{d^2I}{dl^2} = Y_1Z_1I. \tag{353}$$

It will be observed that Eqs. (352) and (353) are alike in form, and must therefore have the same general solution. Whatsoever differences appear in the values of E and I will result from constants of integration, *i.e.*, from assumed terminal conditions.

Multiplying both sides of Eq. (353) by $\dfrac{dI}{dl}$,

$$\frac{dI}{dl}\cdot\frac{d^2I}{dl^2} = Y_1Z_1I\cdot\frac{dI}{dl}. \tag{354}$$

Integrating Eq. (354),

$$\frac{1}{2}\left(\frac{dI}{dl}\right)^2 = \frac{Y_1Z_1I^2}{2} + \frac{c^2}{2} \tag{355}$$

the last term representing the constant at integration.

Let

$$c = Y_1Z_1c_1^2.$$

Then, from Eq. (355),

$$\frac{dI}{dl} = \sqrt{Y_1Z_1}\sqrt{I^2 + c_1^2} \tag{356}$$

or

$$\frac{dI}{\sqrt{I^2 + c_1^2}} = \sqrt{Y_1Z_1}\cdot dl. \tag{357}$$

Equation (357) may be integrated by making the following substitutions:

Let

$$I = c_1 \sinh u$$

Then

$$dI = c_1 \cosh u \cdot du$$

and

$$\sqrt{I^2 + c_1^2} = c_1\sqrt{\sinh^2 u + 1} = c_1 \cosh u.$$

Making these substitutions in Eq. (357) and solving for u,

$$\int du = \sqrt{Y_1Z_1}\int dl + c_2$$

or

$$u = \sqrt{Y_1Z_1}l + c_2 \tag{358}$$

and

$$I = c_1 \sinh\ (\sqrt{Y_1Z_1}l + c_2). \tag{359}$$

Differentiating Eq. (359),

$$\frac{dI}{dl} = \sqrt{Y_1Z_1}c_1 \cosh\ (\sqrt{Y_1Z_1}l + c_2). \tag{360}$$

Substituting Eq. (349) in Eq. (360) and solving for E,

$$E = \sqrt{\frac{Z_1}{Y_1}}c_1 \cosh\ (\sqrt{Y_1Z_1}l + c_2). \tag{361}$$

Equations (359) and (361) are the desired equations for I and E in which it remains to determine the constants c_1 and c_2 from known terminal conditions.

In power transmission problems the current and voltage at the receiver end of the line are usually known or may readily be estimated. If l be taken as positive in the direction from receiver to generator, the assumed known terminal conditions (for $l = 0$) are

$$E = E_r \text{ the receiver vector voltage}$$

and

$$I = I_r \text{ the receiver vector current.}$$

Expanding Eq. (359),

$$I = c_1 (\sinh \sqrt{Y_1Z_1}l \cdot \cosh c_2 + \cosh \sqrt{Y_1Z_1}l \cdot \sinh c_2) \tag{362}$$

or, since

$$I = I_r \text{ for } l = 0$$
$$I_r = c_1 \sinh c_2$$

and

$$\sinh c_2 = \frac{I_r}{c_1}. \tag{363}$$

Expanding Eq. (361),

$$E = c_1\sqrt{\frac{Z_1}{Y_1}}(\cosh \sqrt{Y_1Z_1}l \cosh c_2 + \sinh \sqrt{Y_1Z_1}l \sinh c_2) \tag{364}$$

and since

$$E = E_r \text{ when } l = 0$$

$$E_r = c_1\sqrt{\frac{Z_1}{Y_1}} \cosh c_2$$

or

$$\cosh c_2 = \frac{E_r}{c_1\sqrt{\frac{Z_1}{Y_1}}}. \tag{365}$$

Substituting Eqs. (363) and (365) in Eqs. (362) and (364),

$$E = E_r \cosh \sqrt{Y_1Z_1}l + I_r\sqrt{\frac{Z_1}{Y_1}} \sinh \sqrt{Y_1Z_1}l \tag{366}$$

$$I = I_r \cosh \sqrt{Y_1Z_1}l + E_r\sqrt{\frac{Y_1}{Z_1}} \sinh \sqrt{Y_1Z_1}l. \tag{367}$$

Since E_r and I_r, respectively, represent the voltage to neutral and the current in a line conductor at the receiver or load end of the line, E and I of Eqs. (366) and (367) are the corresponding quantities at any point on the line a distance l from the receiver end. If the assumed known conditions be taken at the supply end of the line as E_s and I_s, instead of at the receiver end, then, since the distance along the line is now measured in the direction of the flow of energy, that is, from generator towards receiver, l becomes negative. Remembering that

$$\cosh \sqrt{Y_1Z_1}(-l) = \cosh \sqrt{Y_1Z_1}l$$

and

$$\sinh \sqrt{Y_1Z_1}(-l) = -\sinh \sqrt{Y_1Z_1}l$$

it is apparent that, with the new set of terminal conditions, the general equations of the line become

$$E = E_s \cosh \sqrt{Y_1Z_1}l - I_r\sqrt{\frac{Z_1}{Y_1}} \sinh \sqrt{Y_1Z_1}l \tag{368}$$

$$I = I_s \cosh \sqrt{Y_1Z_1}l - E_r\sqrt{\frac{Y_1}{Z_1}} \sinh \sqrt{Y_1Z_1}l. \tag{369}$$

These equations yield the current and voltage of the line at any point distant l units from the supply end and in terms of the current and voltage at that end.

If the impedance and admittance of the entire line be used instead of the values per unit of length, the substitutions given below are required.

$Z = Z_1 l =$ the total series impedance on one conductor.
$Y = Y_1 l =$ the total shunted admittance of one conductor.

When these substitutions are made, Eqs. (366) and (367) become

$$E_s = E_r \cosh \sqrt{YZ} + I_r \sqrt{\frac{Z}{Y}} \sinh \sqrt{YZ} \tag{370}$$

$$I_s = I_r \cosh \sqrt{YZ} + E_r \sqrt{\frac{Y}{Z}} \sinh \sqrt{YZ}. \tag{371}$$

Equations (370) and (371) are the general equations for the entire line in terms of the assumed known values of E_r and I_r. The equations are not only exact, but are very convenient to use in the solution of transmission-line problems.

The notation of Eq (370) and (371) may be somewhat simplified by writing

$$E_s = E_r A + I_r B \tag{372}$$

$$I_s = I_r A + E_r C \tag{373}$$

where the complex numbers A, B and C are derived or auxiliary line constants.

Still another variation in the form of these equations is obtained by expressing the various complex numbers involved by their respective cartesian components. Using the vector E_r as the zero or reference vector, the equations become

$$\begin{aligned} E_s &= E_1 + jE_2 \\ &= E_r(a_1 + ja_2) + ({}_rI_1 - j_rI_2)(b_1 + jb_2) \end{aligned} \tag{374}$$

$$\begin{aligned} I_s &= {}_sI_1 + j_sI_2 \\ &= ({}_rI_1 - j_rI_2)(a_1 + ja_2) + E_r(c_1 + jc_2) \end{aligned} \tag{375}$$

where

$$\begin{aligned} A &= a_1 + ja_2 \\ B &= b_1 + jb_2 \\ C &= c_1 + jc_2 \end{aligned}$$

are the complex, derived line constants, and

$$\begin{aligned} E_s &= {}_sE_1 + j_sE_2 \\ I_s &= {}_sI_1 + j_sI_2 \\ E_r &= E_r + j0 \\ I_r &= {}_rI_1 - j_rI_2. \end{aligned}$$

When it comes to the study of line performance, Eqs. (374) and (375) are probably the most helpful of any. With their use one may readily draw the loci of current and voltage vectors for any loads and power factors. The method will be discussed in detail in another chapter.

Derived or Auxiliary Line Constants.—The symbols used in the simplified form of the above equations are interpreted as follows:

$$\begin{aligned} A &= \cosh \sqrt{YZ} \\ &= a_1 + ja_2, \text{ a complex number} \end{aligned} \tag{376}$$

$$\begin{aligned} B &= \sqrt{\frac{Z}{Y}} \sinh \sqrt{YZ} \\ &= Z_0 \sinh \sqrt{YZ} \\ &= b_1 + jb_2, \text{ a complex number} \end{aligned} \tag{377}$$

$$\begin{aligned} C &= \sqrt{\frac{Y}{Z}} \sinh \sqrt{YZ} \\ &= Y_0 \sinh \sqrt{YZ} \\ &= c_1 + jc_2, \text{ a complex number.} \end{aligned} \tag{378}$$

It may be noted also that

$$Z_0 = \sqrt{\frac{Z}{Y}}, \text{ a complex number, called the } \textit{surge impedance} \text{ of the line,} \tag{379}$$

$$\begin{aligned} Y_0 &= \sqrt{\frac{Y}{Z}} \\ &= \frac{1}{Z_0}, \text{ a complex number called the } \textit{surge admittance} \text{ of the line,} \end{aligned} \tag{380}$$

and

$$\begin{aligned} \sqrt{YZ} &= \theta \\ &= \alpha + j\beta, \text{ a complex number called the } \textit{complex angle} \text{ of the line.} \end{aligned} \tag{381}$$

The complex numbers A, B and C are called the *derived* or *auxiliary constants* of the line. They are hyperbolic functions of the complex line angle $\sqrt{YZ}$ subtended by the line, and depend only upon the *fundamental* line constants r, g, L and C and the frequency. Since the values of r, g, C and L are directly proportional to the length of the line, the derived constants also are a function of the length. The surge impedance and the surge admittance are both independent of the length of the line. They depend only upon the values of resistance, inductance, conductance and capacitance *per unit length* of line.

The Complex Line Angle $\theta = \sqrt{YZ}$.—Since, in general, Z and Y are complex numbers, it follows that the same is true of the line angle θ. Also, since $\sqrt{YZ} = \sqrt{Y_1 l \cdot Z_1 l}$, the unit line angle $\theta_1 = \theta \div l = \sqrt{Y_1 Z_1}$. That is, associated with each unit length of a smooth line having uniformly distributed constants r_1, g_1, L_1 and C_1 per unit length of line, and operating at a fixed frequency, is a unit complex angle $\theta_1 = \sqrt{Y_1 Z_1}$. It consists of a real part or hyperbolic component, and an imaginary part or circular component. Custom has established the use of the symbols α_1 and β_1 respectively, to represent these components. Hence the unit line angle may be written as

$$\theta_1 = \sqrt{Y_1 Z_1} = \alpha_1 + j\beta_1. \tag{382}$$

Multiplying this equation through by the length of line and writing $\alpha = \alpha_1 l$ and $\beta = \beta_1 l$, the complex angle of the whole line is

$$\theta = \sqrt{YZ} = \alpha + j\beta. \tag{383}$$

The real part α is measured in hyperbolic radians, while the imaginary part β is measured in circular radians. β is reduced to degrees by multiplying by 57.296, the number of degrees in a radian of circular measure.

Surge Impedance.—The surge impedance of a smooth, alternating-current line is defined by Eq. (379) as

$$Z_0 = \sqrt{\frac{Z}{Y}}$$
$$= \sqrt{\frac{r + jx}{g + jb}}. \tag{384}$$

The operation indicated by the right-hand member of Eq. (384) is most readily performed when the polar form of expression is employed. This notation yields the expression,

$$
\begin{aligned}
Z_0 &= \sqrt{\frac{z/\underline{\theta_z}}{y/\underline{\theta_y}}} \\
&= \sqrt{\left(\frac{r^2+x^2}{g^2+b^2}\right)^{\frac{1}{2}} \Big/ \underline{\tfrac{1}{2}(\theta_z - \theta_y)}} \\
&= \sqrt[4]{\frac{r^2+x^2}{g^2+b^2}} \Big/ \underline{\gamma} \text{ vector ohms} && (385) \\
&= z_0 (\cos\gamma + j \sin\gamma) \text{ vector ohms} && (386)
\end{aligned}
$$

where

$$\theta_z = \tan^{-1} \frac{x}{r}$$

$$\theta_y = \tan^{-1} \frac{b}{g}$$

and

$$\gamma = \frac{1}{2}(\theta_z - \theta_y) \text{ radians.} \tag{387}$$

It is apparent from Eq. (379) that the surge impedance also is independent of the length of line, but is determined solely by the values of r, L, g and C per unit length of line, and by the frequency. For 60-cycle lines of usual construction, the size of Z_0 usually lies between 350 and 400 ohms. The angle γ, called the *characteristic phase angle* of the line, has a small, negative value of from 3 to 6° for very heavy, high-voltage lines, while for lighter lines and lower voltages it may be as high as 15 to 20°.

The surge impedance of a smooth line is equal to the impedance which would be measured at the generator end of the line if its length approached infinity. For, since by Eq. (368),

$$E_r = E_s \cosh \sqrt{YZ} - I_s Z_0 \sinh \sqrt{YZ}$$

the receiver voltage approaches zero as the length of line approaches infinity. Hence, for very large values of l,

$$E_s \cosh \sqrt{YZ} = I_s Z_0 \sinh \sqrt{YZ}$$

and

$$
\begin{aligned}
\frac{E_s}{I_s} &= Z_0 \tanh \sqrt{YZ} \\
&= Z_0 \\
&= z_0 (\cos\gamma + j \sin\gamma)
\end{aligned}
$$

since

$$\tanh \infty = 1$$

The surge impedance of a line may be calculated from measurements of the line impedance taken at the supply end, (*a*) with the receiver end open, and (*b*) with the receiver end short-circuited. It is found to be the geometric mean of the supply end impedances calculated from the above two sets of readings. The proof of this relation follows.

When the receiver circuit is open the receiver current is zero, and hence, for this condition, by Eqs. (366) and (367),

$$E_s = E_r \cosh \sqrt{YZ}$$
$$I_s = E_r Y_0 \sinh \sqrt{YZ}$$

or

$$Z_{Oc} = \frac{E_s}{I_s} = Z_0 \coth \sqrt{YZ}.$$

Similarly, when the line is short-circuited at the receiver, the receiver voltage is zero and

$$E_s = I_r Z_0 \sinh \sqrt{YZ}$$
$$I_s = I_r \cosh \sqrt{YZ}$$

or

$$Z_{sc} = \frac{E_s}{I_s} = Z_0 \tanh \sqrt{YZ}$$

whence

$$Z_0^2 = Z_{Oc} \times Z_{sc}$$

or

$$Z_0 = \sqrt{Z_{Oc} \times Z_{sc}}.$$

It should also be noted that if the leakage conductance and the resistance are negligibly small as compared with the susceptance and the inductive reactance, respectively, the numerical value of the surge impedance reduces to

$$Z_0 = \sqrt{\frac{L}{C}} \text{ vector ohms}$$

and its angle approaches zero. For heavy power lines using conductors of large areas, the resistance per mile is very low, while the values of the reactance and susceptance per mile are practically independent of the size of the conductor employed; for a given frequency they are nearly constant for all lines, since the spacing of the conductors is approximately proportional to the line voltage.

Surge Admittance.—The surge admittance is the reciprocal of the surge impedance. It is

$$Y_0 = \frac{1}{Z_0} = \sqrt[4]{\frac{g^2 + b^2}{r^2 + x^2}} \underline{/-\gamma} \text{ vector mho.}$$
$$= y_0(\cos\gamma - j\sin\gamma) \text{ vector mho.} \quad (388)$$

Its size is the reciprocal of the size and its slope the negative of the slope of the surge impedance, *i.e.*,

$$y_0 = \frac{1}{z_0}$$

and

$$\theta_{y0} = -\theta_{z0}.$$

Solution for α and β.—In terms of the line constants r, x, g and b, the values of α and β may be obtained as follows:[1]

$$\alpha + j\beta = \sqrt{YZ}$$
$$= \sqrt{(r + jx)(g + jb)}. \quad (389)$$

Squaring Eq. (389),

$$\alpha^2 + 2j\alpha\beta - \beta^2 = gr - bx + j(br + gx)$$

or

$$\alpha^2 - \beta^2 = gr - bx \quad (390)$$

and

$$2\alpha\beta = br + gx. \quad (391)$$

Squaring Eqs. (390) and (391) and adding,

$$(\alpha^2 + \beta^2)^2 = (r^2 + x^2)(g^2 + b^2)$$
$$= z^2y^2$$

or

$$\alpha^2 + \beta^2 = zy. \quad (392)$$

Adding Eqs. (390) and (392) and solving

$$\alpha = \sqrt{\frac{1}{2}(zy + gr - bx)} \text{ hyperbolic radians.} \quad (393)$$

Subtracting Eq. (390) from Eq. (392) and solving,

$$\beta = \sqrt{\frac{1}{2}(zy - gr + bx)} \text{ circular radians.} \quad (394)$$

If the insulation of the line is so good that the leakage conductance may be taken equal to zero, the condition which usually

[1] STIENMETZ, C. P., "Transient Phenomena," 3d ed., p. 292.

prevails in high-tension power lines, then $g = 0$ and $y = b$. On the basis of this assumption, the equations for α and β reduce to

$$\alpha = \sqrt{\frac{b}{2}(z - x)} \tag{395}$$

$$\beta = \sqrt{\frac{b}{2}(z + x)}. \tag{396}$$

Since, for power lines, z and x are usually nearly equal in size, the square root of their difference becomes a somewhat uncertain quantity. Accordingly, it is not always convenient or desirable to use Eqs. (395) and (396) to evaluate α and β. In such cases it is preferable to obtain these quantities directly by performing the operation indicated by $\sqrt{YZ}$. The polar expression for the angle will serve best for this purpose.

Representing the impedance and admittance angles of the line by

$$\theta_z = \tan^{-1}\frac{x}{r}$$

and

$$\theta_y = \tan^{-1}\frac{b}{g}$$

and putting

$$\delta = \frac{1}{2}(\theta_z + \theta_y)$$

the polar expression for the line angle becomes

$$\alpha + j\beta = \sqrt{YZ} = \sqrt[4]{(r^2 + x^2)(g^2 + b^2)}\,\Big/\tfrac{1}{2}(\theta_z + \theta_y) \tag{397}$$

$$= \theta\underline{/\delta} \text{ vector radians} \tag{398}$$

where, in size,

$$\theta = \sqrt[4]{(r^2 + x^2)(g^2 + b^2)}. \tag{399}$$

Upon again reducing Eq. (398) to rectangular coordinates and equating reals and imaginaries separately, the values of α and β are found to be

$$\left.\begin{aligned} \alpha &= \theta \cos \delta \text{ hyperbolic radians} \\ \beta &= \theta \sin \delta \text{ circular radians} \end{aligned}\right\} \tag{400}$$

Again, when g is approximately equal to zero, the values of θ and δ are, approximately,

$$\theta = \sqrt{bz} \tag{401}$$

and

$$\delta = \frac{\pi}{4} + \frac{\theta_z}{2}. \tag{402}$$

Variation of α and β with Line Constants and Frequency.—It is of interest to consider the manner in which α and β vary with the constants r, L, g and C and the frequency. If the frequency is zero, then $(x = \omega L) = 0 = (b = \omega C)$,

whence

$$\begin{aligned} \theta &= \sqrt{(r + j0)(g + j0)} \\ &= \sqrt{rg} \underline{/\delta} \end{aligned}$$

where

$$\begin{aligned} \delta &= \frac{1}{2}\left(\tan^{-1}\frac{x}{r} + \tan^{-1}\frac{b}{g}\right). \\ &= 0 \end{aligned}$$

Therefore,

$$\alpha = \sqrt{rg} \text{ hyperbolic radians}$$

and

$$\beta = 0 \text{ circular radians.}$$

This condition prevails in a leaky, direct-current line. In such lines the voltage and current are attenuated in size but not in phase.

On the other hand, if $r = 0 = g$ and the frequency is not zero,

$$\begin{aligned} \theta &= \sqrt{j^2 bx} \\ &= j2\pi f\sqrt{LC} \end{aligned}$$

and

$$\delta = \frac{\pi}{2}.$$

Therefore,

$$\begin{aligned} \alpha &= 0 \text{ hyperbolic radians} \\ \beta &= 2\pi f\sqrt{LC} \text{ circular radians.} \end{aligned}$$

While the resistance in a power line is never negligible, and α, therefore, is not zero, yet as compared with β, α is very small. and β is approximately equal to the value given.

For example, consider a 60-cycle line built of 500,000-cir. mil copper cable, for which the constants per mile are $r_1 = 0.022$, $x_1 = 0.82$, $b_1 = 5.2 \times 10^{-6}$, $g = 0$ (approximately). The resistance, being very small as compared with the reactance, will not materially affect the value of β_1. The latter is

$$\beta_1 = \sqrt{x_1 b_1}$$
$$= 0.00206 \text{ radian per mile}$$
$$= 0.118° \text{ per mile.}$$

This means that the component voltage and current vectors swing through an angle of 0.118° per mile of line.

Forms of Expressions for the Constants A, B and C.—The equations of the long line may be put into a number of different forms, depending upon the expressions which are used for the constants A, B and C and upon the grouping of terms. One useful form is illustrated in Eqs. (366) and (367). Other useful forms are considered below.

a. The Hyperbolic Forms.—In Eqs. (366) and (367) the constants are expressed in terms of hyperbolic functions of the complex angle of the line. They are

$$\left.\begin{aligned} A &= \cosh\sqrt{YZ} = \cosh(\alpha + j\beta) \\ B &= Z_0 \sinh\sqrt{YZ} = Z_0 \sinh(\alpha + j\beta) \\ C &= Y_0 \sinh\sqrt{YZ} = Y_0 \sinh(\alpha + j\beta) \end{aligned}\right\} \quad (403)$$

If charts[1] and tables[2] of complex hyperbolic numbers are available, the constants may be computed from Eqs. of (403) as they stand. In the absence of such aids, and particularly for precise calculations, more convenient forms of expression to use are those in which only circular and hyperbolic functions of real numbers appear. These are obtained by the expansion of the sinh and cosh of the complex variable. They are

$$\left.\begin{aligned} A &= \cosh\alpha\cos\beta + j\sinh\alpha\sin\beta \\ B &= Z_0(\sinh\alpha\cos\beta + j\cosh\alpha\sin\beta \\ C &= Y_0(\sinh\alpha\cos\beta + j\cosh\alpha\sin\beta \end{aligned}\right\} \quad (404)$$

With the aid of tables of hyperbolic functions and circular functions of real numbers and a computing machine, precise calculations are easily made from these equations.

b. Convergent Series Form with Complex Numbers.—By substituting for the hyperbolic functions of Eqs. of (403) the appropriate series in $\sqrt{YZ}$, writing for Z_0 and Y_0 the equivalents $\sqrt{\frac{Z}{Y}}$ and $\sqrt{\frac{Y}{Z}}$, respectively, and simplifying the resulting expressions,

[1] KENNELLY, A. E., "Chart Atlas of Complex Hyperbolic and Circular Functions."

[2] KENNELLY'S "Tables of Complex Numbers."

the complex convergent series forms for A, B and C are obtained. They are

$$\left.\begin{aligned} A &= 1 + \frac{YZ}{2} + \frac{Y^2Z^2}{24} + \frac{Y^3Z^3}{720} + \cdots \\ B &= Z\left(1 + \frac{YZ}{6} + \frac{Y^2Z^2}{120} + \frac{Y^3Z^3}{5{,}040} + \cdots\right) \\ C &= Y\left(1 + \frac{YZ}{6} + \frac{Y^2Z^2}{120} + \frac{Y^3Z^3}{5{,}040} + \cdots\right) \end{aligned}\right\} \tag{405}$$

c. Convergent Series Form with Real Numbers.—Very convenient convergent series forms of expression, using real numbers only, are obtained as follows:

$$\left.\begin{aligned} A &= \cosh (\alpha + j\beta) \\ &= \frac{1}{2}(\epsilon^{\alpha+j\beta} + \epsilon^{-\alpha-j\beta}) \\ &= \frac{1}{2}(\epsilon^{\alpha}\epsilon^{j\beta} + \epsilon^{-\alpha}\epsilon^{-j\beta}) \\ \text{and similarly,}\qquad & \\ B &= \frac{Z_0}{2}(\epsilon^{\alpha}\epsilon^{j\beta} - \epsilon^{-\alpha}\epsilon^{-j\beta}) \\ C &= \frac{Y_0}{2}(\epsilon^{\alpha}\epsilon^{j\beta} - \epsilon^{-\alpha}\epsilon^{-j\beta}) \end{aligned}\right\} \tag{406}$$

If, due to the excellence of line insulation, it may be assumed that $g = 0$, then $Y = jb$ and

$$\begin{aligned} Z_0 &= \sqrt{\frac{Z}{Y}} = \sqrt{\frac{YZ}{Y^2}} = \frac{\alpha + j\beta}{jb} \\ &= \frac{\beta - j\alpha}{b} \end{aligned} \tag{407}$$

$$\begin{aligned} Y_0 &= \frac{1}{Z_0} = \frac{b}{\beta - j\alpha} \\ &= \frac{\beta + j\alpha}{z}. \end{aligned} \tag{408}$$

By substituting in Eqs. of (406) the corresponding equivalent series in place of ϵ^{α} and $\epsilon^{-\alpha}$, for Z_0 and Y_0 the expressions of Eqs. (407) and (408), respectively, and for $\epsilon^{j\beta}$ and $\epsilon^{-j\beta}$ the expressions

$$\begin{aligned} \epsilon^{j\beta} &= \cos \beta + j \sin \beta \\ \epsilon^{-j\beta} &= \cos \beta - j \sin \beta \end{aligned}$$

there results, after simplifying, the rapidly converging series,

$$\left.\begin{aligned}
A &= \left(1 + \frac{\alpha^2}{2} + \frac{\alpha^4}{24} + \frac{\alpha^6}{720} + \cdots\right)\cos\beta \\
&\quad + j\left(\alpha + \frac{\alpha^3}{6} + \frac{\alpha^5}{120} + \frac{\alpha^7}{5{,}040} + \cdots\right)\sin\beta \\
B &= \frac{\beta - j\alpha}{b}\left[\left(\alpha + \frac{\alpha^3}{6} + \frac{\alpha^5}{120} + \frac{\alpha^7}{5{,}040} + \cdots\right)\cos\beta\right. \\
&\quad \left. + j\left(1 + \frac{\alpha^2}{2} + \frac{\alpha^4}{24} + \frac{\alpha^6}{720} + \cdots\right)\sin\beta\right] \\
C &= \frac{\beta + j\alpha}{z}\left[\left(\alpha + \frac{\alpha^3}{6} + \frac{\alpha^5}{120} + \frac{\alpha^7}{5{,}040} + \cdots\right)\cos\beta\right. \\
&\quad \left. + j\left(1 + \frac{\alpha^2}{2} + \frac{\alpha^4}{24} + \frac{\alpha^6}{720} + \cdots\right)\sin\beta\right]
\end{aligned}\right\} \quad (409)$$

The errors introduced by omitting powers of α above the second, or at most the third, will usually be negligible. Hence, for practical calculations the constants are readily evaluated from Eq. (409) if desired.

Generalized Line Constants of Equivalent Networks.[1]—When networks consist of various combinations of series and parallel groupings of their constituent elements, it is frequently desirable to combine the elements into a simple *equivalent network* having only the usual generalized constants. When a single line alone is considered, the supply-end voltage and current are given by the equations,

$$\begin{aligned}
E_s &= E_rA + I_rB \\
I_s &= I_rA + E_rC.
\end{aligned}$$

Frequently, however, a transmission network may consist of two or more lines in parallel; or a line having certain characteristics may be connected in series with another having different characteristics. Transformers form an essential part of a transmission network, and their impedances must be considered in evaluating potential drops, etc., between receiver and supply busses. Sometimes it is desirable to include either generator impedances or load impedances as series impedances in the general equations of the line. Again, it may be required to include the impedance of a shunt circuit with that of the line. The shunt may be at either end of the line or at some intermediate point, depending

[1] Evans and Sels, *Elec. Jour.*, July, 1921: *Trans.*, A. I. E. E., pp. 33 to 35, 1924.

upon the nature of the problem. The problem rapidly becomes very complex and unwieldy as the number of elements increases.

In general, however, *equivalent constants* A_0, B_0, C_0 and D_0 can always be found, such that the supply voltage and current for the network are given in terms of the receiver voltage and current and the equivalent constants, by the equations,

$$\begin{aligned} E_s &= E_r A_0 + I_r B_0 \\ I_s &= I_r D_0 + E_r C_0 \end{aligned} \tag{410}$$

The constants having the zero subscripts are constants of the equivalent simple system, replacing the more or less complex network. The constant D_0 is introduced to take care of the general case, for D and A are equal only in cases in which the equivalent line is symmetrical about its center.

The generalized or equivalent line constants will be worked out for a number of cases.

a. Series Impedance Circuit.—If the circuit under consideration is a simple series impedance Z, as for example, a short transmission circuit in which the capacitance of the cables is negligible, then,

$$E_s = E_r A_0 + I_r B_0 = E_r + I_r Z$$

and

$$I_s = I_r D_0 + E_r C_0 = I_r + 0$$

whence, for this case,

$$\left. \begin{aligned} A_0 &= 1,\ B_0 = Z \\ D_0 &= 1,\ C_0 = 0 \end{aligned} \right\} \tag{411}$$

b. Shunt Admittance Circuit.—When the circuit consists of a shunt admittance Y, only, the voltage is the same on both sides of the admittance, but the currents on the two sides differ. Applying the general equations,

$$\begin{aligned} E_s &= E_r A_0 + I_r B_0 = E_r + 0 \\ I_s &= I_r D_0 + E_r C_0 = I_r + E_r Y \end{aligned}$$

whence

$$\left. \begin{aligned} A_0 &= 1,\ B_0 = 0 \\ D_0 &= 1,\ C_0 = Y \end{aligned} \right\} \tag{412}$$

Using these two simple circuits as a foundation, the generalized constants for various series-parallel combinations of circuits may be evaluated.

c. Networks in Series.—Figure 63 represents two transmission lines or other networks in series. It is assumed that the constants of the separate networks are known, and that it is desired to find the constants of the equivalent simple network. The constants of the two separate components carry the subscripts 1 and 2.

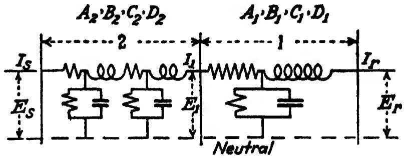


Fig. 63.—Networks in series.

Beginning at the receiver end and writing the equations for voltage and current at the junction of the two networks,

$$\left.\begin{aligned} E_1 &= E_rA_1 + I_rB_1 \\ I_1 &= I_rD_1 + E_rC_1 \end{aligned}\right\} \tag{413}$$

Similarly,

$$\left.\begin{aligned} E_s &= E_1A_2 + I_1B_2 \\ I_s &= I_1D_2 + E_1C_2 \end{aligned}\right\} \tag{414}$$

By eliminating E_1 and I_1 from Eqs. (413) and (414), we find that

$$\left.\begin{aligned} E_s &= E_r(A_1A_2 + B_2C_1) + I_r(A_2B_1 + B_2D_1) \\ &\text{and} \\ I_s &= I_r(B_1C_2 + D_1D_2) + E_r(A_1C_2 + C_1D_2) \end{aligned}\right\} \tag{415}$$

Therefore,

$$\left.\begin{aligned} A_0 &= A_1A_2 + B_2C_1 \\ B_0 &= A_2B_1 + B_2D_1 \\ C_0 &= A_1C_2 + C_1D_2 \\ D_0 &= B_1C_2 + D_1D_2 \end{aligned}\right\} \tag{416}$$

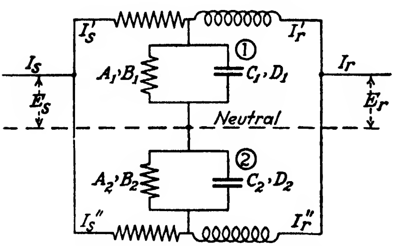


Fig. 64.—Networks in parallel.

d. Networks in Parallel.—Figure 64 represents two transmission lines or other networks in parallel. The constants of the separate networks are designated by the subscripts 1 and 2 indicated in the figure.

The drop in potential between the ends of the circuit is the same *via* either branch; hence

$$E_s = E_rA_1 + I_r'B_1 = E_rA_2 + I_r''B_2. \tag{417}$$

The currents at the supply ends of the two branches are

$$\left.\begin{aligned} I_s' &= I_r'D_1 + E_rC_1 \\ I_s'' &= I_r''D_2 + E_rC_2 \end{aligned}\right\} \tag{418}$$

However,

and

$$\left.\begin{aligned} I_r &= I_r' + I_r'' \\ I_s &= I_s' + I_s'' \end{aligned}\right\} \tag{419}$$

From Eq. (417),

$$I_r' = \frac{I_r B_2 - E_r(A_1 - A_2)}{B_1 + B_2}. \tag{420}$$

Eliminating I_r' from the first part of Eq. (417) and simplifying yields

$$E_s = \frac{E_r(A_1B_2 + A_2B_1)}{B_1 + B_2} + \frac{I_rB_1B_2}{B_1 + B_2}. \tag{421}$$

Adding Eqs. of (418), writing I_r' and I_r'' in terms of I_r and simplifying,

$$I_s = \frac{I_r(B_1D_2 + B_2D_1)}{B_1 + B_2} + E_r\left[C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2}\right] \tag{442}$$

Accordingly, for this case,

$$\left.\begin{aligned} A_0 &= \frac{A_1B_2 + A_2B_1}{B_1 + B_2} \\ B_0 &= \frac{B_1B_2}{B_1 + B_2} \\ C_0 &= C_1 + C_2 + \frac{(A_1 - A_2)(D_2 - D_1)}{B_1 + B_2} \\ D_0 &= \frac{B_1D_2 + B_2D_1}{B_1 + B_2} \end{aligned}\right\} \tag{423}$$

e. Transmission Line and Series Impedance at Supply End.—This is a special case under (*c*). The equivalent constants are given by Eqs. of (416), in which quantities having the subscripts 1 now refer to the line, while those having the subscripts 2 refer to the series impedance at the supply end. For the series impedance, Z_s, at the supply end, the generalized constants have the values given by Eq. (411). Substituting these values in Eq. (416) for the corresponding constants having the subscripts 2, and remembering that $A_1 = A$, $B_1 = B$, etc., yields

$$\left.\begin{aligned} A_0 &= A + CZ_s \\ B_0 &= B + AZ_s \\ C_0 &= C \\ D_0 &= D = A \end{aligned}\right\} \tag{424}$$

f. Transmission Line and Series Impedance at Receiver End.— This is a special case under (*c*). The equivalent constants are again given by Eq. (416), in which constants having the subscripts 1 now refer to the receiver series impedance Z_r, while those having the subscripts 2 refer to the line. Proceeding as before,

$$\left.\begin{aligned} A_0 &= A \\ B_0 &= B + AZ_r \\ C_0 &= C \\ D_0 &= A + CZ_r \end{aligned}\right\} \qquad (425)$$

g. Transmission Line and Series Impedance at Both Ends.— Contained in the circuit between generators and load, the usual transmission circuit includes supply-end and receiver-end trans-

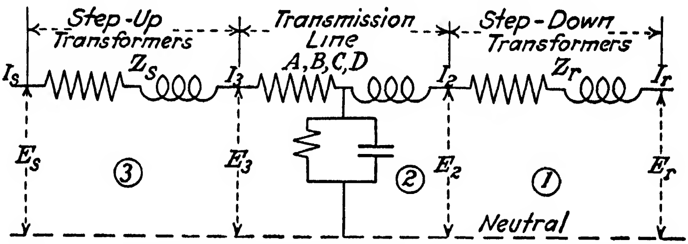


FIG. 65.—Transmission line in series with transformer impedances.

formers in addition to the line itself. It is therefore desirable to have equivalent generalized line constants for this case. If the magnetizing currents and the losses in the transformers be neglected, the equivalent transformer circuit becomes a simple series impedance. We then have the problem of evaluating equivalent constants for a circuit consisting of the transmission line itself, plus a series impedance, Z_r, at the receiver end, plus a series impedance, Z_s, at the supply end. Z_r and Z_s may represent the receiver-end and supply-end equivalent transformer impedances, respectively. The elements of this circuit are represented in Fig. 65. This is also a special case of impedances in series and falls under (*c*).

Following the usual procedure, the equations for each section are written. Thus, for the first section,

$$\begin{aligned} E_2 &= E_rA_1 + I_rB_1 \\ I_2 &= I_rD_1 + E_rC_1 \end{aligned} \qquad (426)$$

for the second section,

$$\left.\begin{aligned} E_3 &= E_2A_2 + I_2B_2 \\ I_3 &= I_2D_2 + E_2C_2 \end{aligned}\right\} \tag{427}$$

and for the third section,

$$\left.\begin{aligned} E_s &= E_3A_3 + I_3B_3 \\ I_s &= I_3D_3 + E_3C_3 \end{aligned}\right\}. \tag{428}$$

Eliminating E_2 and I_2 from Eqs. (426) and (427) will yield two equations in E_3 and I_3. Using these two equations and Eq. (428) and eliminating E_3 and I_3 therefrom yields two equations in E_s, I_s, E_r and I_r. Putting these resultant equations into the usual form yields the equivalent constants

$$\left.\begin{aligned} A_0 &= A_3(A_1A_2 + C_1B_2) + B_3(A_1C_2 + C_1D_2) \\ B_0 &= A_3(B_1A_2 + D_1B_2) + B_3(B_1C_2 + D_1D_2) \\ C_0 &= C_3(A_1A_2 + C_1B_2) + D_3(A_1C_2 + C_1D_2) \\ D_0 &= C_3(B_1A_2 + D_1B_2) + D_3(B_1C_2 + D_1D_2) \end{aligned}\right\} \tag{429}$$

The constants designated by the subscripts 1, 2 and 3, are already known. From (a) the constants with the subscripts 1 and 3 may be identified. They are

$$\begin{aligned} A_1 &= A_3 = 1 \\ B_1 &= Z_r \\ B_3 &= Z_s \\ C_1 &= C_3 = 0 \\ D_1 &= D_3 = 1. \end{aligned}$$

The constants carrying the subscript 2 are the A, B, C and D of the line itself, in which, of course, $A = D$. Substituting the above equivalents in Eqs. of (429), the resultant constants of the equivalent networks are found to be

$$\left.\begin{aligned} A_0 &= A + CZ_s \\ B_0 &= B + A(Z_r + Z_s) + CZ_rZ_s \\ C_0 &= C \\ D_0 &= A + CZ_r \end{aligned}\right\} \tag{430}$$

h. Transmission Line and Transformers at Both Ends Including Exciting Current of Transformers.—This is a case of combined series impedance and shunt admittance at each end of the line, plus the line itself. Constants for this case will not be worked

out here. These constants, if desired, may be found in the reference given at the opening of this article.

Illustrative Calculations.—It will be helpful at this point to illustrate the method of calculating the various constants of a line by the use of a numerical example. Let it be required to find the constants of a 100-mile, 60-cycle line built of No. 000 copper wire, having the following constants per mile of one conductor to neutral:

$r_1 = 0.326$ ohm per mile $\qquad g_1 = 0$ mho per mile

$x_1 = 0.818$ ohm per mile $\qquad b_1 = 5.24 \times 10^{-6}$ mho per mile

$l = 100$ miles.

a. Unit Line Angle $\theta = \sqrt{Y_1 Z_1}$.—

$$Z_1 = r_1 + jx_1$$
$$= 0.326 + j0.818 \text{ vector ohms}$$

or, using the polar form of notation,

$$Z_1 = z_1 \underline{/\tan^{-1} \frac{x_1}{r_1}}$$
$$= \sqrt{r_1^2 + x_1^2} \underline{/\tan^{-1} \frac{0.818}{0.326}}$$
$$= 0.88057 \underline{/68^\circ, 16.26'}.$$

Similarly,

$$Y_1 = 5.24 \times 10^{-6} \underline{/90^\circ, 0.0'}.$$

Then

$$Y_1 Z_1 = 0.88057 \times 5.24 \times 10^{-6} \underline{/68^\circ, 16.26' + 90^\circ}$$
$$= 4.6142 \times 10^{-6} \underline{/158^\circ, 16.26'}$$

and

$$\alpha_1 + j\beta_1 = \sqrt{Y_1 Z_1}$$
$$= \sqrt{4.6142 \times 10^{-6}} \underline{/\tfrac{1}{2}(158^\circ, 16.26')}$$
$$= 0.0021481 \underline{/79^\circ, 8.13'}.$$

In rectangular coordinates,

$$\alpha_1 + j\beta_1 = 0.0021481 [\cos (79^\circ, 8.13') + j \sin (79^\circ, 8.13')]$$
$$= 0.0021481(0.18848 + j0.98208)$$
$$= 0.00040487 + j0.0021096$$

or

$$\alpha_1 = 0.00040487$$
$$\beta_1 = 0.0021096$$

and

$$\alpha = \alpha_1 l = 0.040487 \text{ hyperbolic radian}$$
$$\beta = \beta_1 l = 0.21096 \text{ circular radian.}$$

b. Surge Impedance and Surge Admittance.

$$\frac{Z_1}{Y_1} = \frac{0.88057\underline{/68°, 16.26'}}{5.24 \times 10^{-6}\underline{/90°, 0.00'}}$$
$$= 168{,}050\underline{/-90° + 68°, 16.26'}$$
$$= 168{,}050\underline{/-(21°, 43.74')}$$
$$Z_0 = \sqrt{\frac{Z_1}{Y_1}} = \sqrt{168{,}050}\underline{/-\tfrac{1}{2}(21°, 43.74')}$$
$$= 409.9\underline{/-(10°, 51.87')} \text{ vector ohms.}$$

In rectangular coordinates,

$$Z_0 = 409.9 [\cos (10°, 51.87') - j \sin (10°, 51.87')]$$
$$= 409.9(0.98208 - j0.18850)$$
$$= 402.6 - j77.27 \text{ vector ohms.}$$

The surge admittance is

$$Y_0 = \frac{1}{Z_0} = \frac{1}{409.9}\underline{/10°, 51.87'}$$
$$= 0.002439\underline{/10°, 51.87'}.$$

In rectangular coordinates,

$$Y_0 = 0.002439(0.98208 + j0.18850)$$
$$= 0.0023970 + j0.00045975 \text{ vector mho.}$$

c. Constants A, B and *C.*

$$A = \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta$$
$$B = Z_0 (\sinh \alpha \cos \beta + j \cosh \alpha \sin \beta)$$
$$C = Y_0 (\sinh \alpha \cos \beta + j \cosh \alpha \sin \beta)$$
$$\cosh \alpha = \cosh 0.040487 = 1.00082$$
$$\sinh \alpha = \sinh 0.040487 = 0.040498$$
$$\cos \beta = \cos (0.21096 \times 57.2958)°$$
$$= \cos 12°, 5.22' = 0.97783$$
$$\sin \beta = \sin 12°, 5.22' = 0.20939$$
$$A = 1.00082 \times 0.97783 + j0.040498 \times 0.20939$$
$$= 0.97863 + j0.0084799$$

that is,

$$
\begin{aligned}
a_1 &= 0.97863\\
a_2 &= 0.0084799\\
B &= (402.6 - j77.27)(0.040498 \times 0.97783 + j1.00082 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times 0.20939)\\
&= (402.6 - j77.27)(0.039600 + j0.20956)\\
&= 32.1357 + j81.3090\\
b_1 &= 32.1357\\
b_2 &= 81.3090\\
C &= (0.0023970 + j0.00045975)(0.039600 + j0.20956)\\
&= -0.0000014240 + j0.00052052\\
c_1 &= -0.0000014240\\
c_2 &= 0.00052052.
\end{aligned}
$$

Physical Significance of the Constants α_1 and β_1.—The physical significance of the unit constants α_1 and β_1 is best shown by an interpretation of the exponential forms of the equations of the transmission line. These forms are obtained by substituting the expressions for the constants A, B and C of Eq. (406) in Eqs. (366) and (367). After rearranging, the expressions for the voltage and current l miles from the receiving end are found to be

$$E = \frac{1}{2}(E_r + I_rZ_0)\epsilon^{\alpha_1 l}\epsilon^{j\beta_1 l} + \frac{1}{2}(E_r - I_rZ_0)\epsilon^{-\alpha_1 l}\epsilon^{-j\beta_1 l} \tag{431}$$

$$I = \frac{1}{2}(I_r + E_rY_0)\epsilon^{\alpha_1 l}\epsilon^{j\beta_1 l} + \frac{1}{2}(I_r - E_rY_0)\epsilon^{-\alpha_1 l}\epsilon^{-j\beta_1 l}. \tag{432}$$

Since these equations are exactly alike in form, only one of them will be considered. In Eq. (431) it is to be observed that the voltage E comprises the two vectors

$$E_1 = \frac{1}{2}(E_r + I_rZ_0)\epsilon^{\alpha_1 l}\epsilon^{j\beta_1 l} \tag{433}$$

$$E_2 = \frac{1}{2}(E_r - I_rZ_0)\epsilon^{-\alpha_1 l}\epsilon^{-j\beta_1 l}. \tag{434}$$

For a given line and for any assumed constant values of frequency, receiver voltage and load current vectors, since Z_0 is a function of the fundamental line constants and frequency only, the quantities

$$\frac{1}{2}(E_r + I_rZ_0) = M_1 \tag{435}$$

and

$$\frac{1}{2}(E_r - I_r Z_0) = M_2 \tag{436}$$

are constant vectors. The notation of Eqs. (433) and (434) may therefore be simplified to read

$$E_1 = M_1 \epsilon^{\alpha_1 l} \epsilon^{j\beta_1 l} \tag{437}$$

and

$$E_2 = M_2 \epsilon^{-\alpha_1 l} \epsilon^{-j\beta_1 l}. \tag{438}$$

It is to be recalled that in these equations l is positive in the direction from receiving end to supply end.

At the receiver, where $l = 0$, the exponential terms in Eqs. (437) and (438) each reduce to unity, and hence

$$E_r = E = M_1 + M_2 \text{ (for } l = 0).$$

As l increases in length in the positive direction, that is, towards the supply end of the line, the component voltages E_1 and E_2 are constantly modified by the exponential operators, each in proportion to the distance l from the receiver. The operator $\epsilon^{\alpha_1 l}$ serves to stretch, or increase, the length of the component, while the operator $\epsilon^{j\beta_1 l}$ serves to rotate it in a counter-clockwise direction for positive values of l. The component vector E_2 is modified in the opposite manner. For positive values of l, the operator $\epsilon^{-\alpha_1 l}$ causes the vector to shrink, while $\epsilon^{-j\beta_1 l}$ causes it to rotate in clockwise direction, each in proportion to l.

Viewing the phenomenon from the generator end, and traveling along the line in the direction of the flow of energy, that is, towards the receiver, the generator voltage is the vector sum of the two components E_1 and E_2, which vary with the distance x from the generator and from point to point along the line. In the direction of positive values of x the component E_1 is constantly decreasing in length and falling behind in phase position (rotating clockwise), while E_2 is constantly increasing in length and gaining in phase position (rotating counter-clockwise); E_1 is the e.m.f. of a wave advancing in the direction of the energy flow, while E_2 is the e.m.f. of a reflected wave advancing in the opposite direction. Each component wave decreases in amplitude and lags increasingly in phase position as it advances.

A similar discussion holds for the current.

Attenuation and Wave Length.—From the above discussion it is apparent that the amount of shrinkage in the length of the

voltage or current vector in the direction of the traveling wave, per unit length of line, depends only upon the value of α_1. For this reason α_1 has been called the *attenuation constant* of the line.

Similarly, the amount of phase attenuation or phase shift of the vector per unit length of line depends only upon the value of β_1. The length of the line, for which the phase shift of the vector is 2π radians, is evidently one wave length λ so that

$$\lambda = \frac{2\pi}{\beta}. \tag{439}$$

β is therefore called the *wave length constant.*

On page 149 the phase shift for a certain line there considered, was found to be 0.118° per mile. The length of this line, for which the phase shift of the current and voltage vectors would be 360° (one wave length), is

$$\lambda = \frac{360}{0.118} = 3{,}050 \text{ miles (approximately).}$$

At the half wave-length point, or at 1,525 miles, the current and voltage vectors are just opposite their corresponding positions at the beginning of the line.

The complex number $a_1 + j\beta_1$, already referred to as the unit line angle, is also called the *propagation constant,* since it completely determines the manner in which the waves of current and voltage are propagated along the line.

Velocity of Propagation.—Since the number of miles of line for which the component vectors of current and voltage make one complete rotation is λ, it follows that in 1 sec. a wave will cause f complete rotations of the component vectors, and the velocity of propagation of the component waves is

$$\begin{aligned} v &= f\lambda \\ &= \frac{2\pi f}{\beta_1} \end{aligned}$$

For a line in which there are no losses, *i.e.*, for which $r = 0 = g$, $\beta_1 = 2\pi f\sqrt{L_1C_1}$—for such a line the velocity of propagation is

$$\begin{aligned} v &= \frac{2\pi f}{2\pi f\sqrt{L_1C_1}} \\ &= \frac{1}{\sqrt{L_1C_1}} \text{ miles per second} \end{aligned} \tag{440}$$

where L_1 is the inductance per mile of one line conductor, exclusive of the inductance due to the linkages within the conductor itself, and C_1 is the capacitance to neutral, per mile of line. This is the maximum attainable velocity of propagation and is equal to the velocity of light $= 3 \times 10^{10}$ cm. per second, or, approximately, 186,300 miles per second. It is also the approximate velocity of propagation of electrical impulses in ordinary well insulated high-power transmission lines, since in these lines r and g are small as compared with x and b.

The truth of the above may be verified by substituting in Eq. (440) the values of L_1 and C_1 for the case of one conductor of a parallel-sided loop suspended in air. For such a loop the inductance per mile of one conductor, neglecting internal linkages, is

$$L_1 = 0.00074113 \log_{10} \frac{D}{r}$$

while the capacitance per mile of one conductor to neutral is

$$C_1 = \frac{0.03883 \times 10^{-6}}{\log_{10} \frac{D}{r}}$$

whence, substituting and solving

$$v = 186{,}400 \text{ miles per second.}$$

Natural Frequency.—When for any given frequency the energies stored per unit length in the magnetic and dielectric fields of a line, during alternate half cycles are equal, the line is said to be in *resonance*. The frequency at which resonance occurs is called the *natural frequency* of the line.

Steinmetz has shown that for a line having uniformly distributed constants, the natural frequency is given by the equation

$$f = \frac{1}{4\sqrt{LC}} \tag{441}$$

where L is the inductance in henries of the line due to the linkages of magnetic flux with current, of the magnetic lines threading the air loop only (neglecting those within the conductors themselves), and C is the capacitance in farads of one conductor to neutral. If the constants L and C be reduced to their corresponding per mile values, the above equation becomes

$$f = \frac{1}{4l\sqrt{L_1C_1}}.$$

Remembering that the velocity of propagation is $v = \frac{1}{\sqrt{L_1C_1}}$, one may write

$$f = \frac{v}{4l} \tag{442}$$

and, since the wave length is $\lambda = V \div f$, it follows that, for the frequency of resonance,

$$l = \frac{\lambda}{4}. \tag{443}$$

That is, a line having a length equal to one-quarter wave length for the frequency f will be in resonance to this frequency.

In order to preclude the possibility of the production of dangerously high, induced voltages, such as naturally result from resonance, it is necessary to investigate the above relations for any proposed line in order to determine whether the line is of such length that any frequency likely to be encountered during operation would be apt to produce resonance. If investigation points to the possibility of the existence of dangerous frequencies, means should be found either to eliminate or suppress them.

For example, the frequency of the fifth harmonic of a 60-cycle circuit is 300 cycles per second. Resonance for this harmonic takes place in a line of length

$$l = \frac{186{,}300}{4 \times 300} = 155 \text{ miles, approximately.}$$

Therefore, a 60-cycle line, 155 miles long, would possess the possibility of giving trouble unless the line were artificially loaded, thus alternating its constants.

In transmission-line practice good engineering usually requires the use of transformer connections of a type that will make it impossible for third harmonics to exist in the line even though they may be quite prominent in the generated voltage wave. Grounded neutral systems permit triple-frequency currents to flow. Third harmonic voltages result from the magnetizing currents of transformers when Y-Y connected, or from auto transformers. For this reason Y-Y connected transformers should not be used. Triple-frequency voltages of dangerous values may be avoided by the use of suitable tertiary windings in transformers, through which the triple-frequency voltages are short-circuited, or by the use of delta-connected auxiliary transformers.

PROBLEMS

1. A 150-mile, 60-cycle, three-phase line is built of three 350,000-cir. mil, concentric-lay copper cables. The line is transposed to balance phases, and the equivalent spacing is $D' = 12.5$ ft. If the resistivity of copper at working temperature is 10.5 ohms per mil foot, find the constants, r, x, b, Z_0, Y_0, A, B and C for the line.

2. The line in Problem 1 is operated with a constant receiver voltage of 110 kv. and a constant supply voltage of 120 kv. In transformers, 36,000 kva. are connected at the receiver end and 40,000 kva. at the supply end. All transformers have 0.6 per cent resistance and 10 per cent reactance. Neglecting the exciting current of the transformers, calculate the equivalent line constants A_0, B_0, C_0 and D_0 of line and transformers, referred to the high sides of the transformers.

3. If the connected generating capacity is 40,000 kva., and the generator synchronous reactance may be assumed constant and equal to 90 per cent, what are the generalized line constants including generators?

4. A certain three-phase, 60-cycle line has the following constants per mile:

$$r = 0.25 \text{ ohm} \qquad g = 0 \text{ mho}$$
$$x = 0.81 \text{ ohm} \qquad b = 5.1 \times 10^{-6} \text{ mho}$$

Calculate the generalized line constants per mile of line, (*a*) By the hyperbolic form of equations; (*b*) By the convergent series equations, using real numbers, and discarding all quantities involving powers of α higher than the second.

5. A line 300 miles long has the same constants per mile of line as the 150-mile line of Problem 1. The supply voltage is 100 kv. Calculate the open-circuit current and voltage for each 60-mile interval between supply-end and receiver. Plot open-circuit current and voltage as ordinates against miles from generating station as abscissas.

6. What is one wave length of a line having the constants of Problem 1? Assume a line having these constants to be 3,000 miles long. Let the receiver voltage to neutral, and the current be

$$E_r = 30{,}000 + j0 \text{ volts}$$
$$I_r = 60 - j45 \text{ amp.}$$

Calculate the voltage to neutral and the current for each 200-mile interval from receiver to generator. Plot these values against miles from receiver, (*a*) in rectangular coordinates, and (*b*) in polar coordinates.

CHAPTER IX

VOLTAGE CONTROL OF TRANSMISSION LINES

Generally speaking, alternating-current transmission systems are operated with a fixed receiver voltage. This is necessary, since approximately constant voltage is required at the distribution center at all loads. Slight variations in voltage between the distribution center and the ultimate consumer, due to load changes, are taken care of by feeder regulators.

Voltage Control by Generator Excitation.—Constancy in the value of receiver-end voltages may be maintained, within practical operating limits, by varying the excitation of the generators to suit the load and power factor conditions. This method is quite generally used in short lines. As the load in the line increases, the generator excitation is increased to the point where the additional line drop is compensated for by the additional voltage generated. Under falling loads the reverse action takes place. By the use of automatic voltage regulators, properly compounded to compensate for changes in loads, and operating on the exciters of the generators, the operation of the line may be made quite automatic.

While this method of voltage control is satisfactory for relatively short lines, it is not applicable to lines of great length because the required difference in voltage between no-load and maximum-load conditions becomes excessive. Considerable swings in voltage at the generator end of the line are undesirable in themselves, but even aside from this objectionable feature, a practical limit is set by saturation in the generator field.

Line Regulation.—The regulation of a transmission line is the change in voltage at the receiving end of the line between rated, non-inductive load and no load, with constant voltage impressed upon the supply end. The regulation is usually expressed as a percentage of the terminal rated voltage at the load, and, when so expressed, it is the percentage change in the receiver voltage with respect to its normal rated value.

Constant Voltage, Variable Power-factor Control.—This method of voltage control, sometimes referred to as *phase control*, is commonly used in long-line practice, as well as in some of the more important short lines. It may also be combined with the variable voltage method of control already discussed where this is found to be advantageous.

The receiver-end voltage and the supply-end voltage are both kept constant for all loads, but both receiving-end and supply-end power factors vary, as demanded by the constant voltage requirements. This is made possible by the use of synchronous reactors connected to the receiving end of the line in parallel with the load, the excitation of the reactors being automatically controlled by means of the usual type of voltage regulators.

This method of control has the advantage of maintaining constant voltages at both receiving and supply ends of the line, independent of transformer characteristics and the variable drops introduced by them and by the line, with variable loads. Furthermore, the controlling equipment, while expensive, is nevertheless simple in its operation.

Among the disadvantages of the system, the following are important: When synchronous reactors are employed at the receiving end only, as is here assumed, the voltage at any point in the line other than one of its ends, varies with the load. No constant-voltage taps are therefore possible at any intermediate point In long lines, during light-load periods, if for some reason the synchronous reactors should fall out of step and be disconnected from the line, the voltage at the receiving end of the line would rise to dangerously high values. For this reason, the line insulation of long lines should be built with considerably higher factors of safety than short-line practice would require The high cost of the control equipment is another item of importance. Very long lines are entirely inoperative with this system of control. Such lines, where they are now operated, are usually operated as tie lines having load points connected to them at more or less frequent intervals, rather than as straight, single, transmission circuits. For very long, straight transmission lines it would be advantageous to connect synchronous reactors at one or more points along the line, as well as at the receiving end. In this way the line voltage could be held at fixed values at several points in the line, and the possibility of dangerous voltage rises would be largely avoided.

Reactive Power Required for Phase Control. Approximate Equations for Short Lines.—The discussion following is strictly applicable only to short lines for which either the so-called "impedance circuit" or the "load-end condenser circuit" applies.

The equation relating to the supply and receiver voltages to neutral for these circuits is Eq. (302). If the receiver current is

$$I_r = I_1 - jI_2 \text{ vector amperes}$$

the supply voltage is

$$E_s = E_r + rI_1 + xI_2 + j(xI_1 - rI_2) \text{ vector volts.}$$

This equation applies to either the simple impedance circuit or the load-end condenser circuit, so long as I_2 is interpreted to mean the reactive component of the receiver current. Thus, for the impedance line, I_2 is the load reactive component of current, while for the load-end condenser circuit I_2 is the difference between the lagging quadrature component of the load current and the charging current of the load-end condenser. The constants r and x may be taken as the equivalent resistance and inductive reactance, respectively, of the line, including raising and lowering transformers.

Since the receiver voltage is constant and the value of the in-phase current I_1 is fixed by the receiver load, there remain only two variables in the equation, namely, the generator voltage and the reactive component I_2, of the receiver current. If the latter is determined by the load alone it is beyond the control of the operator, and the receiver voltage can be held constant only by controlling the supply voltage through excitation. This gives rise to the method of voltage control by generator excitation already discussed.

On the other hand, if means are provided at the receiver end of the line by which the reactive component of the receiver current may be controlled, it is possible to keep both receiver and the supply voltages constant at predetermined values, and for all loads. Such control is brought about by the installation of synchronous reactors at the receiver-end of the line, and the method of control employing them is called *phase control.*

The operation is about as follows: At light loads the synchronous reactor is underexcited, thus increasing the lagging reactive power of the receiver circuit and the value of I_2. The line drop is thereby increased. As the active power in the receiver circuit

rises, the lagging reactive power taken by the synchronous reactor is gradually reduced by increasing the excitation, until, at about half load for the receiver, the excitation of the reactor is increased to a point such that the reactive power of the reactor becomes leading instead of lagging, and remains leading for all greater loads. For heavy loads, therefore, much of the lagging reactive power demanded by the load is supplied by the synchronous reactor. For increasing loads the reactive component of current flowing in the line is accordingly gradually reduced from a large lagging value, and may even pass through zero and become leading if the load is increased sufficiently. The vector line drop is therefore increased under light loads and reduced under heavy loads, thus making it possible to keep the absolute values of E_s and E_r both constant for all loads.

The relation which must exist between the active load current I_1 and the reactive component I_2 of the receiver current, in order to maintain constancy in receiver-end and supply voltages, is found as follows:

$$E_s = E_r + rI_1 + xI_2 + j(xI_1 - rI_2).$$

Or, since

$$E_s = E_1 + jE_2 \text{ vector volts}$$

$$E_1 = E_r + rI_1 + xI_2 \text{ volts absolute} \tag{444}$$

and

$$E_2 = xI_1 - rI_2 \text{ volts absolute.} \tag{445}$$

Solving the simultaneous Eqs. (444) and (445) yields the relation

$$\left(I_2 + \frac{E_r x}{z^2}\right)^2 + \left(I_1 + \frac{E_r r}{z^2}\right)^2 = \frac{E_s^2}{z^2}$$

or

$$I_2 = -\frac{E_r x}{z^2} + \sqrt{\frac{E_s}{z^2} - \left(I_1 + \frac{E_r r}{z^2}\right)^2}. \tag{446}$$

Equation (446) may be converted to a power equation by substituting for the two components of receiver current their equivalent power expressions. These are

$$\left.\begin{aligned} I_1 &= \frac{P_r}{E_r} \\ I_2 &= \frac{Q_r}{E_r} \end{aligned}\right\} \tag{447}$$

where P_r and Q_r are the true and quadrature components, respectively, of the receiver volt-amperes. Making these substitutions and dividing through by E_r yields

$$\left(\frac{P_r}{E_r^2} + \frac{r}{z^2}\right)^2 + \left(\frac{Q_r}{E_r^2} + \frac{x}{z^2}\right)^2 = \frac{E_s^2}{E_r^2 z^2}. \tag{448}$$

Equation (448) is the equation of a family of circles in which the coordinates of the center are[1]

$$\left.\begin{aligned} l &= \frac{P_r}{E_r^2} = -\frac{r}{z^2} \\ m &= \frac{Q_r}{E_r^2} = -\frac{x}{z^2} \end{aligned}\right\} \tag{449}$$

and the radius is

$$n = \sqrt{\frac{E_s^2}{E_r^2 z^2}} = \frac{E_s}{E_r z} \tag{450}$$

where

$$z = \sqrt{r^2 + x^2}$$

the numerical value of the line impedance.

For the purpose of numerical calculations it is somewhat more convenient to multiply through by 10^6. The constants of Eq. (448) then become

$$\left.\begin{aligned} l' &= 1{,}000l \\ m' &= 1{,}000m \\ n' &= 1{,}000n \end{aligned}\right\} \tag{451}$$

and Eq. (448) takes the form,

$$\left(\frac{1{,}000P_r}{E_r^2} + l'\right)^2 + \left(\frac{1{,}000Q_r}{E_r^2} + m'\right)^2 = \frac{10^6 E_s^2}{E_s^2 z^2}. \tag{452}$$

Or, if P_r and Q_r be expressed in kilowatts and kilovolt-amperes, respectively, and E_r and E_s are in kilovolts, one may write

$$\left(\frac{P_r}{E_r^2} + l'\right)^2 + \left(\frac{Q_r}{E_r^2} + m'\right)^2 = \frac{10^6 E^2}{E_r^2 z^2}. \tag{453}$$

From Eqs. (448) or (453) the amount of reactive power Q_r, required at the receiver end to maintain fixed voltages at both ends of the line and bearing any assumed ratio $E_s \div E_r$ to each other, may readily be found for any assumed load, P_r.

[1] In order to make the circle diagram represent leading and lagging reactive kilovolt-amperes in the first and fourth quadrants, respectively, the centers of all circle diagrams in this book similar to this one are located at $+l$, $-m$ instead of at $+l$, $+m$, as required by Eq. (449).

Example.—As an example to illustrate the use of these equations, consider the following problem:

A line 38 miles long, built of 250,000-cir. mil, stranded, copper cable delivers a maximum load of 50,000 kw. at a power factor of 85 per cent current lagging. The receiver line voltage is 110 kv. The line constants of one conductor are

$$r = 9.03 \text{ ohms}$$
$$x = 31.0 \text{ ohms}$$
$$b = 1.98 \times 10^{-4} \text{ mho.}$$

The receiver voltage to neutral is constant and equal to $110 \div \sqrt{3} = 63.5$ kv. Assume (*a*) no charging current, and (*b*) the charging current of the entire line flowing over its full length. It is required to find the minimum synchronous reactor which will keep voltages at both receiver and supply ends of the line constant over the entire range of load for each condition, and to find the corresponding values of supply voltages.

Solution.—

$$z = \sqrt{\overline{9.03}^2 + \overline{31.0}^2} = 32.3 \text{ ohms.}$$
$$z^2 = 1{,}043$$
$$\frac{1}{z^2} = 958.8 \times 10^{-6}$$
$$E_r^2 = \left(\frac{110}{\sqrt{3}}\right)^2 = 4{,}034 \text{ (for } E_r \text{ in kv.).}$$

The maximum load per phase is

$$P_r = \frac{50{,}000}{3}$$
$$= 16{,}670 \text{ kw.}$$

The maximum load reactive kilovolt-amperes is

$$Q_r = 16{,}670 \tan^{-1} (\cos 0.85)$$
$$= 10{,}330 \text{ kva. lagging.}$$

The center of the receiver power circle is at

$$l' = \frac{1{,}000 \times 9.03}{1{,}014} = -8.66$$
$$m' = \frac{1{,}000 \times 31}{1{,}043} = 29.7$$

and the radius is

$$n' = \frac{E_s}{E_r}\sqrt{958.8} = 30.96\frac{E_s}{E_r}.$$

For maximum load,

$$\frac{1{,}000P_r}{E_r^2} = \frac{16.67 \times 10^9}{4.033 \times 10^9} = 4.13.$$

For zero load,

$$\frac{1{,}000P_r}{E_r^2} = 0.$$

Substituting the above values in the equation for reactive kilovolt-amperes in the receiver circuit,

$$\frac{1{,}000Q_r}{E_r^2} = -29.7 + \sqrt{958.8\frac{E_s^2}{E_r^2} - (4.13 + 8.66)^2} \text{ for maximum load}$$

$$= -29.7 + \sqrt{958.8\frac{E_s^2}{E_r^2} - (8.66)^2} \text{ for zero load.}$$

Using values of $E_s \div E_r = 0.95$, 1.0, 1.05 and 1.1 in the above equations and solving for each of these yield the values of Q_r and Q_{sr}, as given in columns (5) and (6) of Table 9, for the case of negligible charging current.

Taking the charging current into account, it is found that the leading reactive power, introduced into the receiver circuit by it, is

$$Q_c = bE_r^2$$
$$= 1.98 \times 10^{-4} \times 4.03 \times 10^9$$

or

$$\frac{Q_c}{1{,}000} = 800 \text{ kva. leading.}$$

This has the effect of reducing by 800 kva. each of the values of Q_L in column (5), Table 9. The corrected values of Q_L for this case are shown in column (5)′ and the corresponding reactive kilovolt-amperes required in synchronous reactors in column (6)′.

TABLE 9

(1) $P_r \div 1{,}000$ = receiver kilowatts	(2) $E_s \div E_r$	(3) n'^2	(4) $Q_r \div 1{,}000$ receiver reactive kilovolt-amperes per phase required	$Q_L \div 1{,}000$ = load reactive kilovolt-amperes per phase		$Q_{sr} \div 1{,}000$ = syn. reactor reactive kilovolt-amperes per phase	
				(5) No charging current	(5)′ With charging current	(6) No charging current	(6)′ With charging current
16,670	0.95	865.3	−12,910	10,330	9,530	−23,240	−22,440
0	0.95	865.3	− 6,450	0	0	− 6,450	− 6,450
16,670	1.00	958.8	− 6,050	10,330	9,530	−16,380	−15,580
0	1.00	958.8	0	0	0	0	0
16,670	1.05	1,057.1	+ 810	10,330	9,530	− 9,520	− 8,720
0	1.05	1,057.1	+ 6,450	0	0	+ 6,450	+ 6,450
16,670	1.10	1,160 1	+ 7,660	10,330	9,530	− 2,670	− 1,870
0	1.10	1,160 1	+12,910	0	0	+12,910	+12,910
16,670	1.15	1,268 0	+14,120	10,330	9,530	+ 3,790	+ 4,590
0	1.15	1,268 0	+19,360	0	0	+19,360	+19,360

Note: Plus signs indicate lagging and minus signs leading reactive kilovolt-amperes.

By the method explained in a succeeding article of this chapter for long lines, the best ratio of voltages is found to be $E_s \div E_r = 1.061$, approximately. For this ratio of voltages the synchronous reactor capacity required is about 7,900 kva. per phase or 23,700 kva. total for the line, assuming the reactors to be designed for equal ratings on full leading and lagging loads.

The circle diagram may readily be drawn, for the center is at $l' = -8.66$, $m' = 29.7$, and for a ratio $E_s \div E_r = 1$ the radius is

30.96. The center remains the same for all circles, and the radii for the different circles are proportional to $E_s \div E_r$. It will be noted that the sign of m' has been changed from $-$ to $+$, for the reason already given.

Reactive Power Required for Phase Control in Long Lines.—The mechanism of phase control, and the amount of reactive power, required at any given load to maintain any constant predetermined values of supply and receiver voltages, are readily understood from an examination of the equations applicable to the case. For the long line these equations are developed below:

The general vector equations of the transmission-line circuit are Eqs. (372) and (373). They are[1]

$$\left.\begin{aligned} E_s &= E_rA + I_rB \\ I_s &= I_rA + E_rC \end{aligned}\right\}$$

Using the complex notation of Eqs. (374) and (375), and writing

$$\begin{aligned} \text{Receiver current} &= I_r = {}_rI_1 - j_rI_2 \\ \text{Supply current} &= I_s = {}_sI_1 + j_sI_2 \\ \text{Supply voltage} &= E_s = E_1 + jE_2 \end{aligned}$$

the above equations become

$$E_1 + jE_2 = E_r(a_1 + ja_2) + ({}_rI_1 - j_rI_2)(b_1 + jb_2) \text{ supply vector volts} \quad (454)$$

$$I_1 + jI_2 = ({}_rI_1 - j_rI_2)(a_1 + ja_2) + E_r(c_1 + jc_2) \text{ supply vector amp.} \quad (455)$$

Since the receiver power factor is to vary as required to keep both receiver and supply voltages constant, it is desirable to solve for ${}_rI_2$ in Eq. (454) in terms of the constants of the line, the constant voltages and the active load current ${}_rI_1$. From Eq. (454) the numerical values of the component supply voltages are

$$E_1 = E_ra_1 + {}_rI_1b_1 + {}_rI_2b_2 \text{ volts} \quad (456)$$

$$E_2 = E_ra_2 + {}_rI_1b_2 - {}_rI_2b_1 \text{ volts.} \quad (457)$$

Squaring Eqs. (456) and (457), and adding,

$$E_s^2 = E_r^2(a^2 + a_2^2) + 2E_r\,{}_rI_1(a_1b_1 + a_2b_2) + 2E_r\,{}_rI_2(a_1b_2 - a_2b_1) + ({}_rI_1^2 + {}_rI_2^2)(b_1^2 + b_2^2). \quad (458)$$

[1] If transformer impedances are to be included with the line constants, the derived constants A_0, B_0, C_0 and D_0 should be used in these equations.

Dividing Eq. (458) through by $b_1^2 + b_2^2$ and rearranging, yields

$$\left[{}_rI_1 + E_r\frac{(a_1b_1 + a_2b_2)}{b_1^2 + b_2^2}\right]^2 + \left[{}_rI_2 + \frac{E_r(a_1b_2 - a_2b_1)}{b_1^2 + b_2^2}\right]^2 = \frac{E_s^2}{b_1^2 + b_2^2}. \quad (459)$$

This is the equation of a family of circles, there being one circle for each pair of constant values of E_s and E_r chosen (compare with Eq. (446)). For any given constant supply and receiver voltages E_s and E_r, the circle defines the reactive current ${}_rI_2$ required for any assumed value of active load current ${}_rI_1$.

When solved for ${}_rI_2$, Eq. (459) becomes

$${}_rI_2 = \frac{-E_r(a_1b_2 - a_2b_1)}{b_1^2 + b_2^2} \pm \sqrt{\frac{E^2}{b_1^2 + b_2^2} - \left[{}_rI_1 + \frac{E_r(a_1b_1 + a_2b_2)}{b_1^2 + b_2^2}\right]^2} \quad (460)$$

$$= -E_rm + \sqrt{E_s^2n^2 - ({}_rI_1 + E_rl)^2} \quad (461)$$

where

$$l = \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2} \quad (462)$$

$$m = \frac{a_1b_2 - a_2b_1}{b_1^2 + b_2^2} \quad (463)$$

$$n = \frac{1}{(b_1^2 + b_2^2)^{\frac{1}{2}}}. \quad (464)$$

The coordinates of the center are

$${}_rI_1 = -E_rl \quad (465)$$

$${}_rI_2 = -E_rm \quad (466)$$

and the radius is

$$R = E_sn. \quad (467)$$

Since

$$P_r = E_r\,{}_rI_1 = \text{receiver true power} \quad (468)$$

and

$$Q_r = E_r\,{}_rI_2 = \text{receiver reactive power} \quad (469)$$

by substituting for ${}_rI_1$ and ${}_rI_2$ in Eq. (460) their appropriate equivalents, and dividing through by E_r, the power circle is obtained. It is

$$\left[\frac{P_r}{E_r^2} + l\right]^2 + \left[\frac{Q_r}{E_r^2} + m\right]^2 = \frac{E_s^2n^2}{E_r^2}. \quad (470)$$

The coordinates of the center are $-l$, $-m$ and the radius is $\frac{E_s n}{E_r}$.

Equation (470) is the fundamental equation of phase control, for it determines the amount of reactive power required to maintain a chosen ratio of voltages for any given receiver load P_r. Equation (459) is an exactly similar relation in terms of the component receiver currents.

Example.—An example will serve to illustrate the use of these equations. Assume a 250-mile, three-phase line, transmitting a maximum load of 45,000 kw., or 15,000 kw. per phase, at an assumed constant-load power factor of 85 per cent lagging. The receiver voltage is 89 kv. to neutral, and the construction of the line is such as to yield the following derived line constants:

$$
\begin{aligned}
a_1 &= 0.8702 \\
a_2 &= 0.0282 \\
b_1 &= 41.15 \\
b_2 &= 186.7 \\
c_1 &= -1.26 \times 10^{-5} \\
c_2 &= 125.7 \times 10^{-5}.
\end{aligned}
$$

It is required to find the connected synchronous reactor capacity required at the receiver to maintain the constant supply and receiver voltages in the ratios of 0.8, 0.9, 1.0, 1.1 and 1.2. The amount of capacity required is to be determined for both no-load and maximum-load conditions.

Equation (470) will be put into more convenient form for numerical work by multiplying it through by 10^6. Performing this multiplication and solving for the quantity $\frac{1{,}000Q_r}{E_r^2}$ yields

$$\frac{1{,}000Q_r}{E_r^2} = -1{,}000m + \sqrt{\frac{10^6E_s^2n^2}{E_r^2} - \left[\frac{1{,}000P_r}{E_r^2} + 1{,}000l\right]^2}$$

or, expressing P_r and Q_r in kilowatts and kilovolt-amperes, and E_s and E_r in kilovolts,

$$\frac{Q_r}{E_r^2} = -m' + \sqrt{\frac{E_s^2n'^2}{E_r^2} - \left[\frac{P_r}{E_r^2} + l'\right]^2} \tag{471}$$

where

$$
\begin{aligned}
l' &= 1{,}000l &&= 1.124 \\
m' &= 1{,}000m &&= 4.41 \\
n' &= 1{,}000n &&= 5.23 \\
E_r^2 &= (89)^2 &&= 7{,}921.
\end{aligned}
$$

At maximum load of 15,000 kw. per phase,

$$\frac{P_r}{E_r^2} = 1.89.$$

At zero load,

$$\frac{P_r}{E_r^2} = 0.$$

Substituting the above values of the constants in Eq. (471), and solving for Q_r for each of the assumed values of the ratio $E_s \div E_r$, the values given in the following table are obtained:

TABLE 10

(1) P_r receiver kilowatts per phase	(2) $E_s \div E_r$ assumed ratio	(3) $\frac{E_s^2 n'^2}{E_r^2}$	(4) Q_r receiver reactive kilovolt-amperes required per phase[1]	(5) Q_L load reactive kilovolt-amperes per phase[1]	(6) = (4) − (5) Q' synchronous reactor reactive kilovolt-amperes per phase[1]
15,000.0	0.8	17.50	−11,960	+9,290	−21,250
0.0	0.8	17.50	− 3,010	0	− 3,010
15,000.0	0.9	22.15	− 6,340	+9,290	−15,630
0.0	0.9	22.15	+ 1,270	0	+ 1,270
15,000.0	1.0	27.35	− 1,110	+9,290	−10,400
0.0	1 0	27.35	+ 5,620	0	+ 5,620
15,000.0	1.1	33.09	+ 3,880	+9,290	− 5,410
0.0	1 1	33.09	+ 9,740	0	+ 9,740
15,000.0	1 2	39.38	+ 8,630	+9,290	− 660
0 0	1 2	39.38	+13,940	0	+13,940

[1] Plus signs indicate lagging and minus signs leading kilovolt-amperes.

The load reactive kilovolt-amperes, given in column (5), are:

Load reactive kilovolt-amperes

$$\begin{aligned} Q_L &= P_r \tan \theta_r \\ &= 15{,}000 \times 0.6196 \\ &= 9{,}290 \text{ kva. lagging,} \end{aligned}$$

while the reactive kilovolt-amperes required of the synchronous reactor are found from (4) and (5), as indicated.

For synchronous reactors having equal ratings as generators of leading and lagging reactive kilovolt-amperes, the minimum capacity in reactors is required when the ratio of $E_s \div E_r = 1.06$, approximately. The capacity in reactors required with this ratio is approximately 7,700 kva. per phase or 23,100-kva. total.

Power-circle Diagrams.—The circle diagrams corresponding to the above analytical solution are now readily drawn. Such circles are shown in Fig. 66. They are a great aid in visualizing the line performance. They serve not only as an important check on the accuracy of the analytical work, but, if a sufficiently large scale is used, they may readily serve as a convenient substitute for the analytical solution of long line problems.

Since lagging quadrature currents are usually represented as negative quadrature quantities, while leading quadrature cur-

rents are shown as positive quantities on the quadrature or Y-axis, the corresponding reactive kilovolt-amperes will be similarly represented. Thus, lagging reactive kilovolt-amperes, while appearing in the table with a positive sign prefixed, will be shown in the circle diagram as negative quantities. The reverse will likewise hold for leading reactive kilovolt-amperes.

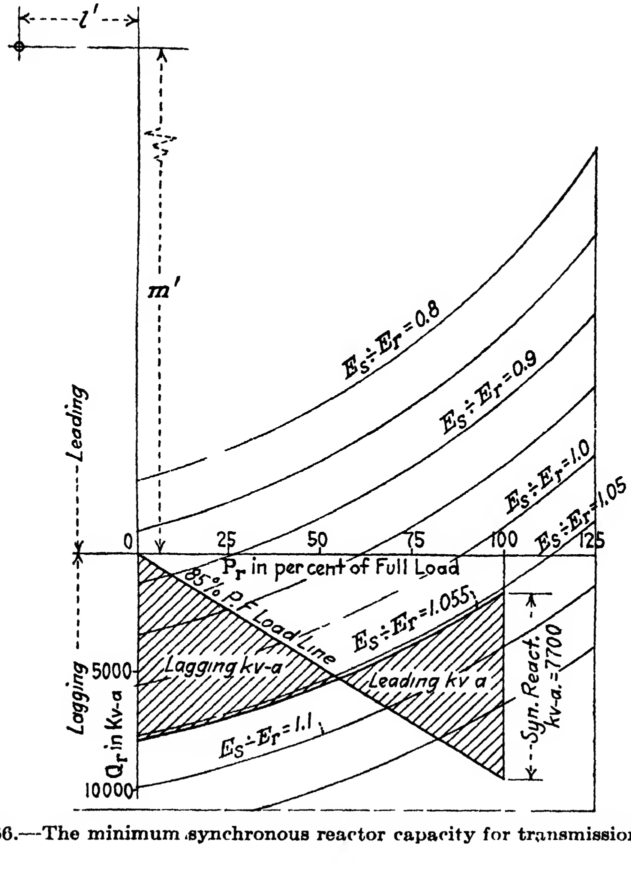


FIG. 66.—The minimum synchronous reactor capacity for transmission lines

To do this requires that the sign of the ordinate to the center of the circle be also reversed. So, instead of the coordinates of the center appearing as $-l'$ and $-m'$, they will be taken as $-l'$ and $+m'$ for the power circles, and as $-E_r l$ and $+E_r m$ for the current circles in all diagrams where they appear.

The power circles for different ratios of supply to receiver voltages are concentric. The radius of the circle for $E_s \div E_r =$

1 is n', while for any other circle the radius is proportional to the ratio assumed; that is,

$$\text{Radius} = n' \times \text{ratio of voltages.}$$

Minimum Synchronous Reactor Capacity.—For any fixed receiver voltage, the synchronous reactor capacity, required to maintain constant voltages at both ends of the line for all loads between zero and maximum values, is a variable depending upon the ratio $E_s \div E_r$, of the voltages. That ratio which requires the minimum reactor capacity, other things being equal, is the most economical ratio to use. This ratio may be found by plotting the algebraic sums of the pairs of values of $Q' \div 1{,}000$ for no load and for maximum load and for a given ratio of $E_s \div E_r$, as a function of the ratio. Where this curve crosses the axis of $E_s \div E_r$ will determine the minimum capacity of reactors, assuming that the reactors have equal ratings as generators of leading and lagging kilovolt-amperes. If the reactors are designed for any other ratio of lagging to leading reactive kilovolt-amperes, as, for example, 60 per cent lagging and 100 per cent leading, then the required values of column (6) Table 10 should be weighted in the ratio of 6 to 10 before the sum is taken and the results are plotted.

That exactly similar diagrams may be drawn for the approximate short-line circuits is obvious.

The same results may be conveniently obtained graphically from the diagram of Fig. 66 by interpolation. For reactors having equal ratings as generators of leading and lagging reactive kilovolt-amperes, the correct ratio is found by locating the circle for which the intercept, on the Y-axis at no load, is equal to the Y-intercept between the circle and the load line at maximum load. For a 60 per cent reactor, for example, the rating, as a generator of lagging kilovolt-amperes, is 60 per cent of its rating when generating leading kilovolt-amperes, hence the above intercepts, for this case, should be in the ratio of 6 to 10 for the most economical ratio of voltages.

In the example given, the most economical ratio of voltages, for a 100 per cent reactor, is found to be 1.055, as indicated in Fig. 66.

If desired, the optimum ratio of $E_s \div E_r$ may be found analytically as illustrated in the problem of Chap. XVI.

CHAPTER X

MECHANICAL DESIGN

SPAN WITH SUPPORTS AT EQUAL ELEVATIONS

The Design of Mechanical Structure.[1]—The burden of this chapter, as indicated by the title, is to consider the transmission-line span, consisting of two supports and a cable suspended between them. To begin with, the theory will be developed for the case which assumes the points of attachment of the cable to the two supports to be at equal elevations. The case of the span, with supports at unequal elevations, will later be discussed as an extension of the theory previously developed.

Suspended Cable a Catenary.—It is well known that a perfectly flexible cord or chain of uniform structure, suspended between two supports at equal elevations, and acted on by the force of gravity only, lies in a vertical plane and assumes the form of curve called the catenary. The conductors of a transmission-line span very closely fulfil the above described conditions, for while the conductor is not entirely inelastic, yet the length of span is great as compared with the conductor diameter, so that whatever stiffness the conductor may possess has little effect in determining its position in space.

In the discussion of theory which follows, it will be assumed that the following conditions are fulfilled:

1. The suspended cable is a cylindrical solid.
2. The suspended cable is of uniform texture.
3. The suspended cable is perfectly flexible.
4. No external force except the force of gravitation acts on the cable.
5. The axis of the cable will assume the form of the catenary.

Let the curve taken by the cable be represented by the curve P_1OP_2 of Fig. 67. Since the mass is uniformly distributed along the axis of the cable, and the active gravitational forces are pro-

[1] Kirsten, F. K., "Transmission Line Design," *Trans.*, A. I. E. E., p. 735, 1917; University of Washington Engineering Experiment Station *Bull.* 17.

portional thereto, in a span having supports at equal elevations, there must exist a condition of symmetry of the shape of the curve P_1OP_2, with respect to a vertical plane, midway between the points P_1 and P_2 and perpendicular to the straight line P_1P_2. This plane will be selected as the reference plane YY. The point of maximum deflection of the cable from the straight line P_1P_2 must lie in this plane. A horizontal plane tangent to the curve P_1OP_2 at the point of maximum deflection is chosen as the reference plane XX. Thus O, the point of maximum deflection of the cable, is the origin of the coordinate axes.

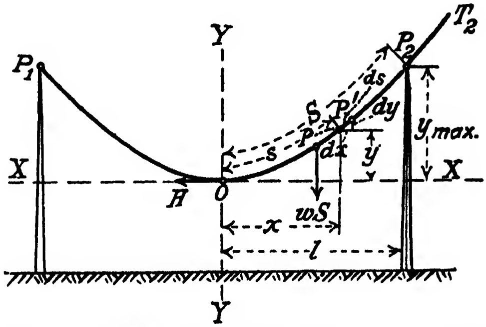


FIG. 67.—Conductor suspended between supports at equal elevations.

Notation.—Letters and symbols, used in the following discussion and in reference to Fig. 67, are as follows:

l = half distance between P_1 and P_2, or half tower spacing.
S = half length of suspended cable.
w = weight per unit length of suspended cable.
x and y are the rectangular coordinates of the point P.
s = the length of cable between O and P.
ds, dx and dy are increments of s, x and y, respectively, which, at the limit zero, will bear the relation $(dx)^2 + (dy)^2 = (ds)^2$.
H = tension in the cable at the point of maximum deflection.
T_2 = tension in the cable at the point of support P_2.
T = tension in the cable at P, whose coordinates are x and y.
α = angle between T and the X-axis.

Derivation of Equations.—Herein it is assumed that all stresses normal to the cross-sectional area, at any point of the cable, are concentrated on the axis of the cable and act in the direction of

the tangent to the axis at that point. Thus the slope of the curve, at any point P, is also the slope of the line of action of the tension T at that point.

The conditions for the equilibrium of the half span require that the vector sum of the forces acting be zero. The three forces in question are the horizontal tension H, the vertical load ws, and the tension T_2. If the point of support were moved to any other point such as P, the conditions for equilibrium would then require that

$$T^2 = H^2 + w^2s^2. \tag{472}$$

From the triangle of forces, it is also apparent that

$$\frac{ws}{H} = \tan \alpha \tag{473}$$

and since T is tangent to the curve at P

$$\frac{dy}{dx} = \tan \alpha \tag{474}$$

or

$$\frac{dy}{dx} = \frac{ws}{H}. \tag{475}$$

Since H is constant for a given span with given loading and at fixed temperature, and since the weight per foot of cable, w, is likewise constant under like conditions for any given material, one may write

$$\frac{H}{w} = c \tag{476}$$

where c does not vary with x and y. The constant c is evidently the length of cable whose weight is equal to the horizontal tension H.

Substituting Eq. (476) in Eq. (475),

$$\frac{dy}{dx} = \frac{s}{c}. \tag{477}$$

Since

$$(ds)^2 = (dx)^2 + (dy)^2$$

or

$$\frac{dy}{dx} = \sqrt{\left(\frac{ds}{dx}\right)^2 - 1} \tag{478}$$

by substituting Eq. (477) in Eq. (478), it follows that

$$\frac{s}{c} = \sqrt{\left(\frac{ds}{dx}\right)^2 - 1} \tag{479}$$

or

$$dx = \frac{cds}{\sqrt{s^2 + c^2}}. \tag{480}$$

By substituting in Eq. (480)

$$s = c \sinh u$$

and

$$ds = c \cosh u \cdot du$$

and integrating, it is found that

$$x = cu + k \tag{481}$$

k being the constant of integration.

Since c and k do not vary with x, the constant k is seen to be zero, whence

$$x = cu$$

$$= c \sinh^{-1} \frac{s}{c} \tag{482}$$

or

$$s = c \sinh \frac{x}{c} \tag{483}$$

and the length of the cable, in a half span, is the value of Eq. (483) when $x = l$

or

$$s = c \sinh \frac{l}{c} \tag{484}$$

From Eq. (475), by substituting $c = H \div w$

$$\frac{dy}{dx} = \frac{s}{c} \tag{485}$$

Substituting the value of s from Eq. (483) in Eq. (485), and transposing dx,

$$dy = c \sinh \frac{x}{c} \cdot \frac{dx}{c} \tag{486}$$

whence, by integration,

$$y = c \cosh \frac{x}{c} + k_2 \tag{487}$$

where k_2 is again the integration constant.

To determine the value of k_2, note that for $y = 0$, $x = 0$ and $\cosh \frac{x}{c} = 1$, whence $k_2 = -c$.

Finally, then,

$$y = c\left(\cosh\frac{x}{c} - 1\right). \tag{488}$$

The deflection or sag y has its maximum value when $x = l$, the half span length, so the maximum sag is

$$d = {}_{\max}.y = c\left(\cosh\frac{l}{c} - 1\right). \tag{489}$$

From Eq. (472), by substituting $c = \frac{H}{w}$ and solving for s,

$$s = \sqrt{\left(\frac{T}{w}\right)^2 - c^2}. \tag{490}$$

Substituting the value of s from Eq. (490) in Eq. (483), putting $x = l$ and solving,

$$\frac{T}{w} = c\cosh\frac{l}{c}. \tag{491}$$

Since the maximum tension occurs at the point of support where $x = l$, the value of $T_2 = {}_{\max}.T$, is

$$\frac{{}_{\max}.T}{w} = c\cosh\frac{l}{c}. \tag{492}$$

Summary of Equations.—The important equations, upon which the solution of problems in span design depend, are those defining the length of cables, the sag and the tension. For convenience they are summarized below:

$$s = c\sinh\frac{x}{c} \tag{493}$$

$$y = c\left(\cosh\frac{x}{c} - 1\right) \tag{494}$$

$$\frac{T}{w} = c\cosh\frac{x}{c}. \tag{495}$$

In the above equations,

x = projection upon a horizontal plane, of the distance between the point of maximum deflection and point P.

c = length of cable whose weight is equal to the horizontal tension H.

T = tension at point P
w = weight per unit length of cable
} expressed in same units.

x, s, y and c are all expressed in the same linear units, which is also the unit that is used with w.

Solution of Problems.—These last three equations show the concepts s, y and $T \div w$ to be hyperbolic functions of x and c, so that their magnitudes could be computed directly if both x and c were the given quantities in a problem of span design. Usually, however, c is not given since it is in reality a more or less fictitious concept, and the solution of span problems, with any two of the remaining concepts given, is accomplished by trial methods.

In order to avoid the loss of time incurred by such methods, Chart I has been devised from which, with any two of the five concepts given, the remaining three may be found at once. The chart is laid out on the basis of the decimal system to permit of easy interpolation. (Chart I is found in the pocket at the back of the book.) The quantities x and c, which form the hyperbolic argument, are the abscissas and ordinates, respectively, of the s, y and $T \div w$ curves.

Interpolation on Chart I.—It can be demonstrated that any straight line, passing through the origin (point O), is divided into intercepts of equal lengths by a set of hyperbolic curves, the indices of which vary in arithmetic progression. For instance, the curves indexed $y = 0.1$, $y = 0.2$, $y = 0.3$ . . . $y = 0.9$, $y = 1.0$ have indices which increase progressively by 0.1, and, in consequence, any straight line passing through the origin will be divided by these ten curves into ten equal intercepts. It follows, then, that, if each intercept of this straight line were again divided into ten equal lengths, the division points would be points on the curves $y = 0.01$, $y = 0.02$, $y = 0.03$ . . . $y = 0.99$, $y = 1.00$. The same reasoning holds true for interpolation between the curves $y = 1$, $y = 2$, $y = 3$, $y = 4$ . . . $y = 9$, $y = 10$ and of the last set $y = 10$, $y = 20$, $y = 30$, $y = 40$ . . . $y = 90$, $y = 100$. Similarly, this system of interpolation is correct for the s and the $T \div w$ curves.

In order to be able to accomplish quick and accurate decimal subdivision of any length of line, it is suggested to trace Fig. 68 on transparent cloth or paper for use on Chart I. Any one of the parallel lines in Fig. 68 is divided into ten, or a multiple of ten, units of equal length.

Example of Interpolation.—It is desired to interpolate for s, y and $T \div w$ a point P, the coordinates of which are $c = 550$, $x = 54$.

Through point P draw a straight line to the origin O. That this line PO is actually subdivided into equal sections by the

curves indexed in arithmetic progression is most clearly shown by the intercepts between the curves. The intercepts OA, AB, BC and CD, of the straight line formed by the curves $y = 1$, $y = 2$, $y = 3$ and $y = 4$, are exactly equal in length. The same holds true of the intercepts formed by the s and $T \div w$ curves.

By placing Fig. 68 on the chart so that the two outside radiating lines pass through points E and F and the parallel lines are at the same time parallel to line EF, point P is interpolated directly and corresponds to the magnitude $s = 54.1$.

Placing Fig. 68 so that the two outside radiating lines pass through points G and H and the parallel lines are parallel to GH, $T \div w$ is read directly and is found to be 553.

In a similar manner, y is read directly and found to be 2.65.

Chart I is independent of any fixed, conventional unit of length, weight or force, and may be used with equal precision for the

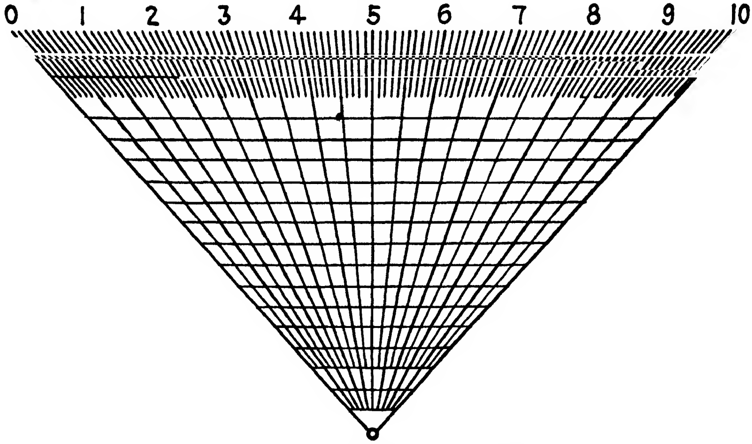


FIG. 68.—Interpolation diagram.

English foot-pound system or the decimal centimeter-gram system.

The use of the chart will now be illustrated by the solution of two representative examples.

Example 1.—A number of 40-ft. poles are to be used for supports of a line consisting of No. 00 hard-drawn, bare, copper wires. The points of support on the insulators are 30 ft. above the ground level when the poles are installed. The minimum clearance of the wire to the ground is to be 20 ft., and the tension on the wire at the point of support is not to exceed 200 lb.

What is the maximum permissible spacing of poles?

The weight per 1,000 ft. of wire is 402.8 lb.

Solution.—If the pound is chosen as the unit of force and the foot as the unit of length, $w = 402.8 \div 1{,}000 = 0.4028$, and $T \div w = 200 \div 0.4028 = 496.5$; $y = 30 - 20 = 10$.

By interpolation between curves $T \div w = 400$ and $T \div w = 500$, curve $T \div w = 496.5$ is drawn. The intersection point of this curve and the curve $y = 10$, has the abscissa $x = 98.3$, which is half the pole spacing in feet.

Hence the maximum permissible pole spacing is $2 \times 98.3 = 196.6$ ft.

Example 2.—A $1\frac{1}{2}$-in. steel cable is to span a river. The foundations for the anchor towers on both shores are 10 ft. above the water level and a distance of 2,000 ft. apart. The maximum tension on the cable is not to exceed 70,000 lb. Weight of cable is 4,700 lb. per 1,000 ft. Clearance between cable and water surface is not to be less than 50 ft.

What is the minimum height of anchor towers?

What is the length of cable between points of support?

What are the vertical and horizontal components of the tension on the points of support in the plane of the suspended cable?

Solution.—Half the tower spacing is 1,000 ft. In order to bring this value within range of Chart I, the unit of length will be chosen as 20 ft. Hence $x = 1{,}000 \div 20 = 50$. For the same unit of length, $w = 4{,}700 \times 20 \div 1{,}000 = 94$. Therefore $T \div w = 70{,}000 \div 94 = 745$. This value is within range of the chart. If this value had exceeded the range of the chart, a unit of length greater than 20 ft. would have had to be chosen.

Between the curves $T \div w = 700$ and $T \div w = 800$, a short length of curve, $T \div w = 745$, is interpolated near the line $x = 50$. The point of intersection of this curve with the line $x = 50$ has the ordinate $c = 742$. Now, a straight line is drawn from the origin through point $x = 50$, $c = 742$, and this point interpolated on the straight line for y and for s.

Interpolation between curves $y = 1$ and $y = 2$ yields $y = 1.68$. Interpolation between $s = 50$ and $s = 60$ yields $s = 50.1$. Hence,

Maximum deflection $= y = 1.68$ units $= 33.6$ ft.

Length of cable $= 2s = 2 \times 50.1$ units $= 2{,}004$ ft.

Minimum height of anchor tower $= 33.6 + 50 - 10 = 73.6$ ft.

Vertical component of tension on point of support $= 2{,}004 \div 2 \times 4.7 = 4{,}720$ lb.

Horizontal component of tension on point of support in plane of cable $= cw = 742 \times 20 \times 4.7 = 69{,}748$ lb.

Approximate Equations for Sag-tension Calculations.—For short-span calculations where the sag is only a small percentage of the span, the error, made by assuming the curve of the suspended cable to be parabolic rather than to be that of a catenary, is negligible. For spans of ordinary sags, the approximate equations applying to the parabola are probably sufficiently accurate up to spans of perhaps 800 ft. in length. For longer spans the error increases quite rapidly. It will therefore be desirable to give the approximate equations which are ordinarily used for short-span calculations, and to show wherein the approximations lie.

The three principal equations for the catenary, representing exactly the conditions prevailing in the suspended cable, have already been derived. They are given by Eqs. (484), (488) and (495).

Since the total vertical stress is that due to the weight of the cable in a half span, the following relations also hold:

The vertical stress is

$$V = ws = wc \sinh \frac{l}{c} \tag{496}$$

and the horizontal stress is

$$\begin{aligned} H &= \sqrt{T_m^2 - V^2} \\ &= wc\sqrt{\cosh^2 \frac{l}{c} - \sinh^2 \frac{l}{c}} \\ &= wc. \end{aligned} \tag{497}$$

Remembering that

$$\sinh \frac{l}{c} = \frac{l}{c} + \frac{l^3}{6c^3} + \frac{l^5}{120c^5} + \frac{l^7}{5{,}040c^7} + \cdots \tag{498}$$

$$\cosh \frac{l}{c} = 1 + \frac{l^2}{2c^2} + \frac{l^4}{24c^4} + \frac{l^6}{720c^6} + \cdots \tag{499}$$

and that

$$c = H \div w \tag{500}$$

the approximate relations given below are readily found.

From Eq. (483), using the first two terms of the series only,
Length of cable in a half span is

$$S = l + \frac{w^2 l^3}{6H^2} \text{ approximately.} \tag{501}$$

From Eq. (489), using only the first two terms of the series,

$$\text{Maximum sag, } d = \frac{wl^2}{2H} \text{ approximately.} \tag{502}$$

Solving Eq. (502) for H yields

$$\text{Horizontal tension, } H = \frac{wl^2}{2d} \text{ approximately.} \tag{503}$$

Substituting the value of H from Eq. (503) in Eq. (501) yields the value of S in terms of deflection.
Length of cable in a half span is

$$S = l + \frac{2d^2}{3l}, \text{ approximately.} \tag{504}$$

Equation (504), solved for d, yields the sag. It is

$$\text{Maximum sag, } d = \sqrt{\frac{3}{2}l(S - l)}, \text{ approximately.} \tag{505}$$

The vertical stress, equal to the weight of cable in a half span, is found by multiplying Eq. (504) by w; whence

$$\text{Vertical stress, } V = \frac{w}{3l}(3l^2 + 2d^2), \text{ approximately.} \tag{506}$$

The maximum stress is the vector sum of the vertical and horizontal stresses; *i.e.*,

$$T_m = \sqrt{H^2 + V^2}. \tag{507}$$

Squaring Eq. (503),

$$H^2 = \frac{w^2l^4}{4d^2}. \tag{508}$$

Squaring Eq. (506),

$$V^2 = \frac{w^2}{9l^2}(9l^4 + 12l^2d^2 + 4d^4). \tag{509}$$

Since d is small as compared with l, and l^2 is negligible as compared with l^4, the last two terms of Eq. (509) may be neglected; whence

$$\begin{aligned} T_m^2 &= \frac{w^2l^4}{4d^2} + w^2l^2 \\ &= \frac{w^2l^2}{4d^2}(l^2 + 4d^2) \end{aligned}$$

and

$$T_m = \frac{wl}{2d}\sqrt{l^2 + 4d^2}, \text{ approximately.} \tag{510}$$

The most useful approximate equations may thus be summarized:

$$\text{Length of cable in half span, } S = l + \frac{2d^2}{3l} \tag{511}$$

$$\left.\begin{aligned} \text{Sag at any point, } y &= \frac{wx^2}{2H} \\ \text{Maximum sag, } d &= \frac{wl^2}{2H} \end{aligned}\right\} \tag{512}$$

$$= \frac{wl^2}{2T_m}, \text{ approximately.}$$

Since for small sags T and H are approximately equal,

$$\text{Maximum sag, } d = \sqrt{\frac{3}{2}l(S - l)} \tag{513}$$

$$\text{Horizontal stress, } H = \frac{wl^2}{2d}. \tag{514}$$

$$\text{Vertical stress, } V = \frac{w}{3l}(3l^2 + 2d^2) \tag{515}$$

$$\text{Maximum tension, } T_m = \frac{wl}{2d}\sqrt{l^2 + 4d^2}. \tag{516}$$

A number of semigraphical methods,[1] for calculating approximately the quantities desired in span design and based upon the above equations, have been devised. None of these will be given here, although the equations have been derived to show wherein the approximations lie.

Average Tension in Cable between Points of Support.—From Eq. (495) the tension, at any point P, is given by the relation

$$\frac{T}{w} = c \cosh \frac{x}{c}. \tag{517}$$

By the well-known rule for obtaining the average value of a variable, the average value of the ratio $T \div w$ is

$$\text{av.}\ \frac{T}{w} = \frac{\int_0^s c \cosh \frac{x}{c} \cdot ds}{\int_0^s ds}$$

and since

$$ds = c \cosh \frac{x}{c} \cdot d\left(\frac{x}{c}\right)$$

$$\text{av.}\ \frac{T}{w} = \frac{\int_0^{\frac{x}{c}} c^2 \cosh^2 \frac{x}{c}\, d\left(\frac{x}{c}\right)}{c \sinh \frac{x}{c}}$$

$$= \frac{c}{2 \sinh \frac{x}{c}} \left[\sinh \frac{2x}{c} + \frac{x}{c}\right]$$

$$= \frac{c}{2}\left[\cosh \frac{x}{c} + \frac{\frac{x}{c}}{\sinh \frac{x}{c}}\right]. \tag{518}$$

Equation (518) will be found useful when considering the influence of changes in temperature and loading upon the tension and sag in a cable.

The Problem of Span Design.—The problem presented for solution in span design may be stated somewhat as follows: The cable in the span must be so strung that during times of severest load conditions, which occur at minimum temperature and under the assumed worst conditions of ice and wind loads, the tension in the cable will not exceed the maximum, allowable value.

[1] IMLAY, L. E., "Mechanical Characteristics of Transmission Lines. Span Formulae and General Methods of Calculation," *Elec. Jour.*, pp. 53 to 57, February, 1925.

This value is usually assumed at from 75 to 100 per cent of the elastic limit for the cable in question. (Further information on this point may be found in the handbooks.)

Since, however, the cable is presumably to be strung on the supports during fair weather, and certainly without being subjected to conditions of maximum wind and ice loads, it is necessary to predetermine the effects of both changes in temperature and changes in loading upon the tension in the cable and upon its sag. An increase in temperature of the cable causes it to elongate, thus increasing its length, reducing the tension and increasing the sag. An increase in loading causes an increase in tension, an increase in length, and an increase in sag. The designer must be able to pass from one condition of load and temperature to any other condition of load and temperature, and to determine what the tension, length and sag of the cable will be under any set of circumstances. More particularly, his problem is to provide the construction foreman with charts from which he may proceed to draw up the cable to predetermined tensions or sags (depending upon which is used as the controlling factor) at known temperatures. These tensions must be such that, under the worst load conditions assumed, the added tension due to the increased load will not bring the total tension in the cable to a value in excess of the maximum tension originally allowed for the cable. Likewise, the sag under maximum load and maximum temperature under which such load may exist (32° F.), must be known so that sufficient clearance to ground may be provided for.

Classes of Loading.—The maximum loadings which are assumed in the calculation of sags generally vary with the severity of climatic conditions, and, to some extent, with the judgment of the designing engineer. Standardization of assumed loadings has been proposed, and certain types of loadings have been suggested. These, however, are intended and are used as guides rather than as standards to which all designs should conform. Assumptions as to loadings for a given locality should be based upon a careful analysis of local weather conditions.

In the suggested standards for loading given below, the wind pressures, per foot of cable, are computed on the basis of the projected area of a foot length of cable. This is numerically equal to the conductor diameter in feet when there is no ice load, or to the conductor diameter plus twice the given thickness of ice, when an ice load is present. In all cases, it is assumed that

the wind load acts in a horizontal plane and is normal to the direction of the span.

Three types of loadings have been suggested by the Joint Committee on Overhead Line Construction of the National Electric Light Association, as follows:

Class A Loading.—A wind pressure of 15 lb. per square foot of projected area of bare conductor plus the dead weight of the conductor.

Class B Loading.—A wind pressure of 8 lb. per square foot of projected area of conductor covered all around with ice ½ in. thick, plus the dead weight of cable and ice load.

Class C Loading.—A wind pressure of 11 lb. per square foot of projected area of conductor covered all around with ice ¾ in. thick, plus the dead weight of cable and ice load.

Another group of three loadings has been suggested by the National Electrical Safety Code. These are offered as appropriate loadings for the corresponding territorial belts, as indicated on the outline map of Fig. 69.

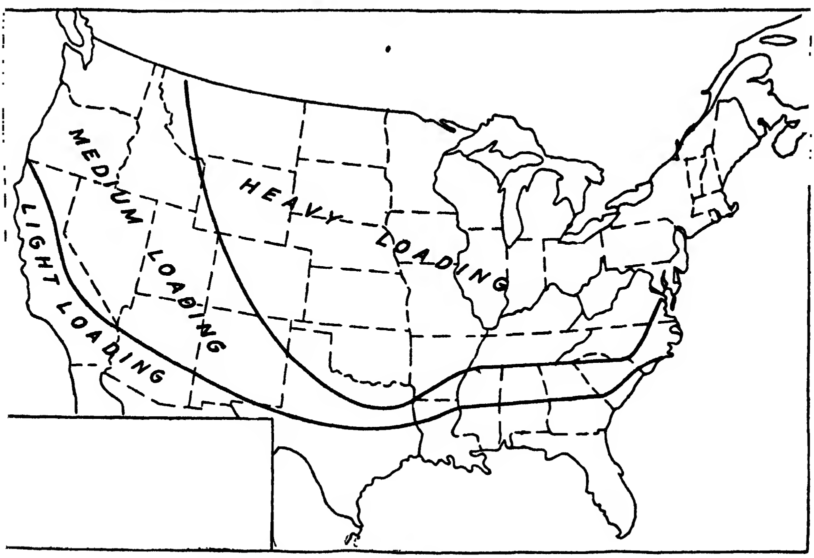


FIG. 69.—Transmission line loading map.

They are:

Heavy Loading.—A wind pressure of 8 lb. per square foot of projected area of conductor covered all around with ice $\frac{1}{2}$ in. thick, plus the dead weight of the cable and ice load. Assumed minimum temperature 0° F.

Medium Loading.—A wind pressure of 8 lbs. per square foot of projected area of conductor covered all around with ice $\frac{1}{4}$ in. thick, plus the dead weight of the cable.

Light Loading.—A wind pressure of 12 lbs. per square foot of projected area of bare conductor, plus the dead weight of the conductor only. The ice loading is assumed to be zero.

Wind Velocities and Pressures.—Transmission-line spans are designed to withstand considerable wind loads, in addition to the weight of the cable itself, and such sleet or snow loads as it may reasonably be expected to accumulate. It is entirely out of the question, however, to design such structures to withstand winds of the highest recorded velocities. Even if such designs were possible, it would probably be uneconomical to employ them. Aside from these exceptional storms, the worst condition prevails when the cables are covered with ice, a high wind is blowing, and the temperature is low. Lowest temperatures, highest wind velocities and snow loads do not occur simultaneously, however. Snow or sleet loads usually accumulate during calm weather. If later the temperature should fall and wind velocities should increase, the load on the structure would reach its greatest value. Very high winds would probably remove much of the snow or sleet load, and thus keep the total load from increasing further.

The wind loads which the designer assumes should take account of the above considerations. Data on wind velocities for any

TABLE 11.—WIND VELOCITIES AND PRESSURES

Velocity, miles per hour		Pressure in pounds per square foot	
Indicated velocity	Actual velocity $= V$	Projected area of cylinder $P = 0.0025V^2$	Flat surface $P = 0.004V^2$
10	9.6	0.23	0.4
20	17.8	0.8	1.3
30	25.7	1.7	2.7
40	33.3	2.8	4.5
50	40.8	4.2	6.7
60	48.0	5.8	9.2
70	55.2	7.6	12.2
80	62.2	9.7	15.5
90	69.2	12.0	19.2
100	76.2	14.5	23.2
110	83.2	17.3	27.7
120	90.2	20.3	32.6

given locality may be obtained from the United States Weather Bureau. From such data, covering a period of years, the designer may judge what velocity it is safe to assume.

The wind velocities, as furnished by the government, are indicated velocities. The true velocities are related to these as shown in Table 11. Here are also shown the corresponding pressures per square foot of projected area on round conductors, and the pressures per square foot of flat surfaces. For flat surfaces the pressure is

$$P = 0.004V^2 \text{ lb. per square foot} \tag{519}$$

where V is the velocity in miles per hour. For smooth cylindrical surfaces the pressure, per square foot of projected area, is one-half the above value, that is, the constant of Eq. (519) is 0.002. For stranded cable H. W. Buck found the constant to be 0.0025. Hence, for stranded cable,

$$P = 0.0025V^2 \text{ lb. per square foot.} \tag{520}$$

Influence of Changes in Temperature.—Within a certain range of temperatures, a given length of cable will change its three dimemsions practically in direct proportion to the amount of temperature change. This range more than covers the extreme range of weather conditions to which transmission spans are subjected. Accordingly, the length of a conductor may be expressed as a function of the temperature by the well known formula

$$s_t = s_0[1 + \alpha(t - t_0)] \tag{521}$$

where

s_0 = length of cable at any given initial temperature.
s_t = length of cable at the final temperature.
t_0 = the initial temperature.
t = the final temperature.
α = the coefficient of expansion for the particular material in question.

While, as intimated above, not only the length of the cable changes with changes in temperature, but also its cross-sectional area and its weight per foot, yet the total changes in the latter two quantities, over the range of temperatures experienced, is so small, that, for practical purposes, they may be neglected.[1] In the following discussion, therefore, it will be assumed that the weight per foot and the cross-sectional area are constants, and that the length only changes with changes in temperature.

[1] For a discussion of theory, in which these variables are taken into account, see F. K. Kirsten, University of Washington Engineering Experiment Station. *Bull.* 17.

Influence of Changes in Tension.—If the tension at any point in the cable were proportional to the product of s and w, a change in temperature would not effect the tension, but, according to Eq. (491),

$$T = wc \cosh \frac{x}{c}$$

whereas, by Eq. (483)

$$s = c \sinh \frac{x}{c}.$$

Hence, a change in length, caused by either a change in temperature or a change in loading, is accompanied by a change in tension at every point in the catenary.

The condition

$$T = wc \cosh \frac{x}{c}$$

is always realized during each minute step in this change, however, the constants in the equation being x and w.

For all stresses below the elastic limit of the cable in question, the strain in the cable is proportional to the stress. This relation is expressed by Hook's Law as follows:

$$s_t = s_0\left(1 + \frac{T - T_0}{EA}\right) \tag{522}$$

where

s_0 = length of cable under initial tension
s_t = length of cable under final tension
T_0 = initial tension
T = final tension
A = cross-sectional area of cable (assumed constant)
E = modulus of elasticity for the given cable.

Since the strain is proportional to the stress, the total change in the length of the suspended cable is proportional to the change in average tension along the cable; or

$$s_t = s_0\left[1 - \frac{{}_{\text{av}}T_0}{EA} + \frac{{}_{\text{av}}T}{EA}\right] \tag{523}$$

where

${}_{\text{av.}}T_0$ = the average initial tension
${}_{\text{av.}}T$ = the average final tension.

Influence of Ice and Wind Loading.—In order to provide for the safety of a span, due allowance must be made for the possibility of the accumulation of additional load on the cable, due to

the formation of sleet and snow on the conductor. Furthermore, since wind pressure is likely to increase the load on the cable at any time, an allowance for wind pressure should also be made. The worst condition of loading is evidently that brought about by the simultaneous action of the increased sleet load, together with a wind load acting at right angles to the direction of the span and effective on the increased cable diameter due to the accumulated sleet.

The result of the superposition of ice and wind load upon the weight of the cable itself is to greatly increase the tension and sag in the cable, and to cause the plane of the cable to be deflected from the vertical in the direction of the wind by an angle θ, such that

$$\tan \theta = \frac{\text{wind pressure, pounds per foot of cable}}{\text{(dead weight + weight of ice load) lbs. per foot of cable}}$$

$$= \frac{p}{w + i}.$$

where

i = weight of ice load per unit length of cable
p = wind pressure exerted normal to the direction of span on unit length of cable with ice load
expressed in the same unit.

The sag in the vertical plane then becomes equal to the total deflection in the inclined plane times cos θ, while the deflection horizontally is the total deflection in the inclined plane times sin θ. Unless otherwise stated, where sags are mentioned, the total deflections in the plane of the resultant force are given.

The resultant force, per unit length of cable and under conditions of combined wind and ice load, acts downwards in the plane of the deflected cable, and has the value

$$w_1 = [(w + i)^2 + p^2]^{\frac{1}{2}}. \tag{525}$$

All of the factors which affect the suspended cable under varying conditions of loading and temperature have now been considered, together with the theory required to translate these influences into changes of length, tension and sag in the curve of the suspended cable. It remains only to summarize the results in concise mathematical form for convenient use in the solution of such problems as ordinarily arise in span design.

Let it be therefore required to find all possible catenaries that may be described by a cable under any possible condition of

loading and temperature. The characteristics of the cable are made available from the following data, assumed to be furnished:

$$\left.\begin{aligned} w &= \text{weight per unit length of cable} \\ A &= \text{cross-sectional area of the cable} \end{aligned}\right\} \text{assumed constant at all temperatures.}$$

E = modulus of elasticity.
α = coefficient of linear expansion.
T_m = maximum allowable tension for the cable.

From the records of the local weather bureau, maximum and minimum temperatures and wind velocity to be assumed in making calculations, may be estimated. These are

t_1 = minimum assumed temperature at which sleet and wind loads exist simultaneously.
t_2 = maximum assumed temperature of conductor.
v = maximum assumed wind pressure, per unit of area normal to the direction of wind.

Catenary Covering Conditions at Minimum Temperature. Cable under Ice Load and Wind Pressure.—Subscripts 1 refer to the above described conditions at minimum temperature t_1. Under the conditions described, the cable is assumed to be stressed to its maximum allowable tension at the points of support. By Eq. (491), the maximum tension is

$$_{\text{max.}}T_1 = w_1 c_1 \cosh \frac{x}{c_1}. \tag{526}$$

As already pointed out, the amount of the allowable maximum tension is known from available data, and, since w_1 may be calculated for any assumed ice and wind loads, Eq. (526), may be solved for any assumed values of $\frac{x}{c_1}$.

If the ice covering on the cable be assumed to be a units in uniform thickness, then the volume of ice per unit length of cable is

$$\text{Volume of ice per unit length} = \pi a(d_s + a) \tag{527}$$

where d_s is the diameter of stranded cable used.

The weight of ice per unit length of cable is

$$i = u\pi a(d_s + a) \tag{528}$$

where u is the weight per unit volume of ice.

The force, exerted by the wind on unit length of cable with ice envelope, is

$$p = v(d_s + 2a). \tag{529}$$

The result, given in Eq. (529), is based on the assumption that the wind pressure on a unit length of the conductor is equal to that which would be experienced by its projected area on a plane normal to the direction of the wind.

Assuming that the wind pressure acts in a horizontal plane and in a direction normal to the direction of the span, the total resultant weight, per unit length of conductor in the inclined plane and by Eq. (525), is found to be

$$w_1 = \{[w + u\pi a(d_s + a)]^2 + v^2(d_s + 2a)^2\}^{\frac{1}{2}}. \tag{530}$$

By substituting the value of w_1 as calculated from Eq. (530), in Eq. (526), the values of ${}_{\text{max.}}T_1 \div w_1$, corresponding to any number of assumed values of $\frac{x}{c_1}$, may be calculated.

By Eq. (526),

$$c_1 = \frac{{}_{\text{max.}}T_1}{w_1} \div \cosh \frac{x}{c_1},$$

and, since for any assumed values of $\frac{x}{c_1}$ the quantity $\cosh \frac{x}{c_1}$ may be found from tables, the value of c_1 itself is readily computed. The corresponding values of length of cable are obtained from Eq. (483), by which

$$s_1 = c_1 \sinh \frac{x}{c_1} \tag{531}$$

while, from Eq. (488), the sag is

$$y_1 = c_1\left(\cosh \frac{x}{c_1} - 1\right) \tag{532}$$

which may readily be transformed to

$$y_1 = \frac{{}_{\text{max.}}T_1}{w_1} - c_1. \tag{533}$$

Equations (526), (531), and (532) or (533) completely determine the catenary under the conditions of minimum temperature and maximum loading and serve as the starting point in span design. It will presently be shown how the equations of the catenaries, for any other possible condition of loading and for

any other temperature, are derived, using these equations, together with Eqs. (521) and (522) as the basic relations.

Catenary Covering Any Condition of Loading and Any Temperature.—If the temperature of the conductor could increase without at the same time altering the tension in it, the elongation of the conductor would be proportional to the temperature change. Thus, assuming that the temperature increases from the minimum value t_1 to any other value t_n, but that the tension does not change, the new length of conductor becomes

$$s_1' = s_1[1 + \alpha(t_n - t_1)]. \tag{534}$$

Owing to the change in temperature, however, a change in tension takes place throughout the conductor, beginning with the initial value ${}_{av}T_1$ at minimum temperature, and ending with some other value. Furthermore, if the change in temperature is great enough, that is, if the final temperature lies above the freezing temperature, the ice load will drop off, and a further change in tension will take place due to the change in loading. The load may also change due to the dying down of the wind even at temperatures below freezing. But, in any case, if the final loading per unit length of conductor is w_n and the average tension corresponding to the new conditions of loading and temperature is ${}_{av}T_n$, the initial and final conditions must separately satisfy Eq. (518). Therefore, initially

$${}_{av}T_1 = \frac{c_1 w_1}{2}\left(\cosh\frac{x}{c_1} + \frac{\frac{x}{c_1}}{\sinh\frac{x}{c_1}}\right) \tag{535}$$

and, finally,

$${}_{av}T_n = \frac{c_n w_n}{2}\left(\cosh\frac{x}{c_n} + \frac{\frac{x}{c_n}}{\sinh\frac{x}{c_n}}\right) \tag{536}$$

Due to the change in tension which has been induced either by change in loading or change in temperature, or both, a further change in length of the cable takes place, as determined by Eq. (523). That is, the length s_n, under final conditions of temperature and loading, is

$$s_n = s_1'\left(1 - \frac{{}_{av}T_1}{EA} + \frac{{}_{av}T_n}{EA}\right) \tag{537}$$

or, since, from Eq. (483)

$$s_n = c_n \sinh \frac{x}{c_n}, \tag{538}$$

$$c_n \sinh \frac{x}{c_n} = s_1'\left(1 - \frac{{}_{\text{av}}T_1}{EA} + \frac{{}_{\text{av.}}T_n}{EA}\right). \tag{539}$$

The substitution of Eqs. (534), (535), and (536) in Eq. (539) yields the equation from which the catenary under the final conditions of temperature and loading may be calculated.
Making these substitutions yields

$$c_n \sinh \frac{x}{c_n} = s_1[1 + \alpha(t_n - t_1)]\left\{1 - \frac{c_1 w_1}{2EA}\left(\cosh \frac{x}{c_1} + \frac{\frac{x}{c_1}}{\sinh \frac{x}{c_1}}\right)\right.$$
$$\left. + \frac{c_n w_n}{2EA}\left(\cosh \frac{x}{c_n} + \frac{\frac{x}{c_n}}{\sinh \frac{x}{c_n}}\right)\right\}. \tag{540}$$

Dividing Eq. (540) through by x yields

$$\frac{\sinh \frac{x}{c_n}}{\frac{x}{c_n}} = \frac{s_1}{x}\left[1 + \alpha(t_n - t_1)\right] - \frac{w_1 s_1}{2EA}\left[1 + \alpha(t_n - t_1)\right]$$
$$\times \left[\frac{\cosh \frac{x}{c_1}}{\frac{x}{c_1}} + \frac{1}{\sinh \frac{x}{c_1}}\right] + \frac{w_n s_1}{2EA}\left[1 + \alpha(t_n - t_1)\right]\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right] \tag{541}$$

$$= F_n - G_n + H_n\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c}}\right] \tag{542}$$

where, to simplify the notation, the following abbreviations are made:

$$D_n = 1 + \alpha(t_n - t_1) \tag{543}$$

$$N = \frac{s_1}{2EA} \tag{544}$$

$$P_1 = \left[\frac{\cosh \frac{x}{c_1}}{\frac{x}{c_1}} + \frac{1}{\sinh \frac{x}{c_1}}\right]. \tag{545}$$

Thus, in Eq. (542),

$$F_n = \frac{s_1 D_n}{x} \tag{546}$$

$$G_n = w_1 P_1 N D_n \tag{547}$$

$$H_n = w_n N D_n. \tag{548}$$

Recapitulation.—In the design of a span, the engineer must guarantee (*a*) that the tension in the suspended cable will not exceed a predetermined safe value when subjected to the maximum loading and minimum temperature conditions; and (*b*) that the clearance of conductor to ground will never be less than a given predetermined amount.

The first requirement is met by the use of Eq. (526), in which the maximum tension ${}_{\max}T_1$ is the maximum safe stress on the cable, and w_1 is the total resultant unit loading on the cable, including weight of cable, weight of ice load and wind load normal to these. The second requirement must be satisfied from an investigation of the vertical sag of the cable under the extreme conditions. A little reflection will disclose what these extreme conditions are.

Minimum and Maximum Sags.—It is evident that the minimum vertical sag occurs either when the temperature is minimum and the cable is free from wind and ice loads, or when, at minimum temperature and maximum loading, wind pressure deflects the cable from a vertical plane. If, at minimum temperature, the cable receives its maximum assumed ice load and, at the same time, is subjected to its assumed wind load, the tension in the cable is increased and the cable is stretched due to the increased tension. Since the distance between supports remains fixed, the stretch in the cable gives rise to an increased sag. If the loading now remains unchanged but the temperature increases, the cable is further elongated. But as the temperature rises and the cable elongates, the tension is diminished and the cable tends to shorten due to the lessened tension. For all materials now used as transmission-line conductors, however, the increase in length, due to any increment of temperature increase, is greater than the corresponding decrease in length which accompanies the associated decrement of tension. These two conflicting forces thus actually result in an elongation of the cable. If the temperature rise should continue indefinitely and the total load on the cable did not change, a condition of maximum sag would be

found at the highest temperature at which the ice load could exist, that is, at 32° F. Thus, the sag of the cable with maximum load and freezing temperature must be found, since it may prove to be a critical condition. This sag is found from the equation

$$y_m = c_n\left(\cosh\frac{x}{c_n} - 1\right)$$

after $\frac{x}{c_n}$ has been evaluated from Eq. (489) for $t_n = 32°$ F. and $w_n = w_1$.

The sag found in this way will be measured in the plane of the resultant force, unless, as is sometimes done, it be assumed that, at freezing temperature, the wind has died down to a negligible value, but that the snow and ice load may have increased to an amount such that the total load per unit length of cable remained unchanged. The sag would then be in a vertical plane. In any case, if y_m is the sag in the plane of the resultant force, the vertical component of sag is given by the equation

$$_{\text{max.}}y_v = {}_{\text{max.}}y \cos\theta$$

where

$$_{\text{max.}}y_v = \text{the vertical component of sag}$$

and

$$\theta = \text{the angle which the plane of the resultant deflection makes with a vertical plane.}$$

As soon as the temperature rises above 32° F., the ice load and a large part of the wind load vanish, resulting at once in decreased deflection. If the temperature should now increase to its maximum, the deflection will again increase to a new maximum, which may be either smaller or larger than the deflection at freezing point and maximum load, depending upon the relative magnitudes of ice and wind loading existing at the freezing point, and upon the maximum temperature. Solving Eq. (541) for the value of $\frac{x}{c_n}$ corresponding to $t_n = {}_{\text{max.}}t$ and $w_n = w$ (assuming no wind load to exist at the maximum temperature), the value of the sag at maximum temperature may be found from the equation for sag, namely,

$$y_n = c_n\left(\cosh\frac{x}{c_n} - 1\right).$$

Critical Catenary.—A comparison of the deflection at maximum temperature, with the deflection at freezing temperature and

maximum load, will determine which sag is the greatest. The catenary having the greatest deflection is called the *critical catenary.* In warm climates the critical catenary will tend to exist at the maximum temperature, whereas in cold climates it is likely to be found at the freezing point. The diameter of the cable used is also a factor in the determination of the critical catenary, as are also the wind and ice loadings assumed. The smaller the cable diameter the greater is the ratio of the combined unit length wind and ice load to the weight of a unit length of the cable itself. Accordingly, in a given locality the critical catenary may occur at the maximum temperature for a large cable, and at the freezing point for a smaller cable. It is therefore desirable to calculate the deflections for both conditions in order to be certain which one it is that controls.

Method of Solving Eq. (541) for $\frac{x}{c_n}$.—While Eq. (541) appears rather cumbersome and difficult to manipulate, yet when viewed in the light of its more simplified form, as given in Eq. (542), and not losing sight of the fact that the problem to be solved is a very complex one, the relative simplicity of the solution will become apparent

The solution of this equation is easily and rapidly obtained by the use of the curves of Chart II, which is found in the pocket on the inside of the back cover. The curve indexed

$$\left[\sinh\frac{x}{c} \div \frac{x}{c}\right] - 1$$

gives values of the magnitude

$$\left[\sinh\frac{x}{c} \div \frac{x}{c}\right] - 1$$

for arguments of $x \div c$ ranging from 0 to 0.25. The curve indexed

$$0.0001 \times \left[\frac{\cosh\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh\frac{x}{c}}\right]$$

represents magnitudes of

$$H \times \left[\frac{\cosh\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh\frac{x}{c}}\right]$$

for the same range of the arguments, and $H = 0.0001$. Corresponding curves, for values of H ranging from 0 to 0.0002, are indexed 0.00001, 0.00002, . . . 0.00019, 0.00020. A straight line, normal to the axis of $x \div c$ is divided into equal lengths by these curves indexed in arithmetic progression. Consequently, direct interpolation may be effected along such lines perpendicular to the axis of $x \div c$.

For all positive arguments, the ratio of the hyperbolic sine to the argument is always greater than unity. By subtracting unity from this ratio, the two sets of curves of Chart II are located nearer the axis of $x \div c$. The equation is again balanced by subtracting unity from the constants. Hence, for convenience. Eq. (542) is changed in form to

$$\frac{\sinh \frac{x}{c_n}}{\frac{x}{c_n}} - 1 = F_n - G_n + H_n\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right] - 1.$$

Supposing $F_n = 1.002821$, $G_n = 0.000525$ and $H_n = 0.0000315$, then

$$F_n - G_n - 1 = 0.002296.$$

The magnitude

$$H_n\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right] = 0.0000315\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right]$$

is given by a curve interpolated between curves indexed 0.00003 and 0.00004 for any argument $x \div c_n$ between the limits 0 and 0.25. The value 0.002296 added to an ordinate of this curve must, according to the above equation, be an ordinate of curve

$$\left[\sinh \frac{x}{c_n} \div \frac{x}{c_n}\right] - 1.$$

Hence, the correct argument $x \div c_n$, which gives the same length of ordinate for the curves

$$\left[\sinh \frac{x}{c_n} \div \frac{x}{c_n}\right] - 1$$

and

$$0.002296 + 0.0000315\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right]$$

can be found by marking the length 0.002296 on the straight edge of a sheet of paper and by moving the point marked on the sheet along the curve 0.0000315, keeping the straight edge perpendicular to the $x \div c_n$ axis, until the other point touches the curve

$$\left[\sinh \frac{x}{c_n} \div \frac{x}{c}\right] - 1.$$

The straight edge in this position indicates, on the $x \div c$ axis, the value 0.1294, which, if substituted for $x \div c_n$ in the above equation, will satisfy this equation.

The curves of Chart II have been drawn with great care from hyperbolic functions computed accurately to the required number of decimal places. With a little practice in interpolation, the argument of Eq. (542) can be found accurately to the fourth decimal place. Values of

$$0.0001\left[\frac{\cosh \frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh \frac{x}{c}}\right]$$

with a given argument, can be read directly to the sixth decimal place from the curve bearing this index. For most practical design problems, the degree of accuracy possible with the use of the chart is more than ample.

If greater accuracy is required than can be obtained by direct reading of Chart II, this chart may be magnified to any desired degree by the use of the tables (Appendix C) of hyperbolic functions of $\frac{x}{c}$, calculated accurately to the tenth decimal place. The functions

$$\frac{\sinh \frac{x}{c}}{\frac{x}{c}} - 1$$

and

$$\frac{\cosh \frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh \frac{x}{c}}$$

are tabulated so that small portions of a chart similar to Chart II may be quickly drawn with ordinates of any desired magnitude, and used for the solution of Eq. (542), with much greater precision than can be obtained by the use of Chart II only. Chart II should be used to determine the approximate value of $\frac{x}{c}$ to the third decimal place. From the tables two or three values, greater and smaller than $x \div c$, should be plotted for the functions

$$\frac{\sinh \frac{x}{c}}{\frac{x}{c}} - 1 \text{ and } H\left[\frac{\cosh \frac{x}{c}}{\frac{x}{c}} + \frac{1}{\sinh \frac{x}{c}}\right] \pm (F - G - 1)$$

upon coordinate paper with ordinates of a magnitude which will yield the desired precision. Since greatly magnified, short portions of these curves appear to be almost straight lines, the intersection point of the curves may be very accurately located.

Temperature-tension Stringing Charts.—The construction engineer should have a means of quickly finding the required stringing tension for any temperature and for the loading corresponding to the weight of conductor alone. The stringing tension, as already explained, should be such that under minimum temperature and maximum loading the cable will not be stressed beyond its maximum safe allowable value. Information as to the proper stringing tension should be available for all lengths of spans which may be required in the construction, and for all temperatures likely to be encountered while the construction work is in progress.

In order to supply this necessary information, the curves which the conductor takes at each of a number of different temperatures should be computed for each of a number of values of the argument $x \div c$. The range in the argument should be great enough to embrace all of the span lengths required. Data for curves of stringing tensions T vs. $x \div c$, and corresponding sags y vs. $x \div c$, are computed for a limited number of different temperatures. The range covered by the extremes in temperature should, of course, include all temperatures likely to be encountered during construction. Curves are then drawn for the constant temperatures chosen. Tensions and sags for other temperatures may be found from the curves by interpolation. The charts, consisting of tension and sag curves vs. span lengths,

similar to Fig. 70, are referred to as the *temperature-tension stringing charts.*

The use of the theory developed in this chapter will best be understood by applying it to the solution of a definite, illustrative problem. Such a problem is stated in the following example.

Example.—A 500,000-cir. mil copper cable is to be suspended from supports at equal elevations in spans of lengths lying between 200 and 1,200 ft. (Loading and other constants are as given in the tabulation of specifications and constants.) It is required to compute the data for and to furnish a temperature-tension stringing chart to include the following curves: (*a*) cable under maximum load at 32° F.; (*b*) cable without ice or wind loads at 10°, 30°, 60° and 100° F.

Solution.—In solving the problem it will be convenient first to tabulate, in systematic order, all of the required data pertaining to it, as illustrated in Table 12, entitled Specifications and Constants, for the 500,000-cir. mil copper cable.

TABLE 12.—SPECIFICATIONS AND CONSTANTS

500,000-Cir. Mil, Stranded, Copper Cable

1	Minimum temperature	$t_m =$	−10° F.
2	Freezing temperature	$t_s =$	+32° F.
3	Maximum temperature	$t_n =$	+100° F.
4	Ice loading (thickness all around cable)	a	0.5 in.
5	Wind pressure normal to span (pounds per square foot)	v	10
6	Weight of ice (pounds per cubic inch)	u	0.0332
7	Modulus of elasticity	E	16,000,000
8	Coefficient of linear expansion	α	9.22×10^{-6}
9	Elastic limit, stranded, copper cable (pounds per square inch)	T_e	28,350
10	Weight of copper (pounds per cubic inch)	W	0.327
11	Cable area in circular mils	cir. mil	500,000
12	Number of strands		37
13	Number of layers around central wire		3
14	Circular mils per strand = (11) ÷ (12)		13,513.513
15	Diameter of strand in inches = $\sqrt{(14)} \div 1{,}000$		0.11625
16	Outside diameter of cable in inches = [2 × (13) + 1] × (15)	d_s	0.81375
17	Diameter of equivalent solid rod = $\sqrt{(11)}$	d	0.70711
18	Area of equivalent solid rod = $[(11) \times 10^{-6} \times \pi] \div 4$	A	0.3927
19	Area of strand in square inches = $[(14) \times 10^{-6} \times \pi] \div 4$		0.0106135
20	Elastic limit per strand in pounds = (19) × (9)		300.8927
21	Elastic limit of cable in pounds = (12) × (20)		11,133.02
22	Maximum allowable tension assumed = 0.75 × (21)	T_m	8,349.76
23	Total modulus of elasticity for cable = (7) × (18)	EA	6,283,200
24	Weight per foot of cable alone = (18) × 12 × (10)	w	1.541
25	Cubic inches ice load per foot of cable = $12\pi[((16)+2a)^2 - (16)^2] \div 4$		24.764
26	Ice load per foot of cable in pounds = (25)u	i	0.822
27	Wind load per foot of cable in pounds = (5) × [(16) + 2a] ÷ 12	p	1.511
28	Total resultant load in pounds per foot of cable = $[(w + i)^2 + p^2]^{\frac{1}{2}}$	w_1	2.804

Next assume a number of arguments, as in column (1) of Table 13, of sufficient range to cover the spans desired. From the tables of Appendix C, column (2) readily follows. Since

$$T_m = w_1 c_1 \cosh \frac{x}{c_1},$$

in which equation all quantities except c_1 are now known, c_1 of column (3) may be computed. The corresponding half-span lengths in column (4) are

$$x = \frac{x}{c_1} \times c_1$$

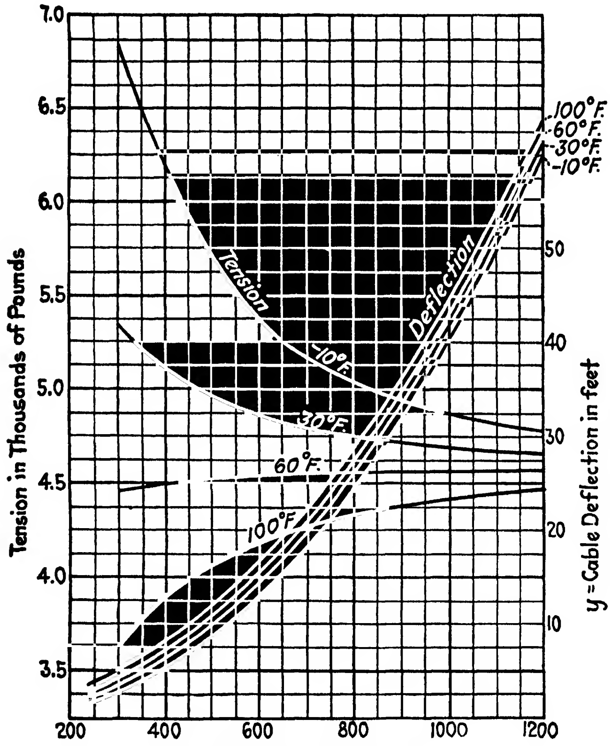


FIG. 70.—Temperature-tension stringing chart for 500,000-circular mil copper cable. (See columns 18 and 19 of Tables 15 to 18.)

and the length of the loaded cable, for each assumed argument, is

$$s_1 = c_1 \sinh \frac{x}{c_1}$$

as in column (5). If desired, the sag under the condition of maximum loading and minimum temperature may also be found from the equation (532)

$$y_1 = c_1\left(\cosh \frac{x}{c_1} - 1\right)\cdot$$

TABLE 13.—CATENARY AT MINIMUM TEMPERATURE

Cable under Maximum Loading Conditions

Data and working equations

T_m = 8,350 lb.
EA = 6,283,200
w = 1 5410 lb.
w_1 = 2.8034 lb.
t_1 = −10° F.

$$\frac{T}{w} = c \cosh \frac{x}{c},$$
$$c = \frac{T}{w \cosh \frac{x}{c}}$$
$$S = c \sinh \frac{x}{c}$$

Sags are not given since they are not critical.

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
Assumed argument	From tables	$\frac{T_m}{w_1} \times (2)$	(1) × (3) $x =$	(3) × sinh (1) $s_1 =$	(5) ÷ (4)	From tables $P_1 =$	[(5) ÷ 2EA] × 10⁶	$w_1 \times (7)$ $w_1P_1 =$
$\frac{x}{c_1}$	$\frac{1}{\cosh \frac{x}{c_1}}$	c_1	$\frac{x}{c_1} \times c_1$	$c_1 \sinh \frac{x}{c_1}$	$\frac{s_1}{x}$	$\frac{\cosh \frac{x}{c_1}}{\frac{x}{c_1}} + \frac{1}{\sinh \frac{x}{c_1}}$	$N \times 10^6$	2.8034 × (7)
0.04	0.999200	2,976.02	119.041	119.073	1.000265	50.013338	0.94755	140.2074
0.06	0.998202	2,973.04	178.380	178.489	1.000613	33.353346	1.42036	93.5028
0.08	0.996808	2,968.89	237.511	237.765	1.001069	25.026698	1.89207	70.1598
0.10	0.995021	2,963.57	296.357	296.851	1.001667	20.033394	2.36226	56.1616
0.12	0.992843	2,957.08	354.850	355.702	1.002401	16.706772	2.83058	46.8358
0.14	0.990279	2,949.45	412.923	414.273	1.003269	14.332549	3.29667	40.1799
0.16	0.987335	2,940.68	470.509	472.519	1.004272	12.553584	3.76017	35.1927
0.18	0.984016	2,930.79	527.542	530.396	1.005410	11.171467	4.22074	31.3181
0.20	0.980328	2,919.81	583.962	587.863	1.006680	10.067155	4.67805	28.2223

TABLE 14.—CATENARY AT FREEZING POINT

Cable under Maximum Loading Conditions

$w_1 = 2.8034$ lb.
$t_n = +32°$ F.
$D_n = 1 + 9.22 \times 10^{-6}(32 + 10)$
$= 1.0003872$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$(6) \times D_n$ F	$(8) \times D_n$ $\frac{10^6 s_1 D_n}{2EA} =$ $10^5 N D_n$	$\frac{(9) \times (11)}{10^5}$ G	$\frac{w_1 \times (11)}{10^6}$ H	$(10) - (12) - 1$ $F - G - 1$	From chart $\frac{x}{c_n}$	$(4) \div (15)$ c_n	cosh (15) $\cosh \frac{x}{c_n}$	$w_1(16)(17)$ $T =$ $w_1 c_n \cosh \frac{x}{c_n}$	$[(18) \div w_1] - (16)$ sag[1] $=$ $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1.000652	0.947917	0.0013290	0.00002657	−0.000677	0.0494	2,409.7	1.00122	6,763.6	2.9
1.001000	1.42091	0.0013286	0.00003983	−0.000329	0.0698	2,555.6	1.00244	7,181.9	6.3
1.001457	1.89280	0.0013280	0.00005306	+0.000129	0.0892	2,662.7	1.00398	7,494.3	10.6
1.002055	2.36317	0.0013272	0.00006625	0.000728	0.1083	2,736.4	1.00587	7,716.3	16.1
1.002789	2.83168	0.0013262	0.00007938	0.001463	0.1275	2,783.1	1.00814	7,865.7	22.7
1.003657	3.29795	0.0013251	0.00009246	0.002332	0.1465	2,818.6	1.01075	7,986.6	30.3
1.004661	3.76163	0.0013238	0 00010545	0.003337	0.1660	2,834.4	1.01381	8,055.7	39.1
1.005799	4.22237	0.0013221	0 00011837	0.004477	0.1858	2,839.3	1.01731	8,097.5	49.2
1.007070	4.67986	0.0013208	0 00013120	0.005749	0.2051	2,847.2	1.02111	8,150.3	60.1

[1] The sags given in this and succeeding tables are in the plane of the resultant force. The sag in the vertical plane is obtained for the above table by multiplying the given sags by $\cos\theta = 0.8427$, where $\tan\theta = \frac{\text{wind load per foot of cable}}{\text{(ice load + weight of conductor) per foot of cable}} = \frac{1.511}{2.363} = 0.6395$.

For tables 15 to 18 inclusive the resultant force is in the vertical plane, since wind load is assumed to be zero.

TABLE 15.—CATENARY AT −10°F.
Cable without Ice or Wind Load

$w = 1.5410$
$t_n = -10°$ F.
$D_n = 1 + 9.22 \times 10^{-6} \times 0$
$= 1$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$(6) \times D_n$ F	$(8) \times D_n$ $\frac{10^6 s_1 D_n}{2EA} =$ $10^6 N D_n$	$\frac{(9) \times (11)}{10^6}$ G	$\frac{w \times (11)}{10^6}$ H	$(10) - (12) - 1$ $F - G - 1$	From chart $\frac{x}{c_n}$	$(4) \div (15)$ c_n	cosh (15) $\cosh \frac{x}{c_n}$	$w(16)(17)$ $T =$ $wc_n \cosh \frac{x}{c_n}$	$[(18) \div w] - (16)$ sag $=$ $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1.000265	0.94755	0.0013285	0.00001460	−0.001063	0.0250	4,761.6	1.00031	7,339.9	1.5
1.000613	1.42036	0.0013281	0.00002189	−0.000715	0.0429	4,158.0	1.00092	6,413.4	3.8
1.001069	1.89207	0.0013275	0.00002916	−0.000259	0.0633	3,752.1	1.00200	5,793.6	7.5
1.001667	2.36226	0.0013267	0.00003640	0.000350	0.0850	3,486.6	1.00361	5,392.2	12.3
1.002401	2.83058	0.0013257	0.00004362	0.001075	0.1066	3,328.8	1.00568	5,158.8	18.9
1.003269	3.29667	0.0013246	0.00005080	0.001944	0.1286	3,210.9	1.00828	4,989.0	26.6
1.004272	3.76017	0.0013233	0.00005794	0.002949	0.1491	3,155.7	1.01114	4,917.1	35.1
1.005410	4.22074	0.0013219	0.00006504	0.004088	0.1708	3,088.7	1.01462	4,829.3	45.2
1.006680	4.67805	0.0013203	0.00007209	0.005360	0.1913	3,052.6	1.01836	4,790.4	56.0

TABLE 16.—CATENARY AT 30°F.
Cable without Ice or Wind Load

$w = 1.5410$
$t_n = 30°$ F.
$D_n = 1 + 9.22 \times 10^{-6}(30 + 10)$
$= 1.0003688$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
(6) $\times D_n$ F	(8) $\times D_n$ $\frac{10^6 s_1 D_n}{2EA} =$ $10^5 ND_n$	$\frac{(9) \times (11)}{10^5}$ G	$\frac{w \times (11)}{10^5}$ H	(10) − (12) − 1 $F - G - 1$	From chart $\frac{x}{c_n}$	(4) ÷ (15) c_n	cosh (15) $\cosh \frac{x}{c_n}$	w(16)(17) $T =$ $wc_n \cosh \frac{x}{c_n}$	[(18) ÷ w] − (16) sag = $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1.000634	0.94790	0.0013290	0.00001461	−0.000695	0.0335	3,553.5	1.00056	5,479.0	2.0
1.000982	1.42088	0 0013286	0.00002190	−0.000347	0.0533	3,346.7	1.00142	5,164.6	4.8
1.001438	1.89277	0.0013280	0.00002917	0.000110	0.0736	3,227 1	1.00271	4,986.4	8.6
1.002036	2.36313	0.0013272	0.00003642	0.000709	0.0941	3,149 4	1.00443	4,874.7	13.9
1.002771	2.83162	0.0013262	0.00004363	0.001445	0.1146	3,096.4	1.00657	4,802.9	20.3
1.003639	3.29789	0.0013251	0.00005082	0.002314	0.1351	3,056.4	1.00914	4,753.0	28.0
1.004642	3.76156	0.0013238	0.00005797	0.003318	0.1559	3,018 0	1.01218	4,707.4	36.8
1.005781	4.22230	0.0013223	0.00006507	0.004459	0.1768	2,983.8	1.01567	4,670.1	46.8
1.007051	4.67978	0.0013207	0 00007212	0.005730	0.1967	2,968.8	1.01940	4,663.7	57.6

TABLE 17.—CATENARY AT 60°F.
Cable without Ice or Wind Load

$w = 1.5410$
$t_n = 60°$ F.
$D_n = 1 + 9.22 \times 10^{-6}(60 + 10)$
$= 1.0006454$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
(6) × D_n F	(8) × D_n $\frac{10^6 s_1 D_n}{2EA} =$ $10^6 N D_n$	$\frac{(9) \times (11)}{10^5}$ G	$\frac{w \times (11)}{10^5}$ H	(10) − (12) − 1 $F - G - 1$	From chart $\frac{x}{c_n}$	(4) ÷ (15) c_n	cosh (15) $\cosh \frac{x}{c_n}$	w(16)(17) $T =$ $wc_n \cosh \frac{x}{c_n}$	[(18) ÷ w] − (16) sag = $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1.000911	0.948161	0.0013294	0.00001461	−0.000418	0.0416	2,861.6	1.00087	4,413.6	2.5
1.001259	1.42128	0.0013289	0.00002190	−0.000070	0.0620	2,877.1	1.00192	4,442.1	5.5
1.001715	1.89329	0.0013283	0.00002918	0.000387	0.0814	2,917.8	1.00331	4,511.2	9.7
1.002314	2.36378	0.0013275	0.00003643	0.000986	0.1012	2,928.4	1.00523	4,536.2	15.3
1.003048	2.83241	0.0013266	0.00004365	0.001721	0.1205	2,944.8	1.00727	4,570.9	21.4
1.003917	3.29880	0.0013255	0.00005083	0.002591	0.1408	2,932.7	1.00993	4,564.2	29.1
1.004920	3.76260	0.0013242	0.00005798	0.003596	0.1607	2,927.9	1.01294	4,570.3	37.9
1.006059	4.22346	0.0013227	0.00006508	0.004736	0.1809	2,916.2	1.01640	4,567.6	47.8
1.007330	4.68107	0.0013211	0.00007213	0.006009	0.2008	2,908.2	1.02023	4,572.2	58.8

TABLE 18.—CATENARY AT 100°F.
Cable without Ice or Wind Load

$w = 1.5410$
$t_n = 100°$ F.
$D_n = 1 + 9.22 \times 10^{-6} (100 + 10)$
$= 1\ 001014$

(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)
$(6) \times D_n$ F	$(8) \times D_n$ $\frac{10^5 s_1 D_n}{2EA} =$ $10^5 N D_n$	$\frac{(9) \times (11)}{10^5}$ G	$\frac{w \times (11)}{10^5}$ H	$(10) - (12) - 1$ $F - G - 1$	From chart $\frac{x}{c_n}$	$(4) \div (15)$ c_n	cosh (15) $\cosh \frac{x}{c_n}$	$w(16)(17)$ $T =$ $wc_n \cosh \frac{x}{c_n}$	$[(18) \div w] - (16)$ sag $=$ $c_n\left(\cosh \frac{x}{c_n} - 1\right)$
1.001275	0.94851	0 0013299	0.00001462	−0.000055	0.0540	2,204.5	1.00146	3,402.1	3.2
1.001624	1.42180	0 0013294	0.00002191	0.000295	0.0731	2,440.2	1.00267	3,770.3	6.5
1.002080	1.89399	0.0013288	0.00002919	0 000751	0.0913	2,601.4	1.00417	4,025.5	10.9
1.002679	2.36466	0.0013280	0.00003644	0.001351	0.1099	2,696.6	1.00605	4,180.6	16.3
1.003413	2.83345	0.0013271	0 00004366	0.002086	0.1282	2,767.9	1.00823	4,300.4	22.8
1.004282	3.30001	0.0013259	0 00005085	0.002956	0.1477	2,795.7	1.01092	4,355.2	30.5
1.005286	3.76398	0.0013246	0.00005800	0 003961	0.1667	2,822.5	1.01392	4,410.0	39.3
1.006426	4.22502	0.0013232	0.00006511	0.005103	0.1857	2,840.8	1.01729	4,453.4	49.1
1.007697	4.68279	0.0013216	0.00007216	0 006375	0 2058	2,837.5	1.02125	4,465.5	60.3

This sag is usually not of particular interest, since it does not represent a critical condition, except in the case of supports at different elevations. In this case the minimum sag may have to be investigated to determine whether uplift on the lower support may not occur at minimum temperature.

The remaining columns (7), (8) and (9) of Table 13 are constants of Eq. (542), which are used in the calculation of the remaining curves, and are tabulated here for convenience.

From the data already computed and tabulated in Tables 12 and 13, and with the use of Eq. (542), the remaining curves are all easily calculated. The method is exactly the same for all curves. It will therefore be sufficient to illustrate this method as applied to the data of Table 14.

By subtracting 1 from each member of Eq. (542), which is the abbreviated form of Eq. (541), this equation becomes

$$\frac{\sinh \frac{x}{c_n}}{\frac{x}{c_n}} - 1 = F_n - G_n - 1 + H_n\left[\frac{\cosh \frac{x}{c_n}}{\frac{x}{c_n}} + \frac{1}{\sinh \frac{x}{c_n}}\right].$$

From Eqs. (543) to (545), inclusive,

$$D_n = 1 + \alpha(t_n - t_1)$$

$$N = \frac{s_1}{2EA}$$

$$P_1 = \left[\frac{\cosh \frac{x}{c_1}}{\frac{x}{c_1}} + \frac{1}{\sinh \frac{x}{c_1}}\right].$$

Whence, by Eqs. (546), (547) and (548),

$$F_n = \frac{s_1 D_n}{x}$$

$$G_n = w_1 P_1 N D_n$$

$$H_n = w_n N D_n.$$

From these equations, columns (7), (8) and (9) of Table 13 and columns (10), (11), (12), (13) and (14) of each of the other tables, representing catenaries for various loadings and temperatures, are readily computed. Column (15) in each case is obtained from the chart by the method already explained under the heading, Method of Solving Eq. (541) for $x \div c_n$. Once the new arguments $x \div c_n$ have been found, columns (16), (17), (18) and (19) follow from the relations given below:

$$c_n = x \div (x \div c_n) \tag{16}$$

$\cosh \frac{x}{c_n}$ obtained from tables of hyperbolic cosines (17)

$$T = w_n c_n \cosh \frac{x}{c_n} \tag{18}$$

where w_n is the particular total loading, per unit length of cable, appropriate to the catenary in question.

$$\text{Sag} = c_n\left(\cosh \frac{x}{c_n} - 1\right). \tag{19}$$
$$= \frac{T}{w_n} - c_n$$

The curves of Fig. 70 are the graphs corresponding to the computed values set down in columns 18 and 19 of the above mentioned tables.

PROBLEMS

1. A 0000 medium hard-drawn, copper cable is to be strung on poles along a level right-of-way. The points of attachments of conductors to insulators are 42 ft. above ground, and the minimum permissible clearance of conductors to ground is 30 ft. The conductor tension at the supports under stringing conditions must not exceed 1,400 lb. What is the maximum allowable distance between supports? What is the length of cable between supports?

2. The line of Problem 1 crosses an inlet. At the place of crossing the minimum available span is 800 ft. The minimum allowable clearance between cable and water is 40 ft. Allowing the same tension as before, what is the required height of the point of conductor attachment above water level? What is the sag at the middle of the span? What is the length of cable in the span?

3. Calculate and plot the temperature-tension curve and the sag curve at 100° F., for a 250,000-cir. mil, stranded, copper cable for which the following data apply. Make calculations covering spans up to 800 ft. for a level right-of-way.

Cable diameter = 0.575 in.

Weight of cable per foot = 0.772 lb.

Ice load assumed at maximum loading = 0.5 in. all around cable.

Minimum temperature at which ice load exists = −10° F.

At maximum load and minimum temperature of −10° F., the wind load, assumed at right angles to line, and acting on ice-covered cable, is 10 lb. per square foot of projected area.

Maximum allowable tension in cable = 4,200 lb.

Modulus of elasticity E = 12,000,000 per square inch.

Coefficient of linear expansion = 9.22×10^{-6}.

CHAPTER XI

MECHANICAL DESIGN

SPANS WITH SUPPORTS AT UNEQUAL ELEVATIONS

Spans with Supports at Unequal Elevations.—Where the transmission-line right-of-way passes through very hilly or mountainous country, the length of span is frequently determined more by the limited choice of possible tower locations than by the conditions of maximum economy, which are discussed in a succeeding chapter. The roughness of the country and the greatly restricted number of suitable tower sites make it frequently necessary to construct spans in which there is a considerable difference in elevation of the points of attachment of a given conductor on adjacent towers. In general, the method followed in the design of such spans is similar to that followed for spans with points of conductor attachments at equal elevations. In addition, however, the designer must be assured that under conditions of minimum sag the conductors do not exert an upward pull upon the towers. The tension in the conductors will be unequal at adjacent towers, as will also the vertical distances from the points of support to the lowest point on the suspended conductors. A simple method of finding these required quantities will be outlined.[1]

Theory Outlined.—In Fig. 71, let AD_0B_0 be a catenary representing a conductor supported at A and at B_0, two points whose elevations differ by b_0 units, and let the length of span be a. The conductor is to be strung so that (a) when under maximum loading and minimum temperature conditions the tension at A will not exceed the maximum allowable tension assumed for the conductor; (b) when under minimum sag the conductor will not exert an upward pull on the tower at B_0. This condition is

[1] SMITH, G. S., "Transmission Line Design. Mechanical Design of Spans with Supports at Unequal Elevations," presented before the A. I. E. E. Pacific Coast Convention 1925; also University of Washington Engineering Experiment Station, *Bull.* 29.

fulfilled by the stipulation that the lowest point D_0, on the curve, must at all times lie between B_0 and A.

In Fig. 71 if the catenary AD_0B_0 be extended to the point C_0, having the same elevation as A, a catenary is obtained which is symmetrical with respect to the Y-axis through the lowest point D_0. For this curve, the half span length is x_0 and the maximum deflection is y_0. All quantities, such as tension, deflection and length for the symmetrical curve, may be computed by the methods already described for spans with supports at equal elevations. Remembering that distances a, x and x_0 etc., are measured horizontally between supports, it is apparent that at any point on the curve distant x units from D_0, the point of maximum sag on the symmetrical curve, the deflection is given by Eq. (488), as

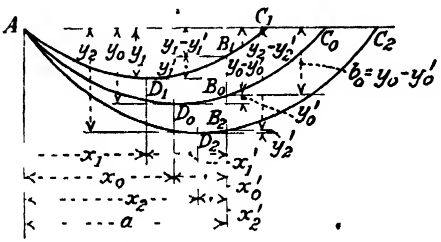


FIG. 71.—Catenaries for spans with supports at unequal elevations.

$$y = c\left(\cosh\frac{x}{c} - 1\right)$$

where the constant c, for any given conditions of temperature and loading, is the ratio $H \div w$ for that set of conditions. If the half span length x_0, of the symmetrical curve AD_0C_0, were known, the distance x_0', from the point of maximum deflection to the nearest support, could readily be found, for

$$x_0' = a - x_0. \tag{549}$$

Once x_0' is known, the maximum deflection y_0 of a symmetrical catenary whose span length is $2x_0$, is available from the equation

$$y_0' = c_0\left(\cosh\frac{x_0'}{c_0} - 1\right) \tag{550}$$

and since

$$b_0 = y_0 - y_0' \tag{551}$$

the difference in elevation of the two supports is expressible in terms of the two symmetrical catenaries having the span lengths $2x_0$ and $2x_0'$. The problem, then, is to find these span lengths.

Interpolation for x_0.—The values x_0 and x_0' of the half span lengths for the long and short spans of the corresponding symme-

trical catenaries may readily be found by interpolation from plotted values of the relation

$$b = y - y'$$

if the range of arguments is so chosen as to embrace the argument corresponding to the desired value $b - b_0$. The following paragraphs will make the truth of this statement more apparent.

Consider a large number of catenaries similar to the one already described, some having spans longer and some shorter than $2x_0$. Let the conductors representing these curves be all alike, let them all be strung to have the same maximum tension under conditions of minimum temperature and maximum loading, and assume that all of them have undergone the same changes of loading and temperature.

Of this series of catenaries AD_1C_1 is one that is a small amount shorter, and AD_2C_2 one a small amount longer than AD_0C_0. For these curves the maximum sags are

$$\left.\begin{aligned} y_1 &= c_1\left(\cosh \frac{x_1}{c_1} - 1\right) \\ y_2 &= c_2\left(\cosh \frac{x_2}{c_2} - 1\right) \end{aligned}\right\} \tag{552}$$

for the symmetrical catenaries of span lengths $2x_1$ and $2x_2$, respectively, and

$$\left.\begin{aligned} y_1' &= c_1\left(\cosh \frac{x_1'}{c_1} - 1\right) \\ y_2' &= c_2\left(\cosh \frac{x_2'}{c_2} - 1\right) \end{aligned}\right\}$$

for the symmetrical catenaries of span lengths $2x_1'$ and $2x_2'$, respectively, by Eq. (488). The above values of half span lengths are related as in Eq. (549); that is,

$$\left.\begin{aligned} x_1' &= a - x_1 \\ x_2' &= a - x_2 \end{aligned}\right\} \tag{553}$$

Since B_1 and B_2 are the points of attachment, the differences in elevation between points of attachment on the two supports for the conductor AD_1C_1 and AD_2C_2 are, respectively,

$$b_1 = y_1 - y_1'$$

and

$$b_2 = y_2 - y_2'.$$

An inspection of the Fig. 71 clearly shows that

$$b_2 > b_0 > b_1$$

and hence it follows that, by assuming a number of arguments $\frac{x}{c}$ and computing the corresponding values of b from Eq. (551), the value of b_0 may be found by interpolation if the series of arguments chosen includes values both larger and smaller than $\frac{x_0}{c_0}$.

The differences b are plotted against their corresponding values of T_m, c or x, and the values of x_1, c_0 and T_0, etc., corresponding to the curve AD_0B_0, readily follow. The method described will be illustrated by an example.

Example.—A 500,000-cir. mil copper cable with the constants and subjected to the loadings and temperature conditions described in the example on page 205, is suspended from supports 600 ft. apart. The difference in elevations of the points of suspension is 40 ft. It is required (*a*) to find the tension at the highest support for each temperature and loading; (*b*) to find the corresponding maximum sags, and (*c*) to test the results of the solution to see whether under minimum sag conditions there is an upward pull on the lowest support. (If an upward pull exists the maximum tension may be reduced or the span may be redesigned for a smaller difference in elevation between supports.)

Solution.—The given data and working equations are

$$a = 600 \text{ ft.} \qquad x' = a - x$$

$$b_0 = 40 \text{ ft.} \qquad y' = c\left(\cosh\frac{x'}{c} - 1\right).$$

The symmetrical spans (spans with supports at equal elevations), have already been computed for lengths up to nearly 1,200 ft. for this particular conductor. The results of these calculations are given in Tables 12 to 18 inclusive. From these tables the already known values of x_1, c and y of Eq. (552) are copied and set down in columns (21), (22) and (23) of Table 19, for the corresponding values of $\frac{x}{c_1}$. From Eq. (553) the values of x' in column (24) follow. After computing columns (25) and (26), the values, of y' in column (27) and b in column (28), are readily calculated. These values of b are next plotted against x, c, T and y of columns (4), (16), (18) and (19), respectively, and the values corresponding to $b_0 = 40$ ft. are read off and tabulated as in Table 20.

It will be observed that the vertical sag is at no time less than 40 ft., and hence no upward pull will ever be exerted on the support.

Stringing Conductors.—If the cable is to be strung by the use of a dynamometer, the tensions of Table 20 will furnish the required

TABLE 19.—SUPPORTS AT UNEQUAL ELEVATIONS

Span = 600 ft.

b_0 = 40 ft.

Temperature and load	(20) Assumed values from (1) $\frac{x}{c_1}$	(21) From (16) c_n	(22) From (19) y	(23) From (4) x	(24) 600 − (23) $x' = a - x$	(25) (24) ÷ (21) $\frac{x'}{c_n}$	(26) cosh (25) $\cosh \frac{x'}{c_n}$	(27) (21)[(26) − 1] $y' = c_n (\cosh \frac{x'}{c_n} - 1)$	(28) (22) − (27) $b = y - y'$
32° F., cable loaded	0.12	2,783.1	22.7[1]	354.8	245.2	0.08810	1.00388	10.8[1]	10.0[2]
	0.14	2,818.6	30.3	412.9	187.1	0.06638	1.00221	6.2	20.3
	0.16	2,834.4	39.1	470.5	129.5	0.04569	1.00104	2.9	30.5
	0.18	2,839.3	49.2	527.5	72.5	0.02553	1.00033	0.9	40.7
	0.20	2,847.2	60.1	584.0	16.0	0.00562	1.00002	0.1	50.6
−10° F., cable only	0.12	3,328.8	18.9	354.8	245.2	0.07366	1.00272	9.1	9.8
	0.14	3,210.9	26.6	412.9	187.1	0.05827	1.00170	5.5	21.1
	0.16	3,155.7	35.1	470.5	129.5	0.04104	1.00084	2.7	32.4
	0.18	3,088.7	45.2	527.5	72.5	0.02347	1.00028	0.9	44.3
	0.20	3,052.6	56.0	584.0	16.0	0.00524	1.00001	0.0	56.0
30° F., cable only	0.12	3,096 4	20.3	354.8	245.2	0.07919	1.00313	9.7	10.6
	0.14	3,056.4	28.0	412.9	187.1	0.06122	1.00187	5.7	22.3
	0.16	3,018.0	36.8	470.5	129.5	0.04291	1.00092	2.8	34.0
	0.18	2,983.8	46.8	527.5	72.5	0.02430	1.00030	0.9	45.9
	0.20	2,968.8	57.6	584.0	16.0	0.00539	1.00001	0.0	57.6
60° F., cable only	0.12	2,944.8	21.4	354.8	245.2	0.08327	1.00347	10.2	11.2
	0.14	2,932.7	29.1	412.9	187 1	0.06380	1.00204	6.0	23.1
	0.16	2,927.9	37.9	470.5	129.5	0.04423	1.00098	2.9	35.0
	0.18	2,916.2	47.8	527 5	72.5	0.02486	1.00031	0.9	46.9
	0.20	2,908.2	58.8	584.0	16.0	0.00550	1.00002	0.1	58.7
100° F., cable only	0.12	2,767.9	22.8	354.8	245.2	0.08859	1.00393	10.9	11.9
	0.14	2,795 7	30.5	412.9	187.1	0.06692	1.00224	6.3	24.2
	0.16	2,822.5	39.3	470.5	129.5	0.04588	1.00105	3.0	36.3
	0.18	2,840.8	49.1	527.5	72.5	0.02552	1.00093	0.9	48.2
	0.20	2,837.5	60.3	584.0	16.0	0.00564	1.00002	0.1	60.2

[1] For the loaded cable at 32° F., the values of y and y^1 represent sags in the plane of the resultant force.

[2] For the loaded cable at 32° F., the values in the table are computed values of $(y - y')$ multiplied by $(\cos \theta = 0.8427)$.

stringing data. On the other hand, if sag is to be the basic criterion, two methods are open: (a) The cable may be suspended from its insulators and allowed to sag until the distance AA'' of Fig. 72 is equal to the value of y_0 in column (28) of Table 19, for the temperature at the time of stringing. (b) The cable may be sagged until the distance $AA' = BB'_0$ is such that the line of sight $A'B_0$ (parallel to AB_0) is tangent to the cable. The point D' is located at the intersection of a vertical line, midway between supports, with the catenary, and is taken as the lowest point on the curve. This is not strictly accurate, but the error made by this assumption is so small as to be entirely negligible here. The value of AA' may therefore be easily calculated for

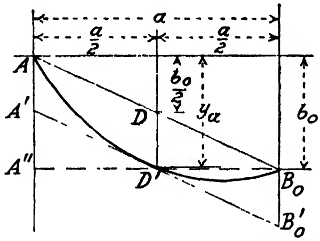


FIG. 72.—Determination of stringing sag.

$$y_a = c\left(\cosh \frac{a}{2c} - 1\right)$$

whence

$$AA' = y_a - \frac{b_0}{2}. \tag{554}$$

TABLE 20

Values of x, c, T_m and y, corresponding to $b_0 = 40$ ft., are obtained from plotted curves using b of column (28) as the independent variable, and each of the quantities sought as the dependent variable.

Temperature and load	x_0	c_0	T_0	y_0
32° F., cable loaded	523.8	2,838.9	8,094	40.8
−10° F., cable only	506.9	3,109.2	4,852	41.4
30° F., cable only	499.4	3,004.8	4,692	41.8
60° F., cable only	494.4	2,922.7	4,568	41.9
100° F., cable only	488 0	2,831 6	4,428	42 3

PROBLEM

The difference in elevation between two adjacent supports in a 700-ft. span, of the line in Problem 3, Chap. X, is 45 ft. It is assumed that the tension at the highest support under maximum loading conditions, the snow load, wind load and all other pertinent data, are as given in the problem above mentioned. Calculate, for a temperature of 110° F., (a) the tension at the highest support; (b) the sag y_0; (c) the length of cable in the span.

CHAPTER XII

ECONOMICS OF SPAN DESIGN

The Most Economical Tower Spacing.—In the design of important lines, and particularly for lines or sections of lines built over more or less level country where spans of uniform length are possible, it is essential that the designer make a careful investigation to determine the length of span which will require minimum total capital outlay. In very rough country where tower locations and lengths of spans are frequently determined within narrow limits by the nature of the terrain, the principles of maximum economy, here discussed, will naturally be applied with greater difficulty. As will be observed from a close study of the problem, the span length of greatest economy is not a sharply defined one on either side of which costs rise rapidly; but, rather, there is a considerable variation in span length over which total costs change rather slowly. Therefore, if so required to suit the contour of the right of way, the span may be varied considerably in length from the one found to be most economical, without seriously affecting the cost.

Economic Principle.—The principles underlying the problem of finding the most economical tower spacing may be stated as follows: It is assumed, as is usually the case, that the minimum permissible clearance between conductor and ground is fixed, either by state law or otherwise. This is the fundamental datum. Furthermore, in any span the minimum clearance occurs when the deflection is maximum. The latter condition, in turn, is defined by the critical catenary.

Thus, with a fixed ground clearance to begin with, the height of the individual support increases with the length of span, whereas their total number decreases. The total cost of the supports is equal to the number of units required times the cost of a unit. With increasing length of span, therefore, the cost of the individual unit is increased on account of its greater height and the greater strength required, while the number of units is, of course, diminished. The most economical spacing to use is the one for

which the total cost of supports is a minimum; that is, it is a spacing such that, if increased by a small increment, the total increased cost for all support, due to the increased cost of a unit, will be exactly equalled by the total reduction in cost resulting from their decreased number.

Methods of Attack.—Cut and try methods are frequently followed in estimating the most economical tower spacing. When this method is used the costs of the several separate items which go to make up the total cost of the unit, such as insulators, foundations, towers, cost of erection, etc., are separately estimated for each of a number of assumed lengths of span. Curves for each of these variable cost items are then drawn with cost expressed as a function of span length. A total cost curve, made up by adding the ordinates to the separate curves, is then constructed. The minimum point on this curve locates the most economical length of span.

In the analysis which follows, instead of the cut-and-try method, a mathematical solution of the problem is offered. This method has the advantage of ease of manipulation, and of furnishing results which may readily be incorporated into a mathematical study of the greater problem having to do with the economics of design involving the line as a whole, as considered in a subsequent chapter.

Basis for Mathematical Solution.[1]—The method by which one may proceed to a mathematical statement of the problem is suggested by the discussion under the caption entitled "Economic Principle," together with a consideration of cost items that enter into the total cost of the supports and how they vary.

The total cost of a tower structure may readily be segregated into various items of cost. For the purpose in hand, the following list of items is used:

1. Cost of tower at place of erection
2. Cost of erection of tower
3. Cost of lease or purchase of tower site
4. Cost of foundation installed
5. Cost of location and inspection of support
6. Cost of insulators at tower location
7. Cost of placing insulators and cable.

In order to make a solution for the most economical tower spacing it is desirable to separate the above items of cost into two

[1] KIRSTEN, F. K. University of Washington Engineering Experiment Station, *Bull.* 17.

groups, one of which is a function of the tower spacing only, and the other of which varies both with the tower spacing and the height of the tower. The total cost will then be expressed in mathematical form in terms of these two groups of cost items. The first question to be answered is, into which class does each of the above items fall?

Item 1.—The weight of the tower is proportional to the maximum cable stress for which it is designed. For a given stress, an increase in the height of the tower increases its weight in proportion to the square of its height. Since the cost of the tower is proportional to its weight, the amount of item 1 is in direct proportion to the square of the height of the tower.

Item 2.—The given stress for which the tower is designed practically fixes the weight per unit length and the lengths of its structural members. Hence an increase in the height of the tower increases the number of its structural members, and, consequently, the cost of erection, in direct proportion to the square of its height.

Item 3.—Since in practical tower design the ratio of the height of the tower to the width of its base is a constant, the area of the tower site and therefore the cost of its lease or purchase will also vary in direct proportion to the square of the height of the tower. (Where an entire right-of-way is purchased, this item should be omitted.)

Item 4.—The cost of the foundation is directly proportional to the tension for which the tower is designed and is practically independent of the height of the tower.

Items 5, 6 and 7.—These items are independent of the mechanical features of the towers and are constants for given transportation rates and market conditions of materials and labor. The magnitude of item 6 is practically proportional to the transmission voltage and is fixed for a given transmission voltage.

Equations Derived.—Let

L = length of the transmission line
$2x$ = tower spacing
h = height of tower
k_1 = minimum clearance of cable to ground
y = maximum deflection of the catenary formed by the cable.

The number of line supports is, then,

$$\text{Number} = \frac{L}{2x} \tag{555}$$

and the height of the support is

$$h = k_1 + y \tag{556}$$

The total cost of the line supports is

$$\text{Cost} = \text{No. } (h^2k_2 + k_3) \tag{557}$$

where

$$h^2k_2 = \text{sum of items 1, 2 and 3}$$
$$k_3 = \text{sum of items 4, 5, 6 and 7.}$$

Substituting Eqs. (555), and (556) in Eq. (557),

$$\text{Cost} = \frac{L}{2x}[k_2(k_1 + y)^2 + k_3]. \tag{558}$$

Substituting for y in the above equation its equivalent from Eq. (488),

$$\text{Cost} = \frac{L}{2x}\left[k_2\left\{k_1 + c\left(\cosh\frac{x}{c} - 1\right)\right\}^2 + k_3\right]. \tag{559}$$

For a minimum or maximum cost the derivative of Eq. (559) with respect to x must be equal to zero:

$$\frac{d(\text{Cost})}{dx} = 0 = \frac{L}{2}\left(\frac{2k_2c\cosh\frac{x}{c}\sinh\frac{x}{c}}{x} - \frac{k_2c^2\cosh^2\frac{x}{c}}{x^2} - \frac{2k_2c\sinh\frac{x}{c}}{x} + \frac{2k_2c^2\cosh\frac{x}{c}}{x^2} - \frac{k_2c^2}{x^2} + \frac{2k_1k_2\sinh\frac{x}{c}}{x} - \frac{2k_1k_2c\cosh\frac{x}{c}}{x^2} + \frac{2k_1k_2c}{x^2} - \frac{k_1^2k_2}{x^2} - \frac{k_3}{x^2}\right)$$

$$= 2cx\cosh\frac{x}{c}\sinh\frac{x}{c} - c^2\cosh^2\frac{x}{c} - 2cx\sinh\frac{x}{c} + 2c^2\cosh\frac{x}{c} - c^2 + 2k_1x\sinh\frac{x}{c} - 2k_1c\cosh\frac{x}{c} + 2k_1c - k_1^2 - \frac{k_3}{k_2}. \tag{560}$$

But $x = c \times \frac{x}{c}$, hence

$$0 = c^2\left(2\frac{x}{c}\cosh\frac{x}{c}\sinh\frac{x}{c} - \cosh^2\frac{x}{c} - 2\frac{x}{c}\sinh\frac{x}{c} + 2\cosh\frac{x}{c} - 1\right) + 2ck_1\left(\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right) - k_1^2 - \frac{k_3}{k_2}. \tag{561}$$

Simplifying,

$$c^2\left(\cosh\frac{x}{c} - 1\right)\left(2\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right) + 2ck_1\left(\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right) = k_1^2 + \frac{k_3}{k_2} \quad (562)$$

from which

$$c = -\frac{k_1\left(\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right)}{\left(\cosh\frac{x}{c} - 1\right)\left(2\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right)} \pm \left[\frac{\left(k_1^2 + \frac{k_3}{k_2}\right)\left(\cosh\frac{x}{c} - 1\right)\left(2\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right) + k_1^2\left(\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right)^2}{\left(\cosh\frac{x}{c} - 1\right)^2\left(2\frac{x}{c}\sinh\frac{x}{c} - \cosh\frac{x}{c} + 1\right)^2}\right]^{\frac{1}{2}} \quad (563)$$

Simplifying Eq. (563),

$$c = -\frac{k_1\left[\frac{x}{c}\sinh\frac{x}{c} - \left(\cosh\frac{x}{c} - 1\right)\right]}{\left(\cosh\frac{x}{c} - 1\right)\left[2\frac{x}{c}\sinh\frac{x}{c} - \left(\cosh\frac{x}{c} - 1\right)\right]} \pm \frac{\left\{\frac{k_3}{k_2}\left(\cosh\frac{x}{c} - 1\right)\left[2\frac{x}{c}\sinh\frac{x}{c} - \left(\cosh\frac{x}{c} - 1\right)\right] + \left(k_1\frac{x}{c}\sinh\frac{x}{c}\right)^2\right\}^{\frac{1}{2}}}{\left(\cosh\frac{x}{c} - 1\right)\left[2\frac{x}{c}\sinh\frac{x}{c} - \left(\cosh\frac{x}{c} - 1\right)\right]}. \quad (564)$$

For a given clearance k_1 of the cable to ground, and a given hyperbolic argument, Eq. (564) contains only one variable, the ratio $k_3 \div k_2$. In practical line design the possible range of this ratio usually falls within the limits 500 and 2,500.

The Most Economical Span S as a Function of Conductor Diameter d_s.—From Eq. (564) the minimum cost of tower may be found if the positive sign before the radical is used. For a given ground clearance k_1 and a given ratio $k_3 \div k_2$, this equation yields the value of c for any assumed argument $\frac{x}{c}$. Since c is

given as a function of $\frac{x}{c}$ it follows that the corresponding tower spacing $2x = S$ is also known as a function of c. Thus, from Eq. (564), curves may be plotted giving values of c as a function of $S = 2x$, for any assumed values of ground clearances k_1 and any ratios $k_3 \div k_2$. For each ratio $k_3 \div k_2$ chosen, there is a family of curves; an individual curve results for each assumed value of k_1. Such a set of curves, applying to stranded, copper cables, is illustrated in the graphs marked 1, 2, 3, etc., of Fig. 73. By assuming values of k_1 and $k_3 \div k_2$ of a range sufficient to cover any conditions likely to be encountered in practice, a single set of curves will suffice for any problem likely to arise. In these curves it must be remembered that $2x$ represents the *most economical span* for the corresponding values of c, k_1, and $k_3 \div k_2$.

If now, on these curves, there be superimposed another set of curves relating the values of c and $2x$, pertaining to the critical catenary for each conductor size of a given conductor material, a ready means is provided for finding the most economical span corresponding to any conductor size and for the material represented. The intersection points of the two sets of curves determine the spans of maximum economy for the various conditions and conductor sizes. The values of c and $2x$ for the critical catenary (the catenary at 100° F. is taken as the critical condition in the work here discussed), are computed by the method already discussed in the previous chapter and there illustrated for a 500,000-cir. mil, copper cable.

Since the circular mil area of a cable is expressible as a function of the overall cable diameter; that is,

$$\begin{aligned} \text{cir. mil} &= d^2 \\ &= d_s^2 \div (1.15)^2 \end{aligned}$$

where the constant 1.151 represents approximately the ratio of outside diameter to the diameter of the equivalent solid rod for stranded cables in sizes above 250,000 cir. mils, the second set of curves of Fig. 73 may be indexed in terms of cable diameters, and the intersections of the two sets of curves may be used to evaluate the most economical span as a function of d_s. That is, by plotting the corresponding values of S and d_s obtained from the intersections, a graphical relation between conductor diameter and most economical span is obtained. This relation is then reduced

to mathematical form by writing an empirical equation to fit the curves over the range covered by the investigation.

As a basis for much of the work of the chapter, computations were made from the results of which a large number of curves

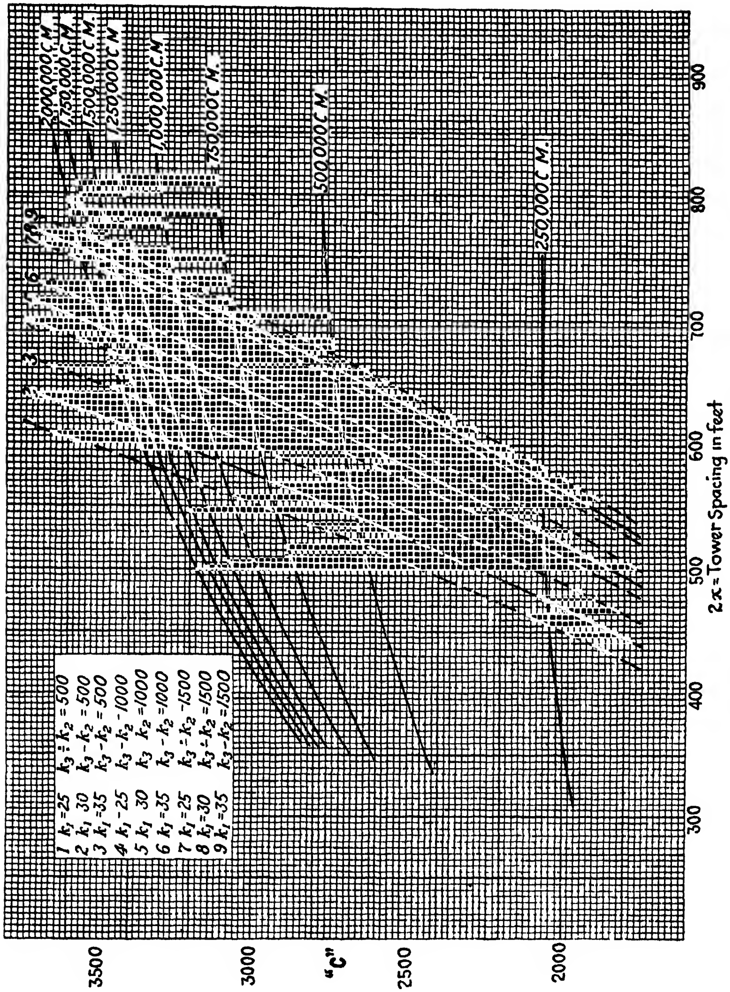


FIG. 73.—Curves for finding the most economical span S.

similar to those of Fig. 73 were drawn. The computations covered the three conductor materials, aluminum, copper and steel for cable sizes between 250,000 and 2,000,000 cir. mils, inclusive. The calculations were repeated for ground clearance

between 25 and 50 ft. for each 5-ft. interval, and for $k_3 \div k_2$ ranging between 500 and 2,500, taken at intervals of 500. From the intersections of the two sets of curves (similar to Fig. 73) resulting from these calculations, an empirical equation was derived for the most economical tower spacing, as discussed in the succeeding article.

Equation of Most Economical Tower Spacing.—An investigation of the most economical tower spacing has been carried out by the method above outlined, for aluminum, copper and steel cables. Curves for composite cables, such as steel-core aluminum and copper-clad steel cables, have not been calculated. For the cables considered, a great number of intersection points were computed for $k_3 \div k_2$ ratios varying from 500 to 2,500, for ground clearances k_1 of from 25 to 50 ft., and for cables varying from 0.5 to 1.7 in. in outside diameter. The curves were plotted with S as abscissas and d_s as ordinates. By making a careful analysis of these curves, it was found that they could be very closely approximated by a rectangular hyperbola of the form

$$(d_s + k_6)(k_7 - S) = k_8 \tag{565}$$

where k_6, k_7 and k_8 are constants for a given curve. Solving Eq. (565) for the most economical span yields

$$S = \frac{k_7(d_s + k_6) - k_8}{d_s + k_6}. \tag{566}$$

From Eq. (566) the most economical span may be found for any conductor size, once the constants are known.

The constants k_7 and k_8 of Eq. (566) have been evaluated for copper, aluminum and steel and for the ranges of $k_3 \div k_2$ and k_1 mentioned above. Those for aluminum and copper are available from the curves of Plates II to V, inclusive. These plates are found at the end of the chapter.

The constant k_6 is independent of k_1, and of the ratio $k_3 \div k_2$, but has a different value for each conductor material. The values of k_6 are

$$\left.\begin{aligned} &= +0.20 \text{ for aluminum} \\ &= -0.15 \text{ for copper} \\ &= -0.30 \text{ for steel} \end{aligned}\right\} \tag{567}$$

The maximum error, involved in the use of Eq. (565), is less than 1 per cent for values of d_s between 0.6 and 2 in. The equa-

tion will still fit the curve closely for values below 0.6 and above 2, but additional constants k_6, k_7 and k_8 are required.

The Most Economical Tower Height.—A study of the maximum sags, for the critical catenaries, corresponding to the spans

TABLE 21.—MAXIMUM CATENARY DEFLECTIONS (IN FEET) FOR MOST ECONOMICAL TOWER HEIGHT

$\frac{k_3}{k_2}$	Conductor material	Conductor diameter d_e						Average cable deflection	
		0.999	1.153	1.289	1.412	1.525	1.631		
500	Al	12.70	12.90	12.72	12.73	12.72	12.70	12.69	$k_1 = 25$
	Cu	12.70	12.72	12.73	12.69	12.66	12.70		
	Fe	12.78	12.68	12.65	12.60	12.54	12.58		
1,000	Al	16.32	16.43	16.32	16.41	16.36	16.33	16.34	
	Cu	16.33	16.35	16.40	16.40	16.40	16.42		
	Fe	16.35	16.36	16.32	16.28	16.25	16.29		
1,500	Al	19.50	19.52	19.49	19.54	19.52	19.47	19.54	
	Cu	19.45	19.51	19.47	19.53	19.58	19.54		
	Fe	19.47	19.60	19.65	19.60	19.61	19.62		
2,000	Al	22.37	22.31	22.33	22.42	22.38	22.34	22.38	
	Cu	22.38	22.35	22.39	22.40	22.37	22.37		
	Fe	22.34	22.42	22.44	22.45	22.42	22.45		
2,500	Al	24.95	24.93	24.98	24.95	24.98	24.96	24.98	
	Cu	25.00	25.00	24.98	24.97	24.99	25.03		
	Fe	24.96	24.96	25.02	25.04	25.04	26.06		
500	Al	13.73	13.96	13.77	13.78	13.75	13.78	13.74	$k_1 = 30$
	Cu	13.75	13.79	13.82	13.76	13.71	13.78		
	Fe	13.78	13.72	13.68	13.62	13.58	13.60		
1,000	Al	17.02	17.08	17.00	17.11	17.06	17.00	17.03	
	Cu	17.02	17.01	17.09	17.11	17.07	17.11		
	Fe	17.00	17.03	17.02	16.97	16.94	16.97		
1,500	Al	19.94	19.98	19.91	20.03	19.96	19.92	19.97	
	Cu	19.96	19.92	19.97	20.02	19.99	20.00		
	Fe	19.90	20.02	20.03	20.01	20.02	20.02		
2,000	Al	22.64	22.58	22.62	22.68	22.63	22.62	22.66	
	Cu	22.63	22.60	22.63	22.64	22.66	22.68		
	Fe	22.60	22.68	22.72	22.73	22.74	22.76		
2,500	Al	25.10	25.10	25.10	25.05	25.09	25.08	25.11	
	Cu	25.10	25.09	25.10	25.10	25.12	25.20		
	Fe	25.09	25.08	25.14	25.18	25.16	25.20		
500	Al	14.96	15.12	14.94	15.03	14.96	14.94	14.94	$k_1 = 35$
	Cu	14.96	15.00	15.00	14.97	14.94	15.00		
	Fe	14.97	14.95	14.89	14.85	14.81	14.83		
1,000	Al	17.90	17.97	17.89	17.98	17.93	17.88	17.92	
	Cu	17.90	17.87	17.95	17.98	17.95	18.00		
	Fe	17.90	17.95	17.92	17.88	17.85	17.89		
1,500	Al	20.63	20.60	20.58	20.67	20.60	20.54	20.63	
	Cu	20.63	20.55	20.61	20.66	20.63	20.63		
	Fe	20.58	20.67	20.69	20.69	20.69	20.69		
2,000	Al	23.12	23.06	23.10	23.17	23.12	23.08	23.13	
	Cu	23.12	23.10	23.11	23.12	23.11	23.11		
	Fe	23.06	23.13	23.15	23.19	23.19	23.20		
2,500	Al	25.33	25.44	25.39	25.39	25.41	25.42	25.43	
	Cu	25.40	25.43	25.41	25.39	25.42	25.48		
	Fe	25.41	25.43	25.49	25.51	25.50	25.53		

TABLE 21.—(*Continued*)

$\frac{k_3}{k_2}$	Conductor material	Conductor diameter d_e						Average cable deflection	
		0.999	1.153	1.289	1.412	1.525	1.631		
500	Al	16.23	16.35	16.23	16.35	16.28	16.23	16.25	$k_1 = 40$
	Cu	16.26	16.28	16.31	16.32	16.29	16.32		
	Fe	16.27	16.25	16.23	16.18	16.12	16.17		
1,000	Al	18.92	18.96	18.90	19.00	18.93	18.89	18.95	
	Cu	18.93	18.86	18.96	19.00	18.98	19.01		
	Fe	18.92	18.99	19.00	18.95	18.96	19.00		
1,500	Al	21.39	21.40	21.44	21.48	21.44	21.38	21.44	
	Cu	21.45	21.40	21.40	21.45	21.42	21.43		
	Fe	21.39	21.46	21.50	21.49	21.49	21.51		
2,000	Al	23.77	23.71	23.71	23.78	23.75	23.71	23.77	
	Cu	23.77	23.76	23.77	23.75	23.73	23.78		
	Fe	23.75	23.80	23.84	23.84	23.85	23.86		
2,500	Al	25.91	25.93	25.91	25.88	25.90	25.91	25.95	
	Cu	25.92	25.93	25.90	25.90	25.94	25.99		
	Fe	25.96	25.96	26.02	26.06	26.01	26.04		
500	Al	17.60	17.68	17.58	17.70	17.67	17.57	17.66	$k_1 = 45$
	Cu	17.60	17.60	17.64	17.69	17.63	17.66		
	Fe	17.62	17.66	17.65	17.58	17.55	17.57		
1,000	Al	20.08	20.09	20.05	20.15	20.08	20.02	20.09	
	Cu	20.08	20.10	20.02	20.10	20.14	20.10		
	Fe	20.02	20.10	20.14	20.12	20.10	20.15		
1,500	Al	22.38	22.41	22.33	22.42	22.38	22.36	22.40	
	Cu	22.39	22.35	22.40	22.40	22.38	22.38		
	Fe	22.35	22.42	22.43	22.46	22.43	22.48		
2,000	Al	24.51	24.53	24.56	24.58	24.51	24.52	24.59	
	Cu	24.58	24.58	24.53	24.54	24.56	24.56		
	Fe	24.50	24.53	24.60	24.55	24.65	24.65		
2,500	Al	26.58	26.62	26.55	26.56	26.65	26.64	26.61	
	Cu	26.60	26.62	26.58	26.58	26.60	26.66		
	Fe	26.59	26.58	26.64	26.68	26.61	26.64		
500	Al	19.00	19.06	18.96	19.08	18.96	19.00	19.04	$k_1 = 50$
	Cu	19.02	18.96	19.04	19.10	19.04	19.09		
	Fe	19.02	19.07	19.11	19.05	19.07	19.05		
1,000	Al	21.28	21.29	21.32	21.32	21.27	21.21	21.30	
	Cu	21.30	21.24	21.26	21.26	21.26	21.28		
	Fe	21.24	21.35	21.38	21.37	21.38	21.42		
1,500	Al	23.48	23.38	23.41	23.45	23.43	23.39	23.45	
	Cu	23.43	23.42	23.41	23.40	23.43	23.42		
	Fe	23.42	23.48	23.50	23.52	23.53	23.55		
2,000	Al	25.40	25.48	25.47	25.45	25.48	25.47	25.49	
	Cu	25.48	25.48	25.47	25.45	25.47	25.51		
	Fe	25.48	25.48	25.53	25.56	25.53	25.57		
2,500	Al	27.32	27.41	27.36	27.36	27.42	27.45	27.40	
	Cu	27.33	27.43	27.39	27.38	27.42	27.45		
	Fe	27.42	27.35	27.43	27.46	27.41	27.42		

of maximum economy as found from Eq. (566), reveals the fact that, for any given ratio of $k_3 \div k_2$ and value of k_1, the maximum deflection of the critical catenary is approximately constant for all conductor sizes and all conductor materials lying within the limits covered by the investigation.

In order to substantiate this rather remarkable conclusion, an investigation was carried out covering conductor sizes of from 500,000 to 2,000,000 cir. mils, for each of the three conductor materials mentioned, in which the maximum sags were determined for the most economical spans under critical conditions, and for each of a number of assumed ratios of $k_3 \div k_2$ and k_1. The maximum temperature assumed was 100° F., at which temperature the critical catenary then exists. The results of this investigation are given in Table 21. The data here recorded were obtained by first finding the span of maximum economy in the manner already explained, and then reading from the critical catenary,[1] of the corresponding conductor size and material, the deflection corresponding to this span.

It is thus possible to express the maximum deflection of the critical catenary in terms of the ratio $k_3 \div k_2$ and k_1, regardless of conductor size or material. This has been done in the curves of Plate I.

Since the height of tower, to the point of attachment of the conductor, is equal to the required minimum ground clearance plus the maximum deflection on the critical catenary, it follows that the height of tower corresponding to the most economical span, or the *most economical tower height* is given by the equation

$$h_e = k_1 + {}_{\max.}y \tag{568}$$

where

h_e = most economical tower height measured from base of tower to point of conductor attachment.

k_1 = minimum, allowable, vertical ground clearance.

${}_{\max.}y$ = deflection of the critical catenary at center of the most economical span.

To illustrate the significance of the foregoing statements, the most economical spans and the most economical tower heights are computed below for aluminum and copper conductors of random sizes and for given assumed values of $k_3 \div k_2$ and k_1.

Example.—Let $k_3 \div k_2 = 1{,}500$ and $k_1 = 30$. It is required to find the most economical span S, and the most economical tower height h_e, for the following conductors:

1,000,000-cir. mil, aluminum cable for which $d_e = 1.153$ in.
500,000-cir. mil, copper cable for which $d_e = 0.815$ in.

[1] Data for the critical catenaries of all conductors falling within the scope of the table are found in University of Washington Engineering Experiment Station *Bull.* 17.

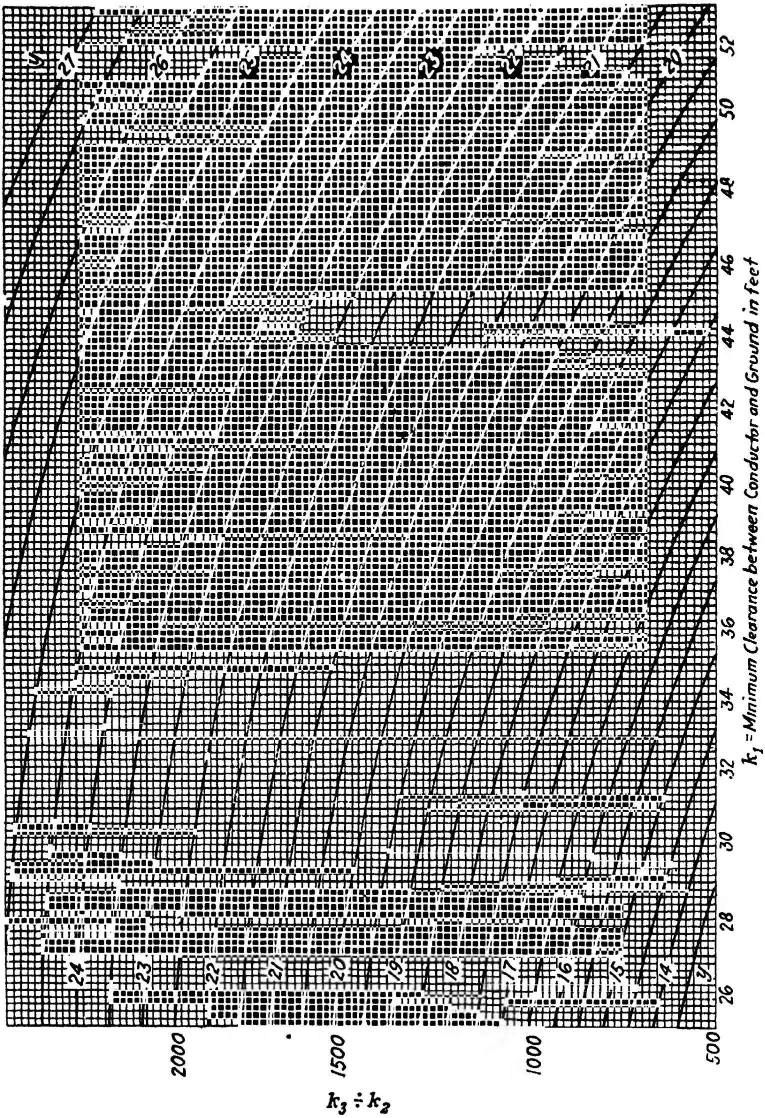


PLATE I.—Curves of maximum conductor sag "y" at center of span for most economical tower spacing.

Solution.—From Plates II to V and from Eq. (567), the constants k_6, k_7 and k_8, for the chosen values of $k_3 \div k_2$ and k_1, are

For aluminum conductor

$$k_6 = +0.20$$
$$k_7 = 968$$
$$k_8 = 425$$

For copper conductor

$$k_6 = -0.15$$
$$k_7 = 832$$
$$k_8 = 112.5$$

Substituting these values in Eq. (566), for each conductor, yields the most economical spans as follows:

1,000,000 cir. mils aluminum	$S = 654$ ft.
500,000 cir. mils copper	$S = 662$ ft.

From the critical catenaries for these conductor sizes and materials, and for the spans found, the maximum sags are found to be

$$_{\max.}y = 20 \text{ ft. for the aluminum cable}$$
$$_{\max.}y = 19.9 \text{ ft. for the copper cable}$$

Turning now to Plate I, it is found that the maximum sag given by the curves for the values $k_3 \div k_2 = 1{,}500$ and $k_1 = 30$, is 20 ft. This checks the results of the calculations.

By Eq. (568), for the above values of $k_3 \div k_2$ and k_1, the most economical tower height is

$$h_c = 30' + 20' = 50 \text{ ft.}$$

for any size conductor and any conductor material within the range covered by the discussion, that is, for aluminum and copper conductors in sizes between 250,000 cir. mils and 2,000,000 cir mils so long as $k_3 \div k_2 = 1{,}500$ and $k_1 = 30$.

Estimating Value of $k_3 \div k_2$.—It will be recalled that the economic study of this chapter is based on the segregation of tower line costs into seven items as given on page 222. By Eq. (557) the total cost of line supports may be expressed in terms of these items by

$$\text{Cost} = \text{No.}(h^2k_2 + k_3)$$

where

No. = total number of line towers (all towers assumed to be alike).

h = height of tower.

k_2 = (item 1 + item 2 + item 3) $\div h^2$.

k_3 = sum of items 4, 5, 6 and 7.

Thus, in order to estimate what the values of the constants k_2 and k_3 will be, it is necessary to secure, from the manufacturer, cost data on towers designed for a definite conductor tension, voltage and height. From such data the constants for another height, voltage and tension may be estimated if no data are available. The process of making such estimate will be illustrated by an example.

Example.—What are the most economical tower spacing and tower height for a 1,000,000-cir. mil, copper cable strung to a maximum tension of 16,700 lb., insulated for 220 kv. between conductors, and having a minimum ground clearance of $k_1 = 30$ ft.?

It is assumed that, as a basis for making estimates, there are available cost data for a 150-kv. tower line in which the tension of the conductors was 6,000 lb. and the towers were 37 ft. high. These data for the latter are as follows:

1. Cost of towers at place of erection	$350.00
2. Cost of erection of towers	70.00
3. Cost of tower site	15.00
4. Cost of foundation	270.00
5. Cost of location and inspection	25.00
6. Cost of insulators at tower site	80.00
7. Cost of placing insulators and cable	20.00

Solution.—Using the above data as a basis, the costs for the proposed line are estimated as follows: The maximum tension for the proposed line is 16,700 lb., whereas the tension for the line to which the above data apply is only 6,000 lb. Since the tower cost is proportional to the tension, the estimated cost of the new towers is

$$\frac{16{,}700}{6{,}000} \times \$350.00 = \$975.00, \text{ approximately.}$$

Likewise, the costs of the foundations are proportional to the tensions, whence the estimated cost of the new foundations per tower is

$$\frac{16{,}700}{6{,}000} \times \$270.00 = \$750.00, \text{ approximately.}$$

If the number of parallel strings, per conductor attachment, remains unchanged, the cost of insulators is roughly proportional to the voltage, whence the cost of insulators per tower for the new line is

$$\frac{220{,}000}{150{,}000} \times 80 = \$117.00, \text{ approximately.}$$

The remaining items of cost remain approximately unchanged. The constants k_1, k_2 and k_3 of Eqs. (556) and (557) may now be found. They are as follows:

k_1 = 30 ft. by conditions of the problem.

k_2 = sum of items 1, 2 and 3 for the new construction, divided by the square of the height of the point of cable support above the tower footing. Thus,

$$k_2 = \frac{975 + 70 + 15}{37^2}$$
$$= 0.774$$
$$k_3 = \text{sum of items 4, 5, 6 and 7}$$
$$= 750 + 25 + 117 + 20$$
$$= 912$$

and

$$k_3 \div k_2 = \frac{912}{0.774} = 1{,}178.$$

It will be observed that the important items are 1, 4 and 6, and that the final result would not be greatly different if all others were neglected entirely. In the present instance, the use of these three items only would yield

$$k_3 \div k_2 = 1{,}210, \text{ approximately.}$$

With the use of Eq. (566), (567) and (568) and the curves of Plates 2 and 3, the most economical span and tower height readily follow, as in the illustrated example already given. They are

$$S = 683 \text{ ft.}$$
$$_{\text{max.}}y = 18.1 \text{ ft.}$$
$$h_e = 48.1 \text{ ft.}$$

PROBLEMS

1. For a given tower line the ground clearance of the attached conductors is 35 ft. The ratio $k_3 \div k_2$ is 2,000. What is the length of the most economical span for a 350,000-cir. mil, copper cable, whose diameter is d_s = 0.682 inch? What is the most economical tower height for type A tower?

2. The costs items 1, 4 and 6 of a certain 165-kv. tower line, designed for a maximum tension of 8,000 lb. and for a height of 40 ft. to the point of conductor attachment, were as follows:

a. Cost of towers at place of erection	$420.00
b. Cost of foundations	300.00
c. Cost of insulators at tower site	75.00

Other items of cost may be considered as independent of tension and voltage. Estimate the most economical tower height, the most economical span and the maximum sag for a 110-kv. line having a ground clearance of 30 ft., and which is designed for a maximum tension of 6,000 lb. on towers.

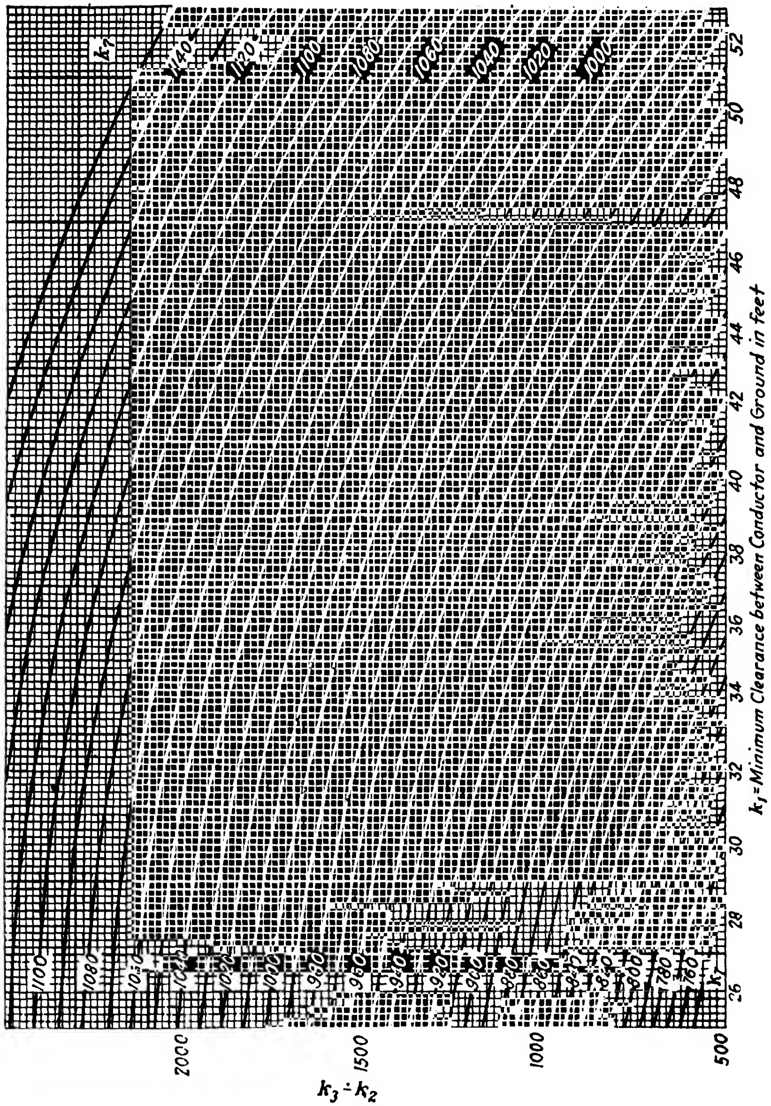


PLATE II.—Curves of constants k_7 for aluminum conductors.

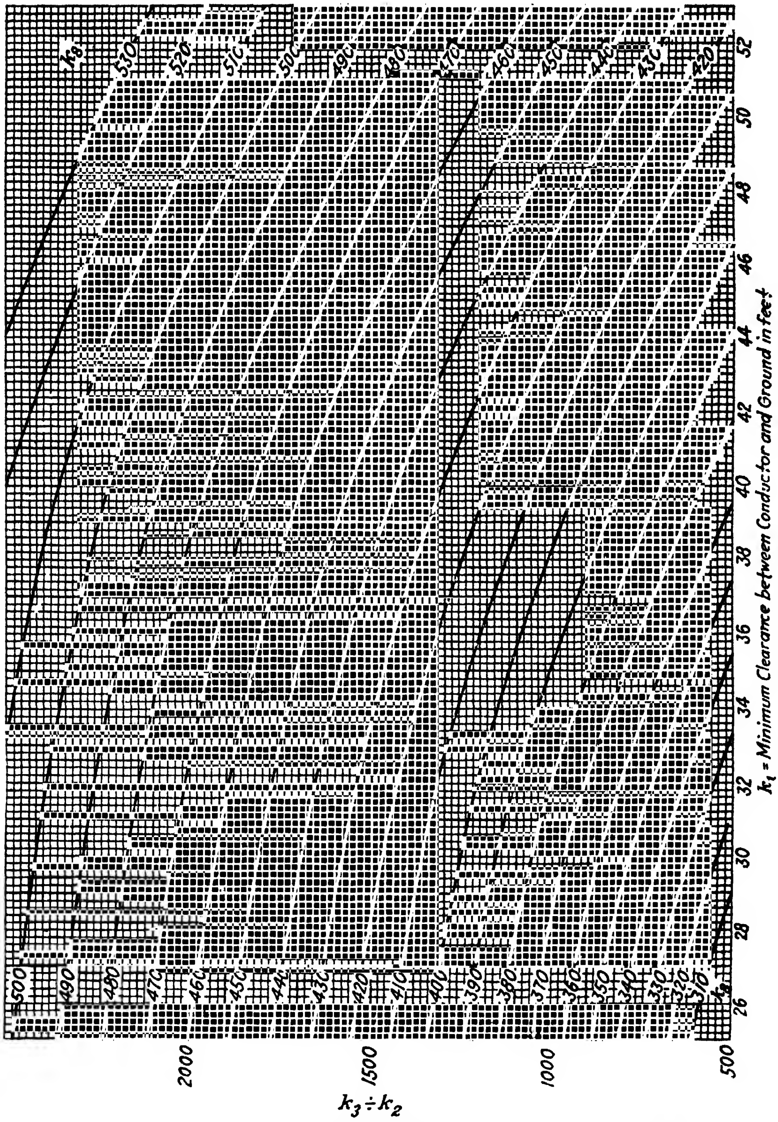


PLATE III.—Curves of constants k_8 for aluminum conductors.

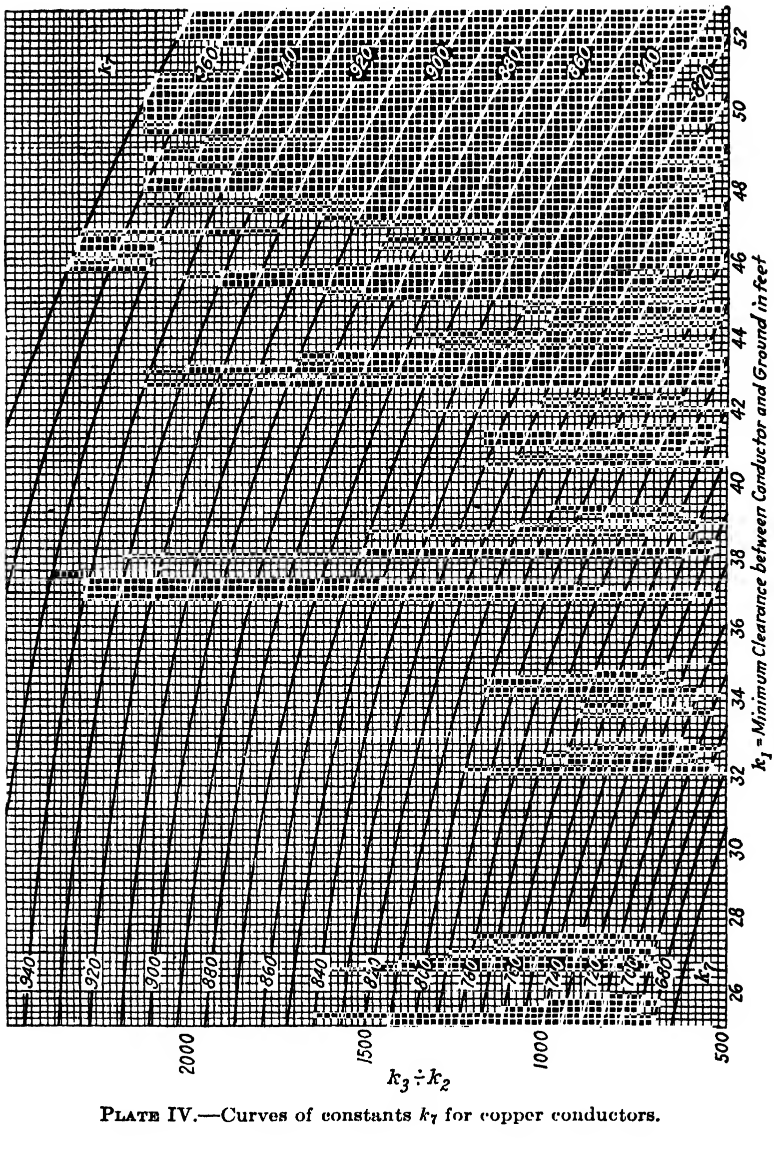


Plate IV.—Curves of constants k_7 for copper conductors.

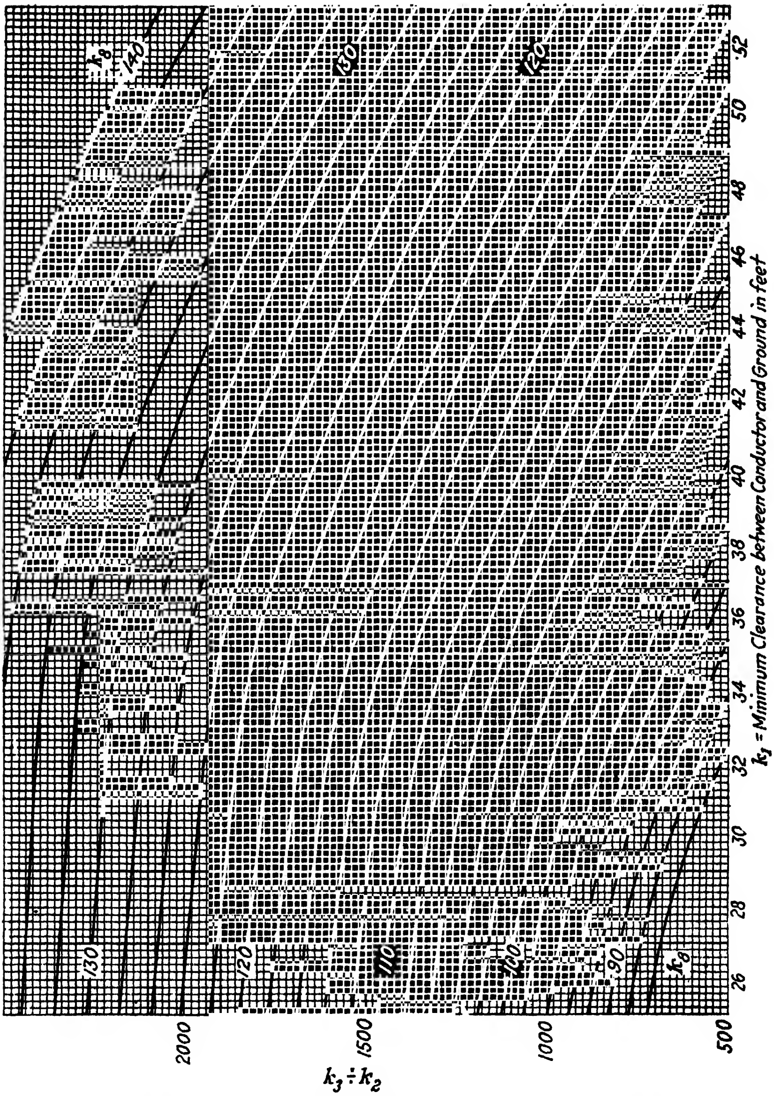


Plate V.—Curves of constants k_8 for copper conductors.

CHAPTER XIII

THE MOST ECONOMICAL VOLTAGE AND CONDUCTOR DIAMETER[1,2]

Given the location of the project, the load to be transmitted and the distance of transmission, the problem of designing a long line divides itself naturally into three parts; namely

1. Determination of the most economical voltage and conductor size.
2. Selection of the most economical tower designs and spans, together with the proper design of span from a mechanical standpoint.
3. Choice of synchronous reactors for line regulation, and the calculation of the electrical performance of the line.

The second of these three items has already received some attention in Chaps. XI and XII. Much of this will serve as a prerequisite to the present chapter. The first item, namely, the formulation of a method by which the most economical voltage and conductor diameter may be found, is the burden of the present chapter.

While the design of important transmission lines is usually placed in the hands of experts of wide experience whose judgment may frequently come to their assistance where rational methods fail or do not exist, there is always danger in relying too much upon judgment which cannot be verified by careful engineering calculations. And, strangely enough, the very part of the problem of calculating the design of a transmission line, where the careful application of the underlying scientific principles is of first importance, is the place where greatest uncertainty usually exists, namely, in the calculation of the most economical voltage and conductor areas for a given project.

Many have contributed towards systematizing and rationalizing the general procedure in transmission-line design, and much

[1] "Transmission Line Design II. The Line of Maximum Economy," A. I. E. E., *Jour.*, 1925, KIRSTEN, F. K., and E. A. LOEW; also University of Washington Engineering Experiment Station *Bull.* 32

[2] "Economy in the Choice of Line Voltages and Conductor Sizes for Transmission Lines." Jour. A. I. E. E. Aug. 1928, p. 561. By E. A. Loew.

has been accomplished in the way of supplying solutions to the various problems encountered, particularly in reference to items 2 and 3 above. While the procedure with reference to these items is quite clearly established and well defined, the same is not true of item 1. The solution here is somewhat more involved, and, while the underlying laws are simple enough, their proper application is quite difficult. Here the usual procedure has followed the method of trial and error, and in the final analysis, much was left to the uncertainty of the engineer's judgment. The object of the present chapter is to rationalize the entire procedure as affecting item 1, so far as is possible, and to provide a method of solution that is based on scientific laws, and in which possible errors, such as are likely to result from placing too great a tax upon the designer's judgment, are reduced to a minimum.

In the past it has been the common practice to treat the electrical and the mechanical features of the line separately as though they were quite unrelated quantities. The size and the material of the conductor were first determined, often by a false application of Kelvin's law, in which no consideration of the most suitable line voltage, of the cost of high-tension equipment, or of tower structures entered. Following this, the proper number of towers, their height and weight necessary to support the conductors, with the required ground clearances, were calculated. Usually the operating voltage of the line was more or less arbitrarily fixed, and the choice of conductor material gave evidence not so much of an intimate knowledge of the economic laws involved as of the relative efficiencies of the sales departments of the conductor manufacturers. This procedure has resulted in the construction of lines which do not give maximum service at minimum cost, although Kelvin's law has apparently been applied in making the design computations.

This chapter undertakes to give the engineer a systematic method of attack upon the problem of finding the most economical voltage and conductor diameter, so that he may be assured of supplying the required service at minimum cost.

Kelvin's Law and Its Modification.—The determination of the most economical size of conductor is based on the well-known principal stated in Kelvin's law. The statement of this law has been somewhat modified since it was first proposed, but, even in its modified form, it is now sometimes improperly inter-

preted. The law was first stated by Sir William Thompson (Lord Kelvin), in 1881, but was later modified by Gisbert Kapp into the following more exact form: "The most economical area of conductor is that for which the annual cost of energy wasted is equal to the interest on that portion of the capital outlay which can be considered proportional to the weight of copper (conductor) used." The term interest used in the above statement of the law should be interpreted to mean interest and depreciation.

The law as stated assumes the line voltage to be fixed; that is, the relative economy of conductors of different sizes is to be compared on the basis of an assumed voltage. When a problem in power transmission is first attacked, however, the voltage is usually as much an unknown as is the size of the conductor to be used. A proper choice of conductor size must accordingly be intimately interlinked with a choice of the most suitable transmission voltage. *The problem, therefore, is to find that conductor which, when used with the most economical voltage, will fulfill the requirements of Kelvin's law.*

The truth of this statement is supported by the following argument. If a fixed loss in the line conductors be assumed, the cross-sectional area of conductor required to transmit a given amount of power is inversely proportional to the square of the transmission voltage. As the voltage is increased, the investment in line conductor is reduced. One cannot go on increasing the voltage and reducing the size of conductor indefinitely, however, because a definite limit is set by two controlling economic conditions. The first limit is set by the fact that for every voltage and conductor spacing, other factors being assumed constant, there is a definite size of conductor which is the minimum that may be used. Any further reduction in size will give rise to corona loss from the conductors and will tend to cause incipient sparking or flashovers, thus endangering the line insulation. A second limitation resides in the fact that, as the voltage is increased, the total investment, in the equipment, required to handle a given amount of power increases very rapidly. This is particularly true of transformers, switches, lightning arresters and insulators. *When that point is reached, beyond which any further increase in voltage will entail an increase in annual fixed charges, on high-tension and line equipment, greater than the corresponding decrease resulting from a reduction in size of line conductor,*

there is no further advantage in increasing the voltage. At the point thus determined, the most economical voltage and the most economical conductor size are both found.

Items of Cost Affecting Choice of Voltage and Conductor.—The final choice of conductor will not necessarily be based on the above considerations alone. Other conditions which may affect the size of conductor are regulation, charging current and the cost of auto-transformers required for linking the line with established systems operating at different voltages. Where synchronous reactors are installed to take care of line regulation, due consideration must also be given to the interdependence of the operating voltage and synchronous reactor capacity required for a desired regulation.

Hence the question, as to what voltage should be used on a transmission system carrying a given amount of energy, can only be answered correctly after investigating all cost items of the line and of its operation which are a function of the voltage, as intimated above. Some of these items are: Heat loss, insulators, transformers, high-tension oil switches and disconnecting switches, lightning arresters, housing space for high-tension apparatus, synchronous condenser capacity, tower dimensions, etc.

In the following analysis the above items are grouped under six main headings, namely

1. Load distribution.
2. Line conductors.
3. Conductor supports and line insulators.
4. High tension apparatus and housing.
5. Regulation.
6. Coordination of existing systems to new project by tie lines.

Load Distribution.—Since the conditions, pertaining to the generating station or stations feeding a transmission line, as well as to the load served, are so variable as to make each line a separate and distinct problem, differing in some features at least, from every other line, no general rules can be laid down for estimating the amount of power to be transmitted over the line, which will apply with equal force to all lines. Each project must be studied separately in the light of all the facts available, and from these an estimate may be reached of the service which the lines should render. In certain sections of the country, where waterpower is abundant, transmission of power is usually

associated with its generation in hydroelectric plants. In fact, a considerable percentage of our principal power transmission projects are fed largely from plants of this kind. A method of estimating the load of a proposed line, based on plants of this kind, will therefore serve to illustrate the procedure for a large number of lines, and, perhaps, at the same time, suggest a suitable approach to the problem of finding the required duty of most others.

a. Rated Ultimate Capacity of Generating Machinery.—The basic data, for the designer of a waterpower project, are the total energy obtainable per year from the generating machinery at the power site, together with the curves indicating its probable distribution from week to week throughout the year. These data are obtained from a careful study of the stream-flow measurements taken over a period of years, and a study of the water storage possibilities for the proposed development. The method of analyzing these data need not be considered here. With a given maximum elevation of the storage dam, and with a knowledge of the conditions of flow into, and the amount of storage available in the storage reservoir, a fairly close estimate of the ultimate output of the generating station can be made.

The capacity of the generating machinery, however, depends not only upon the average amount of energy obtained by averaging the total available yearly energy over the period of one year, but it also depends upon the distribution of the daily demand at the receiving end of the transmission line over the hours of the day. Since the load demand of the distribution system usually fluctuates over a wide range during the day, and since this variation differs from day to day and from month to month, the generating machinery must have enough capacity to supply the maximum annual peak instead of the average daily demand. Where auxiliary plants are provided to supply the peak demand of the distribution system, the required capacity of the generators at the power house is more nearly equal to the average yearly demand.

b. Load Factor.—The ratio, of the average daily power demand for the year to the maximum or half-hour peak demand upon the generators for the year, is called the *average half-hour, yearly load factor* of the power station For western coast cities this factor usually lies between 40 and 65 per cent, depending upon the nature of the load connected to the distribution system.

c. Distribution Curve.—The load factor can only be approximated by the designer after a careful study of the probable, ultimate, daily load distribution from past records of the distribution system. From this study the total ultimate capacity, in kilovolt-amperes of the generating station is obtained by dividing the average daily energy available by 24 times the load factor. If the maximum daily peak demand is of short duration, the generating machinery may be operated under an overload for that period, thereby increasing the load factor.

Having determined the average daily load distribution of the existing distribution system, the average, daily distribution curve for the ultimate system, including the new development, may be found by multiplying the ordinates of the original curve (kilowatts) by a constant, of such magnitude that the area under the new curve, in kilowatt-hours, represents, for the ultimate system, the average daily supply from all power sources feeding the system in question. The constant is quickly determined by integrating the given load-distribution curve with the planimeter and dividing the average daily energy yield of the generating system by this integrated area in kilowatt-hours. The capacity of the generating machinery is found by multiplying the maximum ordinate of the derived curve by the ratio of the maximum peak of the year to the maximum peak of the average day. Other considerations, as auxiliary plants, peak load, power delivered over tie lines from other systems, concurrence of high water conditions with maximum peak load demands, etc., may modify the determination of the generating capacity considerably and must be studied separately for each individual project.

Root-mean-square Kilowatts and Average Heat Loss in the Line.—Having derived the average daily distribution curve for the year, it is now possible to determine the "root mean square" of the average load. The root mean square of the average load is that quantity which, if supplied continuously throughout the year, would give rise to the same total heat loss in the line conductors as is dissipated in them under actual operating conditions. Since the heat energy dissipated in the line conductors is proportional to the square of the line current, assuming constant voltage and power factor, the line loss at any moment is proportional to the square of the power transmitted, and the average line loss for the year is proportional to the average square of the power transmitted, hour by hour throughout the year. The

r.m.s. average load is then found by taking the square root of the average square of the ordinates to the mean daily load curve for the year.

Figure 74 shows in full line a typical mean daily load curve for a western community. The ordinates can be drawn to any arbitrary scale representing kilowatts. From this curve

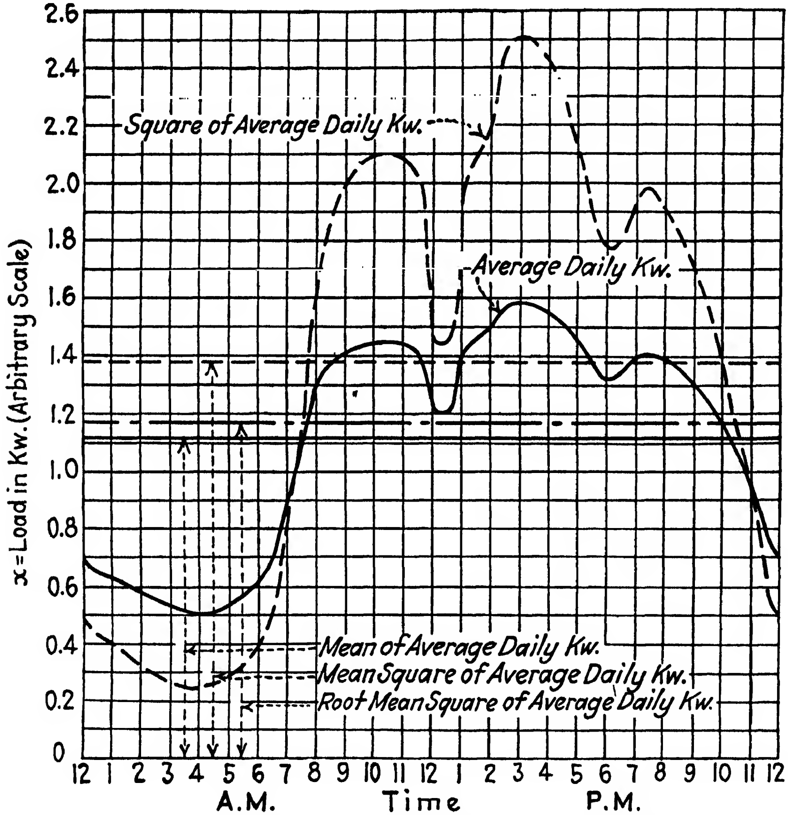


FIG. 74.—R.m.s. kilowatts from average daily load curve.

the $(\text{kw.})^2$ curve, shown dotted, is derived by squaring the ordinates of the kilowatt curve. Applying the planimeter to the latter curve and dividing the integrated area by 24 gives the average $(\text{kw.})^2$ for the day. As a check, the area below the horizontal dotted line should equal the area of the $(\text{kw.})^2$ curve. The square root of the average $(\text{kw.})^2$ is that amount of energy which, if transmitted continuously, will give rise to the same heat loss as the variable load actually transmitted under operating

conditions. The quantity thus derived will hereafter be called the *root-mean-square kilowatts.* (569)

Figure 74 shows that the r.m.s. kilowatts is a quantity greater than the average daily kilowatts.

Line Conductors.—Under this heading should be collected all data pertaining to the conductors, which may be of value in finding the cost of the energy annually wasted as line loss, the annual charge against the first cost of conductor, and the influence of the mechanical features of the conductor upon the cost of the line towers. Since there is usually a choice of several conductor materials, as for instance, copper, aluminum and steel and also a choice of composite cables of steel-core aluminum, copper-clad steel, or of cables of special mechanical construction, such as hollow cables made up of a spiraled inner tube with cable strands spiraled on the surface of the tube, the electrical and mechanical characteristics of all cables which come into consideration should be obtained.

The electrical characteristics of a given cable at once fix the line loss per kilowatt-hour per unit metallic section of the cable. The line loss is determined as follows:

a. Energy Loss.—The total power wasted in the conductors of a three-phase line is

$$P = 3RI^2$$

where

P = watts loss in the line.
R = the total resistance of a single line conductor in ohms.
I = equivalent average current in amperes which will give rise to the same heat loss per year as the integrated losses due to the actual current of the daily variable load.

The energy wasted during the year is

$$\begin{aligned}\text{Watt-hrs.} &= Pt \\ &= 3 \times 8{,}760RI^2 \text{ watt-hrs.}\end{aligned} \tag{570}$$

and the value of the annually wasted energy is

$$\begin{aligned}\text{Cost} &= \frac{3 \times 8{,}760RI^2A}{1{,}000} \\ &= 26.28RI^2A \text{ dollars}\end{aligned} \tag{571}$$

where

A = value of electrical energy at the receiving end of the line, in dollars per kilowatt hour.

If a stranded conductor be used, remembering that its diameter[1] $d_s = 1.153d$, where d_s and d are the diameters of stranded conductor and equivalent solid rod, respectively, in inches,

$$R = \frac{1.02\rho L(1.153)^2 \times 10^{-6}}{d_s^2} \tag{572}$$

where

ρ = resistance of conductor material in ohms per mil-foot
L = actual length, in feet, of the line conductor measured from the generating station to the receiving station along the catenaries formed in suspension.
1.02 = A.I.E.E. resistivity factor introduced to take care of increased resistance due to stranding.

Substituting Eq. (572) into (571),

$$\text{Cost} = 35.64 \times 10^{-6}\rho LA\frac{I^2}{d_s^2} \text{ dollars.} \tag{573}$$

Equation (573) gives the value of the energy lost in transmission in terms of two variables for a given conductor material. These two variables are the equivalent average current I, and the overall diameter of the cable in inches. If composite cables are used, the ratio, of the area of a circle drawn tangent to the outermost elements of the surface layer of strands, to actual metallic section, is $(1.153)^2$ approximately, provided the stranding of these cables is the same as that of cables made of one kind of material. The constant ρ, however, must be the derived electrical resistance per mil-foot of some equivalent material which would yield the same resistance if made up into a cable of the same diameter as the composition cable. If steel is a part of the conductor, the resistance is a function of the frequency. Special attention should here be called to the fact that the constant 1.153 can only be used in connection with cables of standard stranding. The constant, for cables of special mechanical construction, such as hemp-core, tubular conductors and cables of special stranding, must be derived and is the ratio of the diameter of a circle enveloping the extreme outer elements of the cable to the diameter of the equivalent solid rod. It is important to state Eq. (572) in terms

[1] The ratio $d_s \div d$ varies between 1.1508 and 1.1536 for the larger cables as shown on p. 61. The value 1.153 is taken as the correct value for the larger cables.

of the overall diameter of the cable, as will be apparent from subsequent derivations.

b. Fixed Charges against Line Conductors.—The volume of the three cylindrical line conductors is

$$V = \frac{3\pi d_s^2 L}{4 \times 144 \times (1.153)^2} \text{ cu. ft.} \tag{574}$$

and the annual fixed charge against the conductor alone is

$$\text{Conductor fixed charge} = \frac{p_1 VBW}{100} \text{ dollars} \tag{575}$$

where

p_1 = per cent interest and depreciation
B = cost of conductor material delivered, in dollars per pound.
W = weight of conductor material in pounds per cubic foot.

Substituting Eq. (574) in (575),

$$\text{Conductor fixed charge} = \frac{3\pi d_s^2 p_1 BWL}{4 \times 144 \times (1.153)^2 \times 100} \text{ dollars}$$

$$= 12.31 \times 10^{-5}\, d_s^2 p_1 BWL \text{ dollars.} \tag{576}$$

Equation (576) shows that the annual fixed charge against the conductor alone, for a given conductor material, is a function of the square of the overall diameter d_s, which appears as the only variable.

Conductor Supports and Line Insulators. (*a*) *Types of Towers.* For long-distance, high-potential lines, only two general types of towers need be considered by the designer, namely the single-circuit and the double-circuit tower. Figure 75 shows the two types. Type *A* represents the single-circuit tower with the three conductors arranged horizontally on one crossarm, and type *B* the double-circuit tower with the two circuits arranged vertically, one on each side of the tower, supported by three crossarms. On the west coast of America, where timber is still plentiful, the wood-pole tower is still an economic temptation, but with the gradual increase in timber cost and an increasing demand for greater permanency of construction for important trunk lines, this type will soon vanish from the field of long-distance transmission; and, for that reason, is not considered here.

(*b*) *Comparison of Towers Type A and Type B. Mechanical Features.*—For a given conductor clearance to ground, type *B*

tower is much higher than type A. Therefore, the wind load on the conductors and the structure is much greater than for type A tower. This necessitates an increased weight of all members and of the foundation. Furthermore, tower B must be designed for the tension of six conductors, and this tension being very nearly twice that for which tower A is designed, acts, in addition, with a much greater moment about the foundation. In case of

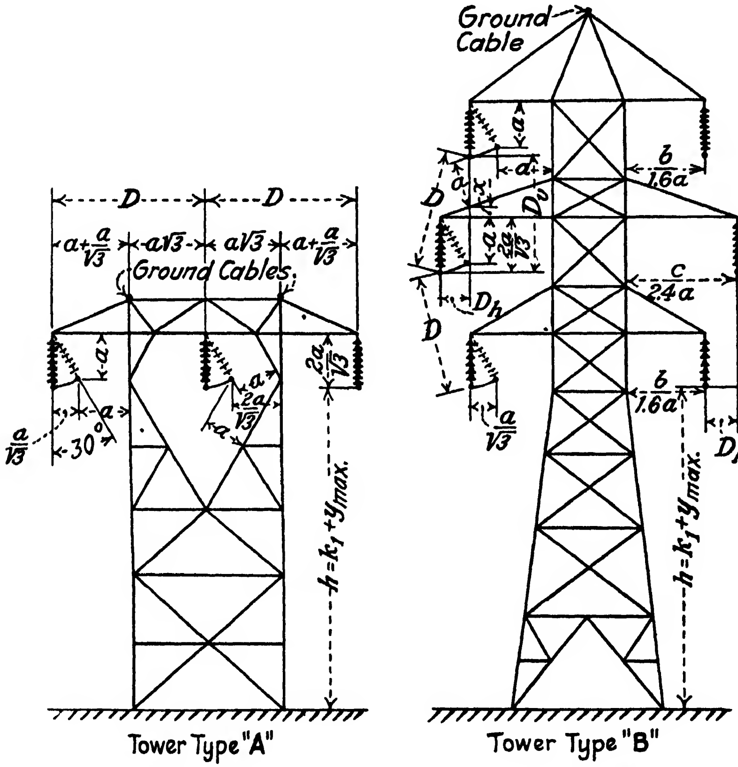


FIG. 75.—Single circuit and double circuit towers.

mechanical failure of one circuit on the double-circuit tower, a possibility which must be considered, the total tension of the remaining three conductors acts as a twisting moment on the tower structure, whereas the total twisting moment for type A tower is that due to the tension on one conductor only. Hence, for the same conductor, the structural members of the double-circuit tower must be heavier than those of the single-circuit tower. The width of the line right-of-way is materially decreased

by using tower type *B*, however, and this item may prove a most important factor in the choice between the two types. Another factor in favor of tower *B* is the need of only one ground cable which, when placed at the apex of the upper crossarm, furnishes protection against lightning disturbances for two circuits, whereas type *A* tower requires two ground cables located on the upper crossarm, as shown in sketch, so that the number of ground wires per line is four times as great for type *A* as for type *B*. Very often only one cable is used for type *A*, but two cables are more desirable from the standpoint of reducing twisting moments on the tower structure and foundations, and distributing these moments over adjacent towers in case of conductor failures. The distribution of such stresses to adjacent towers, by the ground cable of type *B* tower, is less effective in relieving the tower structure, especially in case of failure of the upper conductors, in which case the stresses must pass through half the tower structure before they can be transmitted to other towers by the ground cable.

c. Comparison of Type A and Type B Towers. Electrical Features.—It will be shown later that, for a given line voltage, the spacing between conductors is less for type *B* than for type *A* towers. Hence the inductance per phase is less and the capacitance greater than for type *A*. This may prove an advantage or disadvantage, depending upon the length of line and the character of its electrical load. For short lines of large capacity it is an advantage and for long lines of relatively small capacity it is likely to be a disadvantage.

Due to the offset from a vertical plane of the center conductor, the tower *B* arrangement approaches equilateral conductor arrangement, and hence, the phase performance along the line will be more nearly balanced for the three conductors than for the conductor arrangement of tower *A*. The phase unbalance is further aggravated by tower *A*, inasmuch as the center conductor must pass through a closed iron circuit, giving rise to hysteresis and eddy-current loss, which, however slight, is entirely absent in tower *B*.

Danger of insulator flashovers becoming short circuits between conductors is very improbable for type *A* towers, whereas these flashovers have a natural tendency to spread vertically, thus tending to aggravate the destructive effects of these flashovers for type *B* towers.

For very high potential lines on type *B* towers, repair work on one line may necessitate the shutting down of the other line also, to insure safety to the workmen. This would make the use of type *A* towers not only desirable but almost necessary.

Type *A* tower lends itself very much better than type *B* to a step development of large projects if the magnitude of each step approximates the normal capacity of one line only.

It is not desirable to use different types of transmission towers for parallel lines conveying the energy of a power source to the same distribution center, since, as explained above, the electrical characteristics of these lines would differ.

From the above it is evident that the designer can arrive at the proper choice of tower type only after careful consideration of all factors involved, especially factors of cost; and the cost factors should be analyzed for both type *A* and type *B* towers before giving too much weight to the saving in right-of-way or to the desirability of least possible service interruption. The choice of tower type also influences the size of conductor required to carry a given amount of power, as will be demonstrated in the subsequent discussion.

d. Conductor Spacings.—The clearances which must be provided between the conductor of a line, or between line and ground, are determined largely by considerations of electrical safety. Sufficient clearance should be provided to allow a reasonable factor of safety against sparkover between conductors, or between any one conductor and the tower structure, under the worst weather conditions and with one conductor grounded. In regions where large birds may be the cause of electrical failure the separation of conductors should be sufficient to prevent the likelihood of short circuits from the wings of such birds.

Where two circuits are strung on a single line of towers, the middle conductor is usually offset from the vertical plane containing the upper and lower conductors. The offset is usually away from the tower and amounts to from 2 to 4 ft. This is done to prevent short circuits in a given span due to temporarily decreased clearances which might arise from the dropping of the sleet load from one or more conductors. The swinging of conductors due to the wind is not usually regarded as a serious problem particularly with heavy conductors, since they swing together and maintain more or less constant clearances. The vertical clearances between conductors are usually somewhat less than

the horizontal. The spacing of conductors depends also somewhat upon the length of the span. The clearance between conductor and tower structure must take account of the possibility of the swing of the conductor with the wind. The probable angle of swing is estimated, and the clearances are chosen accordingly. While there is considerable divergence in the practice followed by different companies, the curves of Fig. 76 probably represent present-day average practice fairly well.

e. Clearances to Ground.—Clearances of conductors to ground vary considerably with the location and with the necessity for taking precautions. They should always be sufficient to avoid

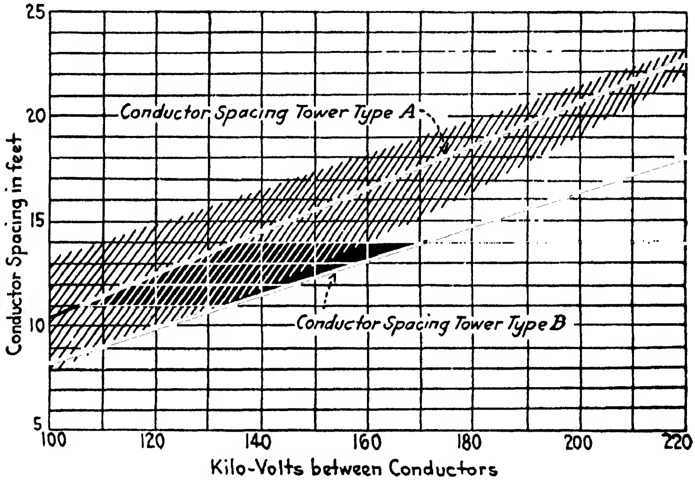


FIG. 76.—Conductor spacings.

the possibility of harm coming to persons, or damage to property from the proximity of other electrical lines. Over highways, railways or waterways special precautions and greater clearances than ordinary are required. In open country from 20 to 25 ft. is usually deemed sufficient. In some states clearances are specified by law. Recommended ground clearances as well as conductor spacings may be found in the National Electrical Safety Code.

Factor of Safety.—The actual sparkover distance between conductors is approximately 1 in. per 10,000 volts. Most systems go so far as to provide for safe operation with one conductor grounded. For that emergency, the voltage between the remain-

ing conductors and the tower structure is not the voltage to neutral, but the full voltage normally between conductors. The grounding to the tower of one of the line conductors naturally decreases the electrical safety factor of the line insulation in the proportion of $1 \div \sqrt{3}$, so that, if the normal safety factor of the line be 6.5, it would be reduced to $6.5 \div \sqrt{3}$ or to 3.75 in case of the grounding of one conductor. Using a factor of safety of 6.5, for long lines under normal operation, will provide 6.5 in. of clearance for every 10,000 volts of operating potential. This will assure sufficient safety for transient potential waves of considerable magnitude.

The choice of a safety factor of 6.5 is rather arbitrary, but seems to fit present practice very well, as shown by Fig. 76. The shaded area represents the extreme ranges of general spacing practice, as found from many present installations, and represents the maximum range as reported in the handbook of the Aluminum Company of America, on page 72. The conductor spacing curves for type A and type B towers were obtained by applying the safety factor 6.5 to both types of towers in connection with actual sparkover distances under extreme wind deflections of the insulators.

f. Empirical Equation for Spacing D.—Referring again to Fig. 75, the minimum clearance of the conductor to the nearest tower member under maximum wind load on the cable is designated a. With the assumption that the maximum deflection of an insulator string is 30° from its normal vertical position, and that, for type A tower, the structural members which form the tower frame on each side of the central conductor are also inclined at an angle of 30° from the vertical, the spacing between adjacent conductors is

$$\begin{aligned} D &= a + \frac{a}{\sqrt{3}} + a\sqrt{3} \\ &= 3.31a \quad \text{for type } A \text{ tower.} \qquad (577) \end{aligned}$$

The diagram of type B tower shows the center conductor projected out a distance designated as D_h from the vertical plane which intersects the other two conductors. This is necessary in order to provide safe clearance near the middle of the cable span in case of unequal sleet loading of the three conductors.

For the type B tower it is assumed that the ratio of the projection upon a vertical plane of the actual conductor spacing D,

designated D_v, to the projection of D upon a horizontal plane, designated D_h, is fixed as $3 \div 1$. Likewise, it is assumed that the slope of the upper member of the middle crossarm with respect to a horizontal plane is $1 \div 3$. From these assumptions, the spacing between adjacent conductors is

$$\begin{aligned} D_v &= \frac{3D}{\sqrt{10}} \\ &= \frac{2a}{\sqrt{3}} + x + \frac{a\sqrt{10}}{3}. \end{aligned}$$

But

$$\begin{aligned} x &= \frac{D_h}{3} \\ &= \frac{D}{3\sqrt{10}} \end{aligned}$$

hence

$$D = 2.62a \text{ for type } B \text{ tower.} \tag{578}$$

Designating the operating voltage to neutral of the line by E_n, and allowing $10{,}000 \div 6.5$ volts per inch,

$$a = \frac{E_n}{1{,}538} \text{ in.} \tag{579}$$

Combining Eq. (579) with Eqs. (577) and (578),

$$\left.\begin{aligned} D &= 0.00215E_n \text{ for type } A \text{ tower} \\ D &= 0.00170E_n \text{ for type } B \text{ tower} \end{aligned}\right\} \tag{580}$$

Any other special arrangement of conductors, on towers different from the general forms, shown by Fig. 75, may be similarly analyzed for a relationship of E and D, resulting, probably, in a change in magnitude of the constants of Eq. (580).

g. Equivalent Spacing.—To equalize the electrical performances of all three phases of a transmission circuit, the conductors of which are unequally spaced, as exemplified by Fig. 75, it is necessary to provide a minimum of two complete transpositions, the transposition points dividing the line into sections of approximately equal length. Additional transpositions may be desirable or even required by law, in order to reduce to a safe working value inductive interference with adjacent telephone circuits. A circuit of unequal conductor spacings, thus transposed, will

perform like a three-phase circuit without transpositions, the conductors of which are equidistant from each other, *i.e.*, arranged so that lines through their axes form an equilateral triangle, provided that

$$D' = \sqrt[3]{D_1 \times D_2 \times D_3} \tag{581}$$

where D_1, D_2 and D_3 are the actual distances, center to center, of the three line conductors properly transposed, and where D' is the equivalent equilateral spacing of the same circuit without transpositions but yielding the same performance. The proof of this relationship is found in Eqs. (160) and (252).

The equivalent triangular spacing, as applied to Fig. 75 is, therefore,

$$\left.\begin{aligned} D' &= \sqrt[3]{2D^3} \\ &= 1.260D \text{ for type } A \text{ tower} \\ \text{and} \qquad & \\ D' &= \sqrt[3]{\frac{6D^3}{\sqrt{10}}} \\ &= 1.238D \text{ for type } B \text{ tower} \end{aligned}\right\} \tag{582}$$

h. Relation of Transmission Voltage and Conductor Diameter.— It has already been stated that in a proper application of Kelvin's law all items of cost, which enter into the completed project and whose values depend upon either the line voltage or the conductor area, must be considered. For convenient use in the solution of problems it is most desirable to express the law in the form of a mathematical equation in which the conductor diameter appears as an explicit function of the load to be transmitted. To do this requires the formulation of the mathematical law by which the voltage and conductor diameter, in an economically designed line, are or should be related. This is so because certain of the cost items in question, such as the cost of transformers, circuit-breakers, towers, insulators, and lightning arresters, are usually supplied by the manufacturer for a given operating voltage. The law, of variation of cost with voltage for these items, is built up from the manufacturers' quotations, as will be explained in greater detail later. The law of variation of cost may then also be expressed as a function of the conductor diameter as soon as the relation which should exist between conductor diameter and line voltage is known.

It was explained at the outset that, if the conductor diameter is assumed to be fixed, the line losses incident to the transmission of a given load over the line are a minimum, when the voltage impressed is the highest practical value, *i.e.*, when the voltage is just under the critical disruptive value. Therefore, the basic assumption is made that *for maximum economy all lines should be operated at an average voltage somewhat below the critical, disruptive value for the conductor used*, but yet as high as is practical.

The interrelation of voltage and conductor diameter sought is expressed by the law of the corona (Eq. (287)) as follows:

$$ {}_cE_n = 2.302 m_0 g_0 \delta \frac{d_s}{2} \log_{10} \frac{2D'}{d_s} $$

where

$_cE_n$ = the critical disruptive voltage to neutral in volts.
m_0 = the irregularity factor.
g_0 = the dielectric strength of air at standard temperature and pressure in volts per inch.
= 53,600 volts (effective value) per inch.
δ = the altitude factor.
d_s = the diameter of stranded conductor in inches.
D' = the equivalent equilateral triangular spacing between line conductors in inches.

If the inverse of the ratio, critical disruptive voltage to actual line voltage, be represented by γ, the relation between the two voltages may be expressed by the equation

$$ E_n = \gamma\, {}_cE_n. \tag{583} $$

For short lines, where control of the generator excitation alone is largely depended upon to maintain constant voltage at the receiving end, γ may have to be as low as 0.8 or 0.85 in order to maintain a suitable difference between actual line voltage and corona voltage at all loads. For longer lines or, in general, lines in which phase control is employed, γ may well be 0.85 to 0.90.

Substituting E_n from Eq. (583), the law of the corona becomes

$$ E_n = 1.151 \gamma m_0 g_0 \delta d_s \log_{10} \frac{2D'}{d_s}. \tag{584} $$

For all practical long-distance transmission circuits the quantity $2D' \div d_s$ lies well within the limits of 100 to 1,000 and, consequently, the logarithm of this quantity is never less than 2 nor greater than 3. Within this limited range, a graph, expressing

the relationship of the logarithm of $2D' \div d_s$ to the quantity $2D' \div d_s$ itself, is almost identical with a parabola of the form,

$$y = 2 + 0.034\left(\frac{2D'}{d_s} - 100\right)^{\frac{1}{2}}. \qquad (585)$$

This statement may readily be verified by calculation as shown in Table 22.

TABLE 22

$\frac{2D'}{d_s}$	$\log_{10}\frac{2D'}{d_s}$	$2 + 0.034\left(\frac{2D'}{d_s} - 100\right)^{\frac{1}{2}}$
100	2.00	2.00
200	2.30	2.34
300	2.48	2.48
400	2.60	2.59
500	2.70	2.68
600	2.78	2.76
700	2.84	2.83
800	2.90	2.91
900	2.95	2.96
1,000	3.00	3.02

The above table shows that the value y computed from Eq. (585) may be substituted for the logarithm of $2D' \div d_s$ for all long transmission lines.

Substituting y from Eq. (585) into Eq. (584) for the value of $\log_{10}\frac{2D'}{d_s}$,

$$1.151\gamma m_0 g_0 \delta d_s\left[2 + 0.034\left(\frac{2D'}{d_s} - 100\right)^{\frac{1}{2}}\right] \qquad (586)$$

and, incorporating the relationships expressed by Eqs. (580) and (582), Eq. (586) becomes

$$E_n = 1.151\gamma m_0 g_0 \delta d_s\left[2 + 0.034\left(\frac{k_9 E_n}{d_s} - 100\right)^{\frac{1}{2}}\right] \qquad (587)$$

where

$$k_9 = 0.0054 \text{ for type } A \text{ tower}$$
$$= 0.00421 \text{ for type } B \text{ tower.}$$

Letting

$$k_{10} = 1.151\gamma m_0 g_0 \delta$$

and solving Eq. (587) for E_n in terms of d_s, there results

$$E_n = d_s k_{10}[2 + 57.7\times10^{-5}k_9 k_{10} \pm \sqrt{2 + 57.7\times10^{-5}k_9 k_{10} - 4.11}]$$
$$= d_s U \qquad (588)$$

TABLE 23

$m_0\gamma\delta$	Tower type A		Tower type B	
	U_A	Difference	U_B	Difference
0.35	53,510	......	51,330	
0.40	62,540	9,030	59,950	8,620
0.45	71,830	9,290	68,790	8,840
0.50	81,320	9,490	77,780	8,990
0.55	91,030	9,710	86,990	9,210
0.60	100,930	9,900	96,340	9,350
0.65	111,070	10,140	105,890	9,550
0.70	121,390	10,320	115,590	9,700
0.75	131,910	10,520	125,460	9,870
0.80	142,620	10,710	135,470	10,010
0.85	153,520	10,900	145,680	10,210
0.90	164,580	11,060	156,020	10,340

where

$$\left.\begin{aligned}U_A &= k_{10}[2 + 3.13 \times 10^{-6}k_{10} + \\ &\quad \sqrt{(2 + 3.13 \times 10^{-6}k_{10})^2 - 4.11}] \text{ for type } A \text{ tower} \\ U_B &= k_{10}[2 + 2.43 \times 10^{-6}k_{10} + \\ &\quad \sqrt{(2 + 2.43 \times 10^{-6}k_{10})^2 - 4.11}] \text{ for type } B \text{ tower}\end{aligned}\right\} \quad (589)$$

Equation (589) furnishes the proportionality factors relating the most economical transmission voltage and the conductor diameter for the types of towers in question. By means of Eq. (589), values of U were computed for various assumed values of the product $m_0\gamma\delta$, as listed in Table 23. To find U for any values of m_0, γ and δ, all that is required is to enter the table with the product $m_0\gamma\delta$ and select the corresponding value of U. Values

TABLE 24.—ALTITUDE CORRECTION FACTOR

Altitude in feet	δ	Altitude in feet	δ
0	1.00	5,000	0.82
500	0.98	6,000	0.79
1,000	0.96	7,000	0.77
1,500	0.94	8,000	0.74
2,000	0.92	9,000	0.71
2,500	0.91	10,000	0.68
3,000	0.89	12,000	0.63
4,000	0.86	14,000	0.58

of the altitude correction factor δ, as given in the standard Handbook, are found in Table 24.

f. Reactance and Susceptance per Mile of Line Relatively Independent of Conductor Diameter and Line Voltage.—For transmission lines which are so constructed that the separation between conductors is a straight line function of the voltage, as for example in Eq. (580), and for which the impressed line voltage is proportional to the conductor diameter, as is assumed in the present discussion (Eq. (588), the inductive reactance and the susceptance per mile of line at a given frequency are both practically independent of the diameter of the conductor used, provided the roughness factor m_0 and the altitude factor δ do not vary, and that the operating voltage is always a constant percentage of the critical disruptive value.

The truth of the above statement is readily verified. Let it be assumed that, for all lines under consideration, the voltage chosen is always 90 per cent of the critical value for the conductor, that the altitude is 1,000 ft., whence $\delta = 0.96$, and that the roughness factor is constant and equal to 0.83. Then

$$\begin{aligned} m_0\gamma\delta &= 0.9 \times 0.96 \times 0.83 \\ &= 0.717. \end{aligned}$$

From Table 23, for this value of $m_0\gamma\delta$, $U_A = 125{,}000$ and $U_B = 119{,}000$ in round numbers.

By Eq. (133) the inductance per mile of one conductor is

$$L = \left(741.13 \log_{10} \frac{D'}{r} + 80.47\right) \times 10^{-6} \text{ henries}$$

and by Eq. (213) the capacitance per mile of one conductor to neutral is

$$C = \frac{0.03883 \times 10^{-6}}{\log_{10} \frac{D'}{r}} \text{ farads.}$$

The corresponding values of inductive reactance and susceptance at 60 cycles are

$$x = 377L$$

and

$$b = 377C.$$

For towers designated as types A and B, respectively, the spacings in inches are given by Eq. (580) as

$$D = 0.00215E_n \quad \text{for type } A$$

and

$$D = 0.00170E_n \text{ for type } B$$

whence, for the spacings designated, the equivalent equilateral spacings, by Eq. (582) are

$$D' = 1.260 \ D \text{ for type } A$$

and

$$D' = 1.238 \ D \text{ for type } B$$

or, in terms of the impressed voltage to neutral,

$$\left.\begin{aligned} D' &= 1.260 \times 0.00215E_n \\ &= 0.002709E_n \qquad \text{for type } A \\ \text{and} \\ D' &= 1.238 \times 0.00170E_n \\ &= 0.002105E_n \qquad \text{for type } B \end{aligned}\right\} \tag{590}$$

By Eq. (588),

$$E_n = d_sU = 2rU$$

whence

$$\left.\begin{aligned} r_A &= \frac{E_n}{2U_A} \qquad \text{for type } A \\ \text{and} \\ r_B &= \frac{E_n}{2U_B} \qquad \text{for type } B \end{aligned}\right\} \tag{591}$$

Substituting the values of U_A and U_B, for the assumed conditions in Eq. (591), and the values of r and D' from Eqs. (590) and (591) in the appropriate equations for x and b yields

$$\left.\begin{aligned} x_A &= 377 \times 10^{-6} \, (741.13 \log_{10} 677.25 + 80.47) \\ &= 0.821 \text{ ohm per mile at 60 cycles for type } A \\ \text{and} \\ x_B &= 377 \times 10^{-6} \, (741.13 \log_{10} 500.99 + 80.47) \\ &= 0.784 \text{ ohm per mile at 60 cycles for type } B \end{aligned}\right\} \tag{592}$$

Similarly,

$$\left.\begin{aligned} b_A &= \frac{377 \times 0.03883 \times 10^{-6}}{\log_{10} 677.25} \\ &= 5.17 \times 10^{-6} \text{ mho per mile at 60 cycles for type } A \\ \text{and} \\ b_B &= \frac{377 \times 0.03883 \times 10^{-6}}{\log_{10} 500.99} \\ &= 5.42 \times 10^{-6} \text{ mho per mile at 60 cycles for type } B \end{aligned}\right\} \tag{593}$$

For 25-cycle lines the corresponding values of x and b are

$$\left.\begin{aligned} x_A &= 0.342 \\ x_B &= 0.327 \end{aligned}\right\} \text{ohm per mile at 25 cycles for type } A \qquad (594)$$

and

$$\left.\begin{aligned} b_A &= 2.15 \times 10^{-6} \\ b_B &= 2.26 \times 10^{-6} \end{aligned}\right\} \text{mho per mile at 25 cycles for type } B \qquad (595)$$

While in practice the spacings will not always be those here assumed, yet for high-voltage lines they will usually not deviate greatly therefrom, and hence, for preliminary calculations, the values of x and b here given are quite satisfactory.

j. Derived Line Constants.—For purposes of estimating, and to serve as a check upon calculated values of the derived line constants a_1, a_2, b_1, b_2, c_1 and c_2, tables of these constants have been worked out for 60-cycle lines up to 500 miles for each of two conductor materials, namely, aluminum and copper, and for both types A and B construction. These tables are based upon the values of x and b derived above and upon values of resistance estimated from the conductor diameters and appropriate resistivities. The tables are found in Appendix D. It is of interest to observe that the constants a_1, b_2 and c_2 are roughly proportional to the length of the line and are relatively independent of the kind of conductor material, and almost entirely independent of the conductor area, for the larger sizes of conductors especially.

Tower Cost and Kelvin's Law.—In order to write a complete mathematical expression for the economic relations involved in the generalized statement of Kelvin's law it is necessary to analyze the cost of every item of expense incurred in the construction of the transmission line, and to pick out those which are functions of the conductor area. The annual charge against the latter is then to be expressed as functions of conductor diameter, if possible, and this expression is to be incorporated as a part of the general statement of the law. Since the cost of the transmission-line towers is an important one of these cost items, it will be investigated with this end in view.

From Chap. XII on the Economics of Span Design, the validity of the following statements is apparent.

1. For a given material and a given cable diameter, there is only one tower spacing which yields minimum cost of tower. This has been called the most economical tower spacing. (596)

2. The most economical tower spacing is a function of the cable diameter, as shown by Eq. (566). (597)

3. The most economical tower spacing is different for different conductor materials as shown by the curves, Fig. 73 and Eq. (566). (598)

4. The most economical height h_e of conductor support, is independent of either cable diameter or conductor material for a given ratio $k_3 \div k_2$, as shown on page 231. (599)

The above statements are important considerations in the analysis which follows, and are here set down for future reference.

Items of Tower Cost Analyzed.—The principal factors affecting the cost of towers are:

1. *Conductor Tension, and Wind and Snow Loads.*—The tower must be designed to withstand the maximum conductor tension plus wind loads on its projected area acting horizontally, together with its own dead weight and assumed snow loads acting vertically. Of these loads, the tension is the principal one, and it is proportional to the conductor area, that is, to d_s^2.

2. *Line Voltage.*—As already pointed out, if a given factor of safety is used, the spacing of transmission line conductors is proportional to the line voltage. The effect of spacing upon cost is not a linear relation, however. Just as tower cost is proportional to the square of its height, so is a portion of its cost also proportional to the square of its spread. Hence, two towers, designed for the same tensions and the same heights, will yet differ in costs if designed for two different voltages, the higher voltage requiring the greater cost. Considering the insulators as a part of the line supports, this item too is a function of the line voltage. Since, however, for most economical design the voltage is a function of d_s, that part of the tower cost, including insulators and their hardware, which varies with the line voltage must be a function of d_s. For the reasons given above, this cost varies in proportion to d_s^n, where n is some number between 1 and 2.

3. *Conductor Height.*—For a given voltage and a given cable tension, the turning moment, about the base of the tower, is proportional to the tower height. Hence, the area of the individual member is proportional to the height, and the weight of the tower as a whole is proportional to the square of its height.

The items into which the total cost of towers was subdivided, it will be recalled, are

1. Cost of tower at place of erection.
2. Cost of erection of tower.
3. Cost of tower site.
4. Cost of foundation installed.
5. Cost of location and inspection of support.
6. Cost of insulators at location of tower.
7. Cost of placing insulators and cable.

These itemized costs may now be used in evaluating the influence of conductor diameter upon the cost of towers. The variation in cost due to conductor tension and line voltage will first be considered.

Influence of Voltage on Cost as a Function of d_b.—Since neither voltage nor tension greatly affects either items 3, 5 or 7, and since they are small and relatively unimportant in any case, only items 1, 2, 4 and 6 are here considered.

In order to evaluate the sum of these cost items as a function of the line voltage, the following procedure is employed:

Obtain from the manufacturer bids on both anchor and suspension towers of the types under consideration, as for example the single-circuit towers of type *A* and the double-circuit of type *B*. Bids should be requested for each type of tower designed for each of three different voltages as, for example, 100, 150 and 200 kv., all towers to be of the *same height,* say 45 ft., and all to withstand the *same tension* per cable, say 6,000 lb. In addition to the above, two more sets of quotations should be obtained on similar towers of the same height (45 ft.), both designed for one of the above voltages, say of 150 kv., but built to withstand different tensions, say of 3,000 lb. and 9,000 lb. These data will serve as the basis for finding the desired relation of conductor diameter to voltage and tension.

The conductor spacings, for towers of type *A* and for the above voltages, may be read directly from Fig. 76. The same figure also specifies conductor spacings and crossarm dimensions for towers of type *B* if general proportions as in Fig. 75 are used.

The expression "height of tower," as here used, means the elevation above ground level of the point of support of the lowest conductors or it is the minimum allowable clearance to ground plus the maximum deflection of the critical catenary. For an anchor tower, it is the elevation above ground of the

lowest crossarm, and, for a suspension tower, it is the elevation above ground of the lowest crossarm minus the length of the insulator string.

The number of insulators per string may be found by dividing the line voltage by the allowable volts per disc; or, approximately,

$$\text{Number of standard suspension discs} = \frac{\text{Line voltage}}{17,000} \tag{600}$$

Number of discs for 100 kv.	= 6	For length of in-
Number of discs for 150 kv.	= 9	sulator string, see
Number of discs for 200 kv.	= 12	Fig. 75.

Thus the manufacturer can be given full specifications for overall tower dimensions for both anchor and suspension towers of both types A and B for an assumed height of 45 ft. and a tension of 6,000 lb.

Assuming that the most economical voltage will be used, it is expressable as a function of d_s. From Table 23, for an elevation of 1,000 ft. above sea level, for example, if the voltage is approximately 90 per cent of the corona forming value for the conductor used, the potential difference between conductors is

$$E = 215,000d_s \text{ for type } A \text{ tower}$$
$$E = 205,000d_s \text{ for type } B \text{ tower.}$$

From this relation, for type A towers,

$$\left.\begin{array}{l} 100,000 \text{ volts corresponds to } d_s = 0.465 \text{ in.} \\ 150,000 \text{ volts corresponds to } d_s = 0.698 \text{ in.} \\ 200,000 \text{ volts corresponds to } d_s = 0.930 \text{ in.} \\ \text{and, for type } B \text{ tower,} \\ 100,000 \text{ volts corresponds to } d_s = 0.487 \text{ in.} \\ 150,000 \text{ volts corresponds to } d_s = 0.731 \text{ in.} \\ 200,000 \text{ volts corresponds to } d_s = 0.975 \text{ in.} \end{array}\right\} \tag{601}$$

It should be noted here that the number of insulators for an anchor tower is at least twice the number used on a suspension tower. It is more than twice that number if high-strength conductors are used in such a way that the mechanical strength of one standard string is not sufficient to carry the maximum conductor tension. In that case, two insulator strings are placed in

parallel. Due to the decreased dielectric strength of a parallel group of insulators below that of a single string, additional units must be added in series, such that a string of 12 insulators is increased to 14 when connected in parallel with another string. For preliminary calculations the assumption that the anchor towers each require twice the number of insulators required by each suspension tower, may be used. After the conductor size is finally found, more refined calculations should be made for finding the insulator requirements.

After the cost data are obtained from the manufacturer and before final tabulation of the separate items is made as in Table 32 page 343, all items of cost except 3, 5 and 7 are recalculated for the *average* support *per line.* This is done by using the relation

Cost item for average support

$$= \frac{\text{Cost item for anchor tower} + \dfrac{T_s}{T_a}\,\text{cost item for suspension tower}}{1 + \dfrac{T_s}{T_a}} \qquad (602)$$

where T_s = total number of suspension towers per line.
T_a = total number of anchor towers per line.

Item 6 should be taken to include the insulator hardware, as well as insulators.

The cost of the *average line support* for each type of tower investigated is thus determined, (*a*) For each of three different voltages and a fixed tension; (*b*) for each of three different tensions and a fixed voltage.

For the two circuit towers of type *B*, the total cost is divided by two to get the cost per average support *per line.* From the relations given in Eq. (601), the conductor diameter d_s, corresponding to each price quoted, is available. The values d_s^2 obtained from Eq. (601) are plotted as abscissas against the sum of items 1, 2, 4 and 6 for the *average tower per line* as ordinates, and a straight line is drawn through the three points thus located. A straight line may not touch all three points, but since it is desired to express the cost as a function of d_s^2, and it is known that the cost is proportional to d_s^n, where n does not differ greatly from 2, it is apparent that the quadratic law can be made to fit the actual curve fairly well over a limited range at least. Hence,

the straight line is drawn to coincide with a smooth curve through the three points *in the region of the values of d_s, within which the value of d_s for the project in question is most likely to fall.* In this way the quadratic law will fairly well fit the curve over the desired range, and the degree of accuracy over the remainder of the curve is immaterial.

The straight line thus drawn is equivalent to the parabola that would result were the cost plotted against the first power of d_s. The equation of the straight line is

Cost items 1 + 2 + 4 + 6 of average support $= k_v + md_s^2$ (603)

where k_v = that part of items 1 + 2 + 4 + 6 which is independent of the line voltage, or conductor diameter, and is the intercept of the straight line on the cost axis. The constant m is the slope of the straight line, in terms of cost and d_s^2.

The above analysis shows how the influence of transmission voltage upon the cost of towers may be found. The constant m, it should be noted, is independent of the tension and of the height of tower, since the tension T_1 and the height h_e are constant factors in the three costs used to locate the straight line. Thus m has the same value whether the tension is 5,000 lb. or 8,000 lb. per conductor; it depends only upon the line voltage. If quotations were obtained for the same three voltages but for each of two other constant tensions T_2 and T_3, the curves of cost *vs.* d_s^2 for the latter would be straight lines parallel to the curve for T_1. Thus, in Fig. 77, if T_1 is the cost curve for tension T_1 = 6,000 lb., T_2 and T_3 would represent the corresponding curves for T_2 = 3,000 lb. and T_3 = 9,000 lb. Since it is known that these lines are parallel it is unnecessary to have more than one point on each, once the slope is known. This point is obtained for each line from the quotations for towers built for the two additional tensions but for the same voltage as one of those specified for the data of curve T_1 as already explained.

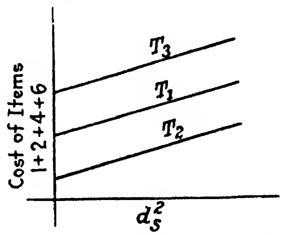


FIG. 77.—Tower costs as functions of conductor tensions.

Influence of Tension on Cost as a Function of d_s.—The influence of tension on cost may now readily be found from the data already assembled, as soon as the relation between conductor area and tension is known for each of the conductor materials to be investigated.

The elastic limit, assuming stranded conductors, for aluminum and copper may be assumed in accordance with the data given in Chap. II as

$$\left.\begin{array}{lll}\text{Aluminum} & T_e = & 14{,}000 \text{ lb. per sq. inch} \\ \text{Copper} & T_e = & 28{,}000 \text{ lb. per sq. inch} \\ \text{Steel} & T_e = & 54{,}000 \text{ lb. per sq. inch}\end{array}\right\} \qquad (604)$$

If it be assumed that the conductor may not be safely stressed beyond 75 per cent of its elastic limit, and that, for the sizes of cable considered, the ratio of $d_s \div d$ may be taken as 1.151, then the relation between the maximum allowable tension T_m and outside cable diameter d_s is

$$\left.\begin{aligned} d_s^2 &= \frac{4 \times (1.151)^2}{\pi} \times \frac{T_m}{0.75 \times 14{,}000} \\ &= 16.1 \times 10^{-5} T_m \text{ for aluminum} \\ d_s^2 &= \frac{4 \times (1.151)^2}{\pi} \times \frac{T_m}{0.75 \times 28{,}000} \\ &= 8.04 \times 10^{-5} T_m \text{ for copper} \\ d_s^2 &= \frac{4 \times (1.151)^2}{\pi} \times \frac{T_m}{0.75 \times 54{,}000} \\ &= 4.17 \times 10^{-5} T_m \text{ for steel} \end{aligned}\right\} \qquad (605)$$

If, as is sometimes done, it is assumed that the conductor may be safely stressed up to its elastic limit, on the theory that, if so stressed, the conductor will simply stretch slightly to relieve the tension without injuring it, then the above figures may be increased by 33 per cent. The exact values to be used will vary somewhat with the judgment of the designing engineer who is responsible for the work. We are here concerned mainly with the principle involved rather than with the precise values to be used.

In Eq. (605), for T_m may be substituted in succession the three values of tension for which bids on towers have already been obtained. These substitutions will yield the three corresponding values of conductor diameters for which these tensions are permissible, or the desired values of d_s^2.

Referring now to Eq. (603) and Fig. 77, it is apparent that each tension curve has a separate value of the intercept k_v, while the slope is the same for all curves. Thus, k_v varies with the tension but not with the voltage. From the three curves of Fig. 77, the law of variation of k_v, with tension or with d_s^2 may be

found by evaluating k_v for each of the three curves and replotting these values, again using k_v (costs) as ordinates and d_s^2 as abscissas. From the argument already presented, it is apparent that these values of k_v are approximately proportional to d_s^2. Hence, the new plot will yield a straight line as before, whose equation is

$$k_v = k_T + nd_s^2 \tag{606}$$

where k_T is the intercept on the k_v (cost) axis and n is the slope of the new curve. By substituting the value of k_v from Eq. (606) in Eq. (603), the cost of items 1 + 2 + 4 + 6, for the average support and as a function of conductor area, is found to be

$$\text{Cost of items } 1 + 2 + 4 + 6 = k_T + (m + n)d_s^2. \tag{607}$$

Evidently, k_T is the same for all conductor materials, since it is independent of T_m, whereas the slope n, which is a function of the tension, has a different value for each material. Thus Eq. (607) takes account of the variation in tower cost which results from variations in voltage and tension. It must be remembered, however, that it was derived on the assumption of a *fixed height* of tower. The influence, of tower height on the cost of tower, will now be considered to determine how, if at all, it should enter into consideration from the standpoint of economics.

Influence of Tower Height on Cost as a Function of d_s^2.—According to item 4 Eq. (599), the height of tower is fixed for a given ratio of $k_3 \div k_2$, no matter what the size of d_s or what the material of the conductor. By definition (Eq. (557)), for type A tower,

$$\frac{k_3}{k_2} = \frac{(\text{Items } 4 + 5 + 6 + 7)h^2}{(\text{Items } 1 + 2 + 3)}. \tag{608}$$

An examination of these items and the way in which they are influenced by conductor diameter shows that the effect of cable diameter on the ratio $k_3 \div k_2$ is very small since the items in the numerator and those in the denominator of Eq. (608), with the exceptions of the small items 5 and 7, are similarly affected. Since 4 + 6 is not greatly different in size from 4 + 5 + 6 + 7, the effect of the two small items 5 and 7 becomes quite negligible. Hence, considering the ratio practically independent of conductor diameter, and since the height of tower is constant for a given ratio, regardless of conductor size or material, the *height of tower does not enter as a factor to be considered in the economics of transmission-line design.*

For the double-circuit towers of Type B, Eq. (608) needs slight modification. This tower carries its circuits in a vertical plane, and, from the standpoint of turning moments about the foundation, the pull of the line conductors may be thought of as being concentrated in a nearly horizontal plane passing through the point of conductor attachment on the middle crossarm. Therefore, the cost of the tower structure is not proportional to the square of its height measured to the lower crossarm, but rather to the square of its height measured to the maximum point on the middle conductor catenary. Hence, when using the general proportions of tower and the spacings, as shown in Fig. 75, the correct equations corresponding to Eq. (608) for the two types of towers are

$$\left.\begin{aligned} \frac{k_3}{k_2} &= \frac{h^2(\text{Items } 4 + 5 + 6 + 7)}{(\text{Items } 1 + 2 + 3)} \text{ for type } A \text{ tower} \\ \frac{k_3}{k_2} &= \frac{\left(h + \dfrac{D_v}{12}\right)^2(\text{Items } 4 + 5 + 6 + 7)}{(\text{Items } 1 + 2 + 3)} \text{ for type } B \text{ tower} \end{aligned}\right\} \quad (609)$$

(For D_v see Fig. 75 and page 255.)

For type A tower the value of k_1, the minimum clearance to ground, is assumed or specified by law. For the double-circuit tower, since the height is measured to the point of attachment of the middle conductor, a value k_1' is used in place of k_1 where

$$k_1' = k_1 + D_v. \quad (610)$$

For example, for a 150 kv. line whose minimum clearance to ground is to be 28 ft., by Fig. 75 and page 255,

$$D_v = \frac{3D}{\sqrt{10}} = 0.947D$$

and by Eq. (578),

$$D = 0.0017E_n$$

whence

$$\begin{aligned} D_v &= 0.947 \times 0.0017 \times 150{,}000 \div \sqrt{3} \\ &= 138 \text{ inches} \\ &= 11.5 \text{ feet} \end{aligned}$$

and

$$k' = 28 + 11.5 = 39.5 \text{ ft.}$$

Equation for Total Cost of Towers.—The total cost of the average line support may now be evaluated. The cost, as

influenced by tension and voltage, has already been expressed as a function of d_s^2 by Eq. (607). The height of tower, as shown above, is not a function of d_s^2. The height of tower does affect the cost of certain items in proportion to its square, however. The items thus affected are 1, 2 and 3. The cost of the average support should therefore be corrected by an amount equal to the difference between the assumed value of items 1 + 2 + 3 and the value found from the equation

$$\text{Corrected cost items } 1 + 2 + 3 \doteq \frac{h_e^2}{h^2} \times (\text{Items } 1 + 2 + 3) \quad (611)$$

where

h_e = the most economical tower height

and

h = actual tower height for which costs were obtained.

The difference between cost items 1 + 2 + 3, as per Eq. (611), and the assumed cost of these items is the correction factor

$$k_4 = (\text{Items } 1 + 2 + 3)\left(\frac{h}{h^2} - 1\right). \quad (612)$$

Items 5 and 7 are independent of voltage, tension and height, and may be represented by the constant k_5; that is,

$$k_5 = \text{Items } 5 + 7. \quad (613)$$

Combining Eqs. (607), (612) and (613), the total cost of the average line support may be written

$$\text{Total cost of average line support} = (m + n)d_s^2 + k \quad (614)$$

where

$$k = k_T + k_4 + k_5. \quad (615)$$

Total Cost of Line Supports.—From Eq. (615), the total cost of line supports may be written, for

L = total length of line in feet

and, since S = length of the most economical span in feet

$$\text{Total cost of line supports} = \frac{L}{S}[(m + n)d_s^2 + k]. \quad (616)$$

By Eq. (566), however,

$$S = \frac{k_7(d_s + k_6) - k_8}{d_s + k_6}.$$

Substituting this value of S in Eq. (616) yields the equation of tower cost as a function of conductor diameter. It is

$$\text{Total cost of line supports} = \frac{J[(m + n)d^2 + k](d_s + k_6)}{k_7(d + k_6) - k_8}. \tag{617}$$

The meaning of the constants in Eq. (617) are summarized below for convenience. They are

$$\left.\begin{array}{ll} m = & \text{slope of cost } vs.\ d_s^2, \text{ from curves of Eq. (603).} \\ n = & \text{slope of cost } vs.\ d_s^2, \text{ from curves of Eq. (606).} \\ k_7 = & \text{constant of Eq. (565), found from Plates II and IV,} \\ & \text{for aluminum and copper respectively.} \\ k_8 = & \text{constant of Eq. (565), found from Plates III and V,} \\ & \text{for aluminum and copper, respectively.} \\ k_6 = & \text{constant of Eq. (565), having the following} \\ & \text{values:} \end{array}\right\} \tag{618}$$

$$k_6 = +0.20 \text{ for aluminum}$$
$$k_6 = -0.15 \text{ for copper}$$
$$k_6 = -0.30 \text{ for steel.}$$

Equation (617) is too complex in form to be of much use as an item to be incorporated in a general statement of Kelvin's law. It was found that, for ranges of d_s between 0.6 and 2 in. and for values of k_1 and $k_3 \div k_2$ lying within the limits of Plates II to V, inclusive, the quantity

$$\frac{[(m + n)d_s^2 + k](d_s + k_6)}{k_7(d + k_6) - k_8}, \tag{619}$$

which is the cost of supports per foot of line, is very closely approximated over a considerable range by the simpler equation of the parabola

$$Md_s^2 + N \tag{620}$$

where M and N are constants.

The constants are found by computing the value of Eq. (619) for a number of values of d_s, and the required values of $k_3 \div k_2$ and k_1, and plotting the results on coordinate paper, using values from Eq. (619) as ordinates against d_s^2 as abscissas. The resulting graph is approximately a straight line. Again, the line should be drawn so as to fit the curve best over the range of values within which the looked-for value of d_s lies. The slope of this

straight line is M, and its intercept on the cost axis is N. The total cost of line supports then finally becomes

$$\text{Cost of line supports} = L(Md_s^2 + N). \tag{621}$$

Tower Cost as a Factor in Kelvin's Law.—Kelvin's law embraces all factors of transmission-line cost which vary with the conductor area. The purpose of the above analysis of tower costs is to derive an expression in which tower costs are given as a function of conductor area, or d_s^2, in order that that portion of such cost, which varies with the conductor area, may be taken into proper account in a mathematical statement of the law of economy. It is to be remembered also that three items or groups of items constitute the cost factors involved, namely, (*a*) the line conductors and ground cables; (*b*) line supports or towers; (*c*) high-tension transformers and other terminal apparatus.

Of these three items, the costs of (*a*) and (*c*) can be very accurately expressed as a function of the conductor area. The evaluation of (*b*) is far more difficult and the results may be somewhat less satisfactory. Since it is only one of three items, however, and by no means the most important one, unavoidable errors or uncertainties that may be involved will not seriously affect the result. As a matter of fact, a good first approximation of conductor diameter is obtained by a solution of Eq. (635) neglecting the tower factor entirely. The value of d_s thus obtained will of course be larger than the correct value.

In Eq. (621) LMd_s^2 is that portion of the total cost of towers which varies with the conductor area, while the part LN is independent of conductor area. Thus,

$$LMd_s^2 \tag{622}$$

is that part of the cost of line supports which must be embraced by Kelvin's law, and the yearly charge against this item, in interest and depreciation, must be added to the fixed charges against the line conductors as given by Eq. (524). The annual fixed charges against the line supports then is

$$\text{Tower fixed charges} = \frac{p_2 LMd_s^2}{100} \tag{623}$$

where p_2 is the per cent interest and depreciation chargeable to towers, expressed as a whole number.

Cost of Ground Cables.—The function of the ground cables suspended from tower to tower above long transmission lines is to

provide protection against lightning surges and to transfer stresses to adjacent towers in case of the mechanical failure of a line conductor. These cables are generally made of steel and are clamped to the tower structure at its highest point so that this grounded steel catenary is situated well above the transmission conductors. For type A tower, if one ground wire is used, its location should be midway between any pair of conductors, so that the ground cable catenary may have the maximum clearance to the conductors. The best construction, however, from a mechanical standpoint, is the use of two ground wires as shown in Fig. 75. This minimizes eccentric stresses upon the tower structure. For type B tower only one ground cable is used, which is clamped to the apex of the tower top, as shown in Fig. 75. Thus, one ground wire is sufficient for two transmission lines of the type B construction, giving adequate protection electrically, whereas two ground wires should be used for each transmission line of type A construction.

From the standpoint of electrical protection, a very small ground wire would be sufficient and its size would largely be determined by analysis of its mechanical loading at minimum temperature and at maximum sleet and wind load for the span in question; but if its duty is also to increase the rigidity of the suspension towers by transferring stresses to the anchor towers, in case of conductor failures, its size should be ample to not only support its own weight and normal loadings, but to also transmit the stresses upon the tower caused by the failure of at least one line conductor.

Accordingly, since the cost of the ground cable is a function of the size of conductor, this cost should be estimated in terms of conductor area and should be included as an item in the mathematical statement of a generalized Kelvin's law. By inspection of Eq. (605) the reader will notice that the maximum allowable stresses for aluminum, copper and steel are in the proportion 1, 2 4, respectively. This simple relationship is responsible for the following assumptions, which, although rather arbitrary, may form a convenient basis for the determination of the proper ground cable sizes to be used for long transmission lines, and an analysis of their cost as a function of conductor area.

Assumption 1.—Steel cables when used as conductors in transmission circuits will be of such size that a mechanical failure of the line may be considered beyond the range of possibility

Hence, the ground cable may be made of the same size as the conductors proper, and it may be strung with the same maximum catenary deflection at the center of the span as the conductors.

Assumption 2.—Copper cables when used as conductors are to be considered within the range of possible mechanical failure. Hence, if the ground wire be made of *the same* metallic cross-sectional area as that of the copper conductor, and if the ground cable be strung with a maximum catenary deflection equal to that of the conductors, the ground cable will be of sufficient strength to transmit to the anchor towers the stresses due to the failure of one conductor. This follows by reason of the fact that the steel ground cable is twice as strong as the copper and will therefore be able to carry its own dead weight and loading plus that of a copper cable of equal area.

Assumption 3.—Aluminum cables when used as conductors, are to be considered within the range of possible mechanical failure. Hence, if the ground wire be made of *half* the metallic cross-sectional area as compared to the aluminum conductor, and if the ground cable be strung with a maximum catenary deflection equal to that of the conductors, it will be of sufficient strength to transmit to the anchor towers the stresses due to the failure of one conductor.

Since the specific gravity of aluminum is less than half that of steel, the weight per unit length of catenary under extreme sleet and wind conditions at minimum temperature is approximately the same as that of a steel cable of half the cross-sectional metallic area. This fact lends additional support to assumption 3. The specific gravities of copper and steel are near enough alike to also strengthen assumption 2. The catenaries formed by the conductors and ground cables, if the sizes specified by the above assumptions are used, will, therefore, perform very nearly alike over the range of weather conditions and loadings to which the transmission line is exposed in a given locality. This will always insure proper clearances of conductors to ground, and the factors of mechanical safety will be practically alike for aluminum, copper and steel of both types *A* and *B* tower construction, thus permitting a fair cost comparison between these materials and tower types to be made.

If two ground cables are used for type *A* tower construction, the combined metallic cross-sectional area of the two cables should be equal to that of the single ground cable for type *A* or *B* towers.

According to the foregoing assumptions the fixed charge against the ground wire, per line (Eqs. (574), (575), and (576)) is

Ground cable fixed charge

$$\left.\begin{array}{lll} = 2.05 \times 10^{-5} d_s^2 L p_{Fe} B_{Fe} W_{Fe} & \text{for aluminum} & \\ = 4.10 \times 10^{-5} d_s^2 L p_{Fe} B_{Fe} W_{Fe} & \text{for copper and steel} & \\ & \text{for type } A \text{ towers} & \\ = 1.03 \times 10^{-5} d_s^2 L p_{Fe} B_{Fe} W_{Fe} & \text{for aluminum} & \\ = 2.05 \times 10^{-5} d_s^2 L p_{Fe} B_{Fe} W_{Fe} & \text{for copper and steel} & \\ & \text{for type } B \text{ towers} & \end{array}\right. \qquad (624)$$

Including in the conductor fixed charge that of the ground cable, Eq. (576) becomes

Conductor fixed charge

$$= 12.31 \times 10^{-5} d_s^2 L (p_1 B W + g P_{Fe} B_{Fe} W_{Fe}) \qquad (625)$$

where

p_{Fe} = per cent interest and depreciation chargeable against the ground cable.

B_{Fe} = cost of steel cable in dollars per pound.

W_{Fe} = weight of steel in pounds per cubic foot.

$$\begin{array}{ll} g = 0.167 \text{ for aluminum conductors} & \left.\right\} \text{type } A. \\ = 0.333 \text{ for copper and steel conductors} & \left.\right\} \text{tower.} \\ = 0.083 \text{ for aluminum conductors} & \left.\right\} \text{type } B \\ = 0.167 \text{ for copper and steel conductors} & \left.\right\} \text{tower.} \end{array}$$

High-tension Apparatus and Housing.—Owing to the greater amount of insulation required, the greater clearances necessary and the increased difficulty of manufacture, etc., the cost of high-tension apparatus increases rapidly with the operating voltage. This fact has an important bearing upon the choice of the most economical transmission voltage, and results, in the selection of a lower voltage than would be used, were these costs independent of the voltage. It is important also to observe that *the total cost of the high-tension apparatus, required for a given project, depends only upon the kilovolt-amperes developed, and the voltage used, and is independent of the distance over which the power is to be transmitted.* Accordingly, for a very short line, this item will have a very important influence upon the choice of the conductor diameter and transmission voltage, since it represents a relatively large percentage of all cost items which influence their choice, while in a very long line the cost of terminal apparatus will be a relatively

smaller part of these costs and will accordingly influence the choice of conductor much less. Expressed in another way, it may be stated that the extent to which these cost items bear upon the selection of the most economical voltage and conductor diameter, is measured by the cost of high-tension equipment per mile of line, or is in inverse proportion to the length of line.

In order to give proper weight to the cost of terminal apparatus, as influencing the choice of conductor diameter and line voltage, the variation of costs of the required equipment is studied for a suitable range of line voltages within which the voltage of the project in question is known to lie. That is, the combined cost of all items of such apparatus is obtained from the manufacturer for each, of say three voltages, covering the desired range and for the type of station layout to be used. Kelvin's law, it will be recalled, is concerned only with such cost items as are a function of the conductor area, *i.e.*, a function of the diameter squared. Furthermore, it has already been shown that the relation which should hold between the conductor diameter and line voltage is given by the constant U, of Table 23. We therefore have a means of expressing the cost of terminal apparatus as a function of d_s, and may introduce the variable part of this cost into a mathematical statement of Kelvin's law.

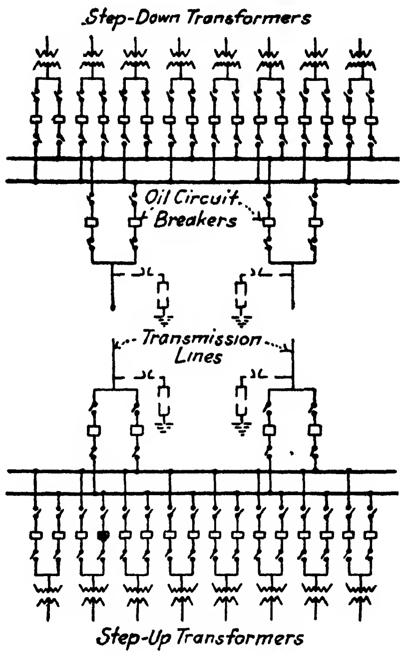


FIG. 78.—Transmission line and station wiring diagram.

Method.—The procedure, suggested for formulating the law of variation of terminal apparatus and housing, as a function of the line voltage, is as follows: Make up a circuit diagram of the high-tension apparatus required for the project, as illustrated in Fig. 78. Most station layouts will be simpler and less costly than the one represented. Whatever type of layout is proposed

should form the basis of this investigation, however. Other types of layouts are shown in Figs. 80 to 83, inclusive. Since the size of the development is known to begin with, the number and rating of each unit is known, or may be decided upon, irrespective of the number of transmission lines, with the exception of the line circuit breakers, disconnecting switches and lightning arresters, if any. The ratings and number of line switches and the number of lightning arresters will obviously depend upon the number of transmission circuits. For reasons which will be developed more fully later, it is apparent that in the ultimate development of any project which is economically designed, the separate transmission circuits, between a given pair of generating and receiving stations, will not exceed two in number unless it should happen that the maximum power limit of the two lines should be exceeded by the load to be transmitted. In such a case, more than two lines would be required. Two transmission circuits may therefore be assumed as the number required, unless conditions are such that a single circuit will answer. That is, if during times when the line is out of service, either due to failure or for the making of necessary repairs, the load can be economically supplied from other sources until service is re-established, a single line will prove to be more economical than two lines. In such cases a single line should be used.

After having prepared the wiring diagram for the power station and lines, indicating the probable number of lines, a print of this diagram together with other necessary data as to the number of units of each kind, their kilovolt-ampere, rating, etc., is submitted to the manufacturer, with a request for quotations on the required apparatus for each of several standard transmission voltages, covering the range within which the project will fall. These quoted prices are then tabulated and totaled for each of the voltages chosen.

Cost of Wiring, Housing, Etc.—Estimates should next be made of the cost of structures required for the proper housing, supporting and connecting the substation equipment. Under this heading are included the costs of wiring, foundations, supports for buses and other wiring, housing, if any, etc. As with the apparatus itself, so here, for any given voltage these costs will vary considerably with the degree of complexity of the substation layout, and, to some extent, with the total kilovolt-amperes of station capacity.

For a given station layout, however, the total costs of the above items are a function of the line voltage used. This is apparent from the fact that the amount of area, required to properly arrange and support the equipment, is proportional to the square of the line voltage, since, for a given factor of safety, the clearances necessary between transformers, buses, switches, etc., increase with the line voltage in two dimensions. The cost of the structures will therefore vary with E^2.

Total Cost of Substation Equipment, Including Wiring and Housing.—After the estimates of the above items of cost have been made for the chosen type of layout and for each of the assumed voltages under consideration, the totals obtained should be added to the totals for the apparatus itself. Assuming that this has been done, we now have a separate total cost for each of a number of voltages. The next step is to plot these totals as ordinates against the squares of their corresponding voltages. Investigation will show that, through the points thus located, a curve approximating a straight line can be drawn; that is, the total cost varies approximately as the square of the line voltage over a considerable range of voltages (Fig. 79). Accordingly, a parabola of the form

$$C = k'_{11}E^2 + k_{12} \tag{626}$$

may always be found which will closely fit the curve of cost *vs.* voltage over the desired range of voltage, where

C = total cost of terminal equipment, wiring, housing, etc.
k_{12} = the intercept of the cost curve on the cost axis, representing that part of the cost which is independent of line voltage.
k'_{11} = that part of the cost which varies with the line voltage and which must be considered in the application of the modified Kelvin's law.
E = the line voltage in volts.

Such a curve of cost *vs.* E^2, for a 50,000-kva. plant of the type shown in the diagram of Fig. 78, is the curve in Fig. 79. The costs for this curve were estimated for 88, 110, 132, 154 and 220 kv. Inspection of this figure shows that while the straight line *a* fits the curve fairly well over the entire range of from 88 to 220 kv., yet, if one knew to begin with that the project considered would probably require a voltage of less that 154 kv., it

would be better to use the line *b*, while, for voltages above 132 kv., the line *c* is more nearly correct. That is, the straight line should be chosen to fit the particular part of the curve which embraces the voltage of the project in question.

It may be argued that the total cost of terminal apparatus, housing, wiring, etc., will not be correctly represented, over a wide range of voltages, by a parabola of the form given in Eq. (626); and that, therefore, the results obtained by using the variable part ($k_{11}'E^2$) as a factor in Kelvin's law, will be in error.

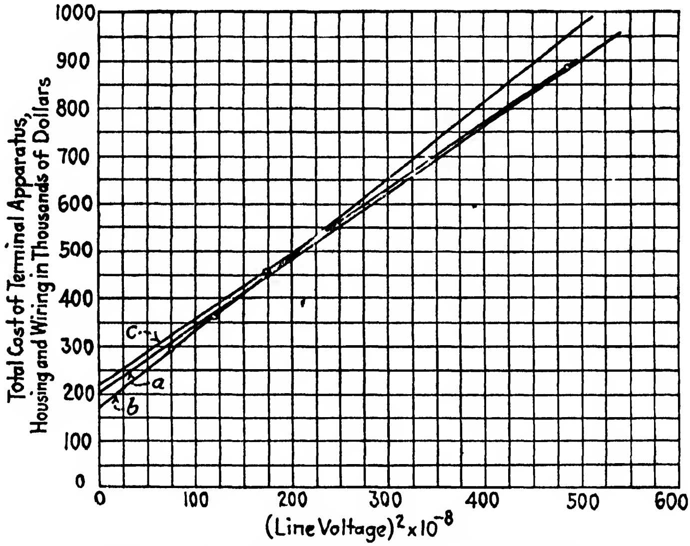


FIG. 79.—Cost estimate for 50,000 kva. substation.

It is true that, were the exponent 1.6 used instead of 2 in Eq. (626), the equation would usually fit the curve better over its entire range. It is not necessary for the purpose in hand, however, that the equation should fit the curve throughout its entire length. Close agreement, over a range of sufficient length to include the voltages of the given project, is sufficient. The far greater simplicity, which results, together with the fact that accuracy in making the estimate is in no way sacrificed if one always selects the most suitable portion of the curve, is at once the reason and the justification for selecting 2 instead of 1.6 as the exponent of E in Eq. (626).

Empirical Equation of Cost. Constants k_{11}' and k_{12}.—Equation (626) is the general form of the empirical equation of cost, the

constants of which have to be found. These are to be obtained from the straight line relations of Fig. 79. To the scale of the curve, k_{12} is the amount of the cost intercept while k'_{11} is the slope of the line. Thus, for the entire range of voltages from 88 to 220 kv., using the line a, the constants are:

$$\left.\begin{aligned} k'_{11} &= 14.3 \times 10^{-6} \\ k_{12} &= 195{,}000 \end{aligned}\right\} \text{88 to 220 kv.}$$

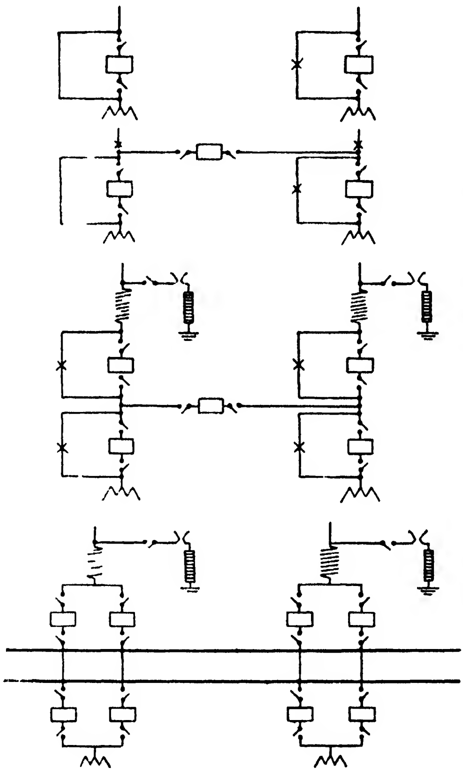


FIG. 80.

FIG. 81.

FIG. 82.

FIG. 83.

FIGS. 80, 81, 82 and 83.—Substation wiring diagrams of various degrees of complexity.

Similarly, for the lines b and c:

$$\left.\begin{aligned} k'_{11} &= 16.2 \times 10^{-6} \\ k_{12} &= 170{,}000 \end{aligned}\right\} \text{88 to 187 kv.}$$

$$\left.\begin{aligned} k'_{11} &= 13.6 \times 10^{-6} \\ k_{12} &= 215{,}000 \end{aligned}\right\} \text{132 to 220 kv.}$$

TABLE 25

Type of substation circuit	50,000-kva. plant						100,000-kva. plant					
	88 to 220 kv.		88 to 187 kv.		132 to 220 kv.		88 to 220 kv.		88 to 187 kv.		132 to 220 kv.	
	$10^6k'_{11}$	$10^{-3}k_{12}$	$10^6k'_{11}$	$10^{-3}k_{12}$	$10^6k'_{11}$	$10^{-3}k_{12}$	$10^6k'_{11}$	$10^{-3}k_{12}$	$10^6k'_{11}$	$10^{-3}k_{12}$	$10^6k'_{11}$	$10^{-3}k_{12}$
Fig. 80	5.9	140	7.1	120	5.5	153	6.0	180	7.7	150	5.4	200
Fig 81	7.5	155	9.2	125	7.1	165	8.5	175	9.9	150	7.8	195
Fig. 82	11.1	170	12.3	155	10.3	195	12.1	195	13.2	180	11.1	220
Fig. 83	14.3	195	16.2	170	13.6	215	15.3	215	17.2	190	14.6	240

As already pointed out, the constants k'_{11} and k_{12} vary with the degree of complexity of the layout and with the kilovolt-amperes rating of the project. In order to make available approximate values of these constants for purposes of estimating, these constants were calculated from estimated total costs of substation equipment housing, etc., for circuits of four different types, as shown in Figs. 80, 81, 82 and 83, and for total capacities of 50,000 and 100,000 kva. The constants for each type and rating are computed for each of three ranges of voltage, as illustrated in connection with Fig. 79. These constants are listed in Table 25.

Cost of Terminal Apparatus and Housing as a Function of Conductor Diameter.—Before the cost of terminal equipment and housing can be introduced as a factor in a mathematical statement of Kelvin's law of economy, the cost must be expressed as a function of conductor diameter. This is readily accomplished, for, by Eq. (588), the voltage to neutral is

$$E_n = d_s U$$

whence

$$E^2 = 3E_n^2$$
$$= 3d_s^2 U^2.$$

Substituting this value of E^2 in Eq. (626) yields the desired equation of cost in terms of conductor diameter. It is

$$\text{Cost of terminal apparatus and housing} = 3k'_{11}U^2d_s^2 + k_{12}$$
$$= k_{11}d_s^2 + k_{12} \qquad (627)$$

where

$$k_{11} = 3k'_{11}U^2. \qquad (628)$$

The Annual Charge.—The only part of Eq. (627), which influences the choice of conductor diameter and voltage, is the first term of the right-hand member. This item of cost is

$$C' = k_{11}d_s^2 \qquad (629)$$

and the annual charge against this item is

$$\text{Annual charge against terminal apparatus and housing}$$
$$= 0.01p_3k_{11}d_s^2 \qquad (630)$$

where

p_3 = weighted average percentage (expressed as a whole number) applicable to the total cost of terminal apparatus and housing, to take care of the annual interest and depreciation on this item.

Equation (630) may now be incorporated in a mathematical statement of the generalized law of economy.

The final equation, which includes all of the factors to be considered, and which balances the selling value of the annually wasted energy in the line against the annual fixed charges on all items whose costs vary with the conductor diameter or voltage, may now be set down.

Kelvin's Law Involving All Cost Factors.—From Eq. (625), the annual fixed charge against the conductor is

$$12.31 \times 10^{-5}d_s^2L(p_1BW + gp_{Fe}B_{Fe}W_{Fe})$$

and from Eq. (623), the annual fixed charge against the line supports is

$$0.01p_2LMd_s^2$$

and from Eq. (630), the annual fixed charge against the housing and apparatus is

$$0.01p_3k_{11}d_s^2.$$

The total annual fixed charges, against all factors which make up the cost of a transmission line, and which are a function of the conductor size, are

$$12.31 \times 10^{-5}\, d_s^2L(p_1BW + gp_{Fe}B_{Fe}W_{Fe}) + 0.01p_2LMd_s^2$$
$$+ 0.01p_3k_{11}d_s^2$$

and these charges, according to Kelvin's law, must be equal to the value of the annually wasted energy of the line, which from Eq. (573) is

$$35.64 \times 10^{-6}\rho LA \frac{I^2}{d_s^2}.$$

Equating the above two statements

$$35.64 \times 10^{-6}\rho LA \frac{I^2}{d_s^2} = 12.31 \times 10^{-5}d_s^2 L(p_1 BW + gp_{Fe}B_{Fe}W_{Fe}) + 0.01p_2 LMd_s^2 + 0.01p_3 k_{11}d_s^2$$

and from this,

$$I = d_s^2\left(\frac{1}{\rho A}\left[3.45(p_1 BW + gp_{Fe}B_{Fe}W_{Fe}) + 280.6p_2 M + 280.6p_3 \frac{k_{11}}{L}\right]\right)^{\frac{1}{2}}. \quad (631)$$

Multiplying the left and right members of Eq. (631) by three times the left and right members of Eq. (588), respectively,

$$3E_n I = 3d_s^3 U\left(\frac{1}{\rho A}\left[3.45(p_1 BW + gp_{Fe}B_{Fe}W_{Fe}) + 280.6p_2 M + 280.6p_3 \frac{k_{11}}{L}\right]\right)^{\frac{1}{2}}. \quad (632)$$

And, from Eq. (632),

$$d_s = \left(\frac{\text{r.m.s. volt-amperes per line}}{\left\{\frac{U^2}{\rho A}\left[31.05(p_1 BW + gp_{Fe}B_{Fe}W_{Fe} + 2{,}525\left(p_2 M + p_3 \frac{k_{11}}{L}\right)\right]\right\}^{\frac{1}{2}}}\right)^{\frac{1}{3}} \quad (633)^*$$

Assuming that, with the proper synchronous reactor equipment connected, the average power factor, along the whole length of the line (for r.m.s. load), is represented by cos θ, Eq. (633) may also be written

$$d_s = \left(\frac{\text{r.m.s. kilowatts per line} \times 1{,}000}{\left\{\frac{U^2 \cos^2\theta}{\rho A}\left[31.05(p_1 BW + gp_{Fe}B_{Fe}W_{Fe}) + 2{,}525\left(p_2 M + p_3 \frac{k_{11}}{L}\right)\right]\right\}^{\frac{1}{2}}}\right)^{\frac{1}{3}}. \quad (634)$$

* In modified form, this equation was first used by F. K. Kirsten in 1916. The term involving the terminal apparatus was later added by the author.

Equation (634), expressed in general terms and simplified form, is

$$d_s = 10 \times \frac{(\text{r.m.s. kilowatts per line})^{\frac{1}{3}}}{[J(F + G + H)]^{\frac{1}{6}}} \tag{635}$$

where

$J = \dfrac{U^2 \cos^2 \theta}{\rho A}$	Factor involving the value of the electrical energy dissipated as heat in the line conductors of one line.
$F = 31.05(p_1 BW + g p_{Fe} B_{Fe} W_{Fe})$	Factor involving the cost of the line conductors of one line.
$G = 2{,}525 p_2 M$	Factor involving the cost of that part of the line towers per line, which is a function of the conductor diameter.
$H = 2{,}525 p_3 \dfrac{k_{11}}{L}$	Factor involving that part of the cost of high-tension apparatus and housing per line, which is a function of the conductor diameter.

Equation (635) is a mathematical interpretation of Kelvin's law, as modified to include all of the factors involved. The conductor having the diameter d_s, as found from this equation, is the most economical conductor for the conductor material investigated, and the line with which it is used may be called the *line of maximum economy.* An inspection of this equation leads to several very interesting conclusions, among which the following are important:

Conclusions from Economic Relations in Eq. (635).—1. The diameter of the conductor, for the line of maximum economy, is practically independent of the length L of the line, except for relatively short lines. Since, for a given elevation, the voltage to be used is assumed to be proportional to the conductor diameter, it follows that *the voltage used is nearly independent of the length of line.* These conclusions follow since the only factor in the equation which involves the length of line is the factor H. H is inversely proportional to the length of line. Therefore for a very short line the cost of high-tension equipment has an important bearing on the conductor diameter, and the line voltage, whereas, for a very long line this item tends to become a negligible factor. *For very long lines the diameter of conductor and the voltage used are both determined largely by the kilovolt-*

amperes transmitted, and are roughly proportional to the cube root of this quantity.

This conclusion is well illustrated in the curves of Fig. 84. These curves were calculated from the equation for the most economical conductor diameter for certain reasonable assumptions as to cost of towers, and are intended to illustrate relative magnitudes rather than exact values. Copper was assumed to cost 20 cts. per pound delivered, and energy losses were calculated on the basis of 4 mils per kilowatt-hour. It will be seen that for a load of 50,000 r.m.s. kw., the variation of the most economical

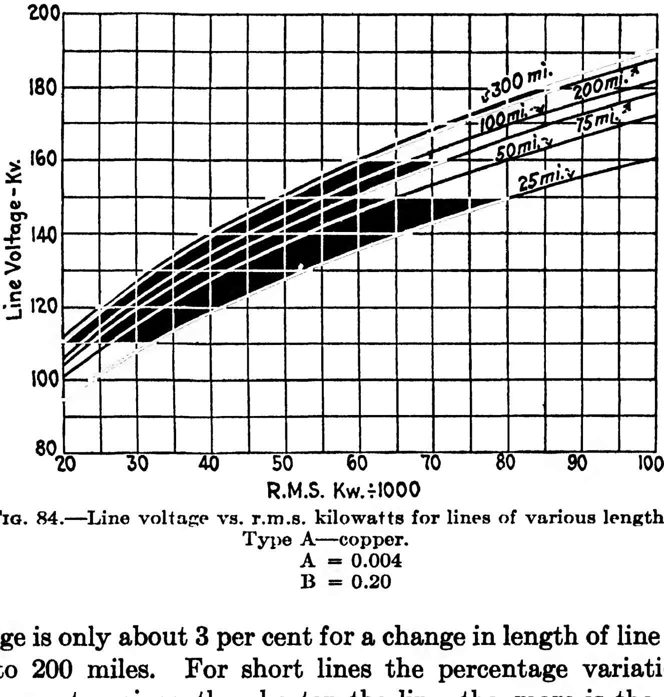


Fig. 84.—Line voltage vs. r.m.s. kilowatts for lines of various lengths.
Type A—copper.
A = 0.004
B = 0.20

voltage is only about 3 per cent for a change in length of line from 100 to 200 miles. For short lines the percentage variation is much greater, since the shorter the line, the more is the cost, and hence the voltage, influenced by the cost of terminal apparatus.

2. Since the line losses are inversely proportional to the square of the conductor diameter, and since, as shown above, for long lines the diameter of conductor is approximately independent of the length of line, it follows from Eq. (635) that *for the line of maximum economy, the losses per unit length of line should be directly proportional to the two-thirds power of the power trans-*

mitted. Or, *for a given amount of power transmitted, neglecting the factor H, the total line losses should vary in direct proportion to the length of the line.* The proof of this statement follows:

$$\text{Loss} = 3I^2R = \text{constant} \times \frac{(kw)^2}{E_n^2 d^2}.$$

But, by equation (588),

$$E_n^2 = \text{constant} \times d_s^2$$

whence

$$\text{Loss} = \text{constant} \times \frac{(kw^2)}{d_s^4}.$$

If in Eq. (634) the power transmitted is the only variable, (and this is approximately true in very long lines), then

$$d_s = \text{constant} \times (kw)^{\frac{1}{3}}$$

and

$$\text{Loss} = \text{constant}\ \frac{(kw)^2}{(kw)^{\frac{4}{3}}}$$

$$= \text{constant} \times (kw)^{\frac{2}{3}}.$$

Good engineering will therefore not permit one to assume a given percentage of line loss, independent of the length of line, as a basis upon which to determine other design features. This method of procedure is nevertheless frequently followed. To illustrate further: Given two lines of the same length, one of which transmits twice as much power as the other. For maximum economy, the permissible losses in the two lines should be in the ratio of $1 \div \sqrt[3]{4}$; or the losses in the heavily loaded line should be about 1.59 times as much as the losses on the lightly loaded line.

3. In Eq. (635) the sixth root of the quantity $J(F + G + H)$ appears in the denominator of the expression, whereas only the cube root of the numerator is involved. *It is apparent therefore that considerable errors may be made in estimating either of the factors F, G or H without seriously affecting the accuracy of the result.* The factors F and H can usually be very closely estimated. Considerable latitude in the variation of G is therefore permissible. The factor J, involving the value of the wasted energy, influences the final choice of diameter to a greater degree than either of the factors F, G or H, and should therefore be carefully estimated. For preliminary computations, approximate values of these constants will suffice to arrive at a general

conclusion as to the size of conductor and the line voltage for a given conductor material.

Figure 85 was prepared to illustrate the influence of the value A (selling price of energy) upon the most economical voltage. These curves show that if dependable results are to be had, care must be exercised in evaluating the selling price of energy.

One other rather important conclusion may be drawn from curves such as these. A similar set of curves, drawn for a line of say 250 miles, would show that there is little real likelihood of

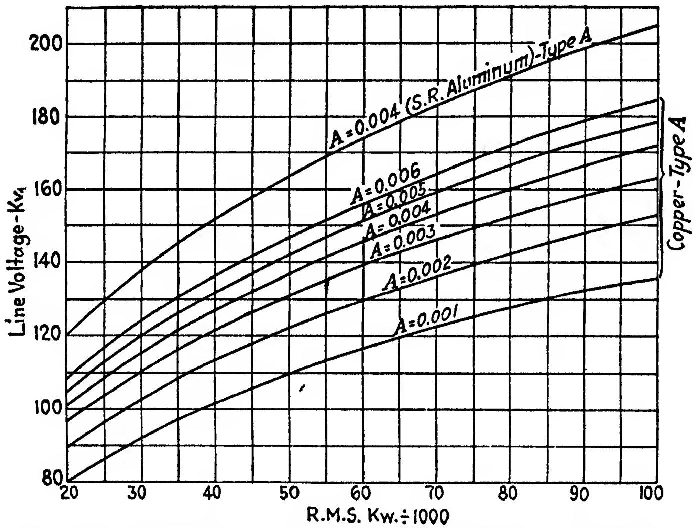


FIG. 85.—Line voltage *vs.* r.m.s. kilowats for a 50-mile line and various values of A.

economic conditions in the near future requiring transmission voltages in excess of 220 kv., where stranded conductors of the usual type are used. Figure 84 illustrates the same idea. The topmost curve in Fig. 85 is for steel-reinforced aluminum conductor. It was included to show the marked difference between the economical voltages for the two kinds of conductors.

4. Since the loss of energy in the line is inversely proportional to the square of the line voltage, the theoretically correct conductor to use is the largest one which Eq. (635) will yield. Evidently, then, from consideration of Kelvin's law as expressed by this equation, one is driven to the conclusion that the most economical development of any project requires that all of the

energy which is to be transmitted between a pair of generating and receiving stations, be delivered over a single transmission line, assuming that the power limit of the line is not exceeded.

There are certain other conditions, however, such as the necessity for guaranteeing continuity of service, facilitating repair and replacements, etc., which make it practically necessary to construct a minimum of two lines. *If, therefore, it be granted that, for reasons of practical operation, a minimum of two lines is necessary, it seems fair to state that there is no good reason for building more then two lines on any project between a given pair of stations*, provided the voltages called for do not exceed values justified by the then existing state of the art, and provided further that the limits of the capacity of a single line are not exceeded and the question of reliability of service of two lines *vs.* more than two does not enter. It is important to note also that since high voltage and large conductor diameters go together, extra high-potential lines will also be lines of rugged mechanical design, having relatively large factors of safety. The important hazards due to ice and wind loads will be reduced to comparative insignificance in such lines, because they will represent a decreasing percentage of the conductor weight as the conductor diameters increase.

Regulation.—The problem of properly controlling the line voltage, for a wide range of load conditions, becomes increasingly complex and costly as the load on the line and the distance of transmission increase. The method of automatic line regulation, now in vogue is the idling of synchronous reactors at the receiver end or at other points of the line. The function of these machines is to adjust the load power factor so that the voltages at both ends of the line may be kept constant for all loads. Automatic voltage regulators, actuated by bus potential coils, operate on the fields of these synchronous motors to change the excitation so that the kilovolt-amperes output of the reactors varies from maximum kilovolt-amperes lagging at no load to maximum leading at maximum line load. The former condition compensates for the condenser characteristics of the line at no load, and the latter compensates for the excessive lagging kilovolt-amperes of the load as it increases to a maximum, the usual load power factor being seldom greater than 0.8 lagging. This system of automatic regulation is the "constant-voltage, variable-power-factor system."

The following discussion will have reference only to the constant-voltage, variable-power-factor system of regulation as applied to lines not exceeding 300 miles in length and carrying blocks of power not in excess of the natural power limit per line.

An exhaustive study was made of required, synchronous condenser capacities for constant voltages at the generating end and the receiving end of lines varying in length from 100 to 300 miles and transmitting blocks of power up to the limit of practical operation. The conductors investigated varied from 250,000 to 2,000,000-cir. mil, metallic cross-section, and the investigation covered three materials, namely, aluminum, copper and steel. The results of this investigation may be summarized as follows:

1. The minimum synchronous reactor capacity required is almost proportional to the kilovolt-amperes carried by the line and varies between 40 and 60 per cent of the line kilovolt amperes depending somewhat upon the conductor diameter.

2. The minimum synchronous reactor capacity required decreases somewhat with an increase in conductor diameter for the same load transmitted and for the same length of line.

3. The minimum synchronous reactor capacity required increases somewhat with an increase in the length of line for the same load transmitted and for a given conductor diameter.

4. The minimum synchronous reactor capacity required is practically independent of the conductor material for the same diameter, load transmitted and line length.

From the above and from consideration of Eq. (635) it follows that the cost factors involving line regulation have little or nothing to do with Kelvin's law. An increase in length of line will slightly increase the conductor diameter for maximum line economy. Hence, since the required minimum reactor capacity decreases slightly with increased conductor diameter, and increases slightly with increased line length, an increase of both conductor diameter and line length will not affect the minimum required reactor capacity to any appreciable extent.

The correct method of procedure, in finding the most economical conductor and voltage, is to omit consideration of the required synchronous reactor capacity and proceed with the solution of Eq. (635) without involving costs of automatic regulation requirements.

Coordination of Existing Systems and New Project by Tie Lines. It often happens that the choice of voltage at the receiver end

of a line is strongly influenced by the voltage of existing lines with which the new project is to be coordinated. It is evident that, for long transmission lines, the transmission voltage is much higher than would be suitable for the distribution system which the lines feed. Hence, step-down transformers must be used between the line of the new project and the distribution bus. The same is true of possible tie lines if their operating voltages approach in magnitude that of the new project. The only object gained by equalizing the tie-line voltage with that of the new project is the possibility of interchangeability of transformers and certain other apparatus between the systems in cases of failure. This item, however, is often insignificant as compared with the factors of maximum economy which determine the choice of line voltage, and it should therefore not be given undue weight. In fact, as a first step, each line should be designed independently of any other line which happens to be tied to the same system of distribution in order to obtain maximum overall economy. Should there arise the necessity or desirability of interlinking different transmission systems outside of the distribution system, the use of auto-transformers may in the end be cheaper than the attempt to violate the requirements of Eq. (635) for each line, in an effort to equalize the transmission-line voltages for all interlinked projects.

PROBLEMS

1. Derive the empirical Eq. (585).

2. A type B, double-circuit line carries 550,000-cir. mil cables of diameter 0.855 in., for which the roughness factor $m_0 = 0.83$. If $\delta = 0.96$ and the line is to operate at 87 per cent of the critical voltage, what is the value of U_B? What is the correct line voltage? Assume that the conductor spacing is given by Eq. (580).

3. To the scale of one unit = 3,000 kw., the ordinates to the average-day load curve for a community are given in the table below.

A. M.		P. M.	
Time	Number of units	Time	Number of units
12	9.2	12:30	13.0
1	8.2	1	15.5
2	6.8	2	15.7
3	6.7	3	15.8
4	6.6	4	16.3
5	6.7	5	18.8
6	9.2	6	19.0
7	13.8	7	15.7
8	16.4	8	14.0
9	16.2	9	13.3
10	15.2	10	12.4
11	15.7	11	11.0
11:55	14.4	11:55	9.4

Calculate the mean kilowatt load and the r.m.s. kilowatt load.

If the average annual load factor of the system is 55 per cent, what is the installed machine capacity?

4. A load equal to that of Problem 3 is to be transmitted over a 125-mile, single-circuit line. The influence of the cost of ground cables may be neglected in calculating the most economical conductor diameter, and the constant G is 25,000. The elevation of the line is 900 ft. Assume: Cost of copper conductor = 18 cts. per pound; $p_1 = p_3 = 10$; selling price of power = \$0.004; weight of copper = 550 lb. per cubic foot; $\delta = 10.5$; cos $\theta = 0.85$; and substations corresponding in cost to type 3 will be used. Find the most economical copper cable diameter and the most economical line voltage.

5. Assume all conditions the same as in Problem 4 except that the steel-reinforced aluminum conductor is to be substituted for copper. On the basis of average percentages of aluminum in the cross-section of the composite cable, we may assume that the effective resistivity of the composite cable is $\rho = 19.1$, and that the weight per cubic foot of composite cable is 210 lb. Find the most economical voltage and conductor diameter for the aluminum line if the composite cable costs 34 cts. per pound.

CHAPTER XIV

VECTOR AND CIRCLE DIAGRAMS OF LINE PERFORMANCE

Vector diagrams for the solution of both short and long transmission lines have appeared from time to time in electrical engineering literature. Those pertaining to short lines usually yield approximate results and cannot therefore be applied with accuracy to lines of great length. Many of those pertaining to long lines are based on approximations, the limits of which must first be carefully investigated in order to satisfy the designer of the degree of accuracy to be expected in a given problem. The series of diagrams[1] given in this chapter are based on the exact equations of the long line developed in Chap. VIII, and contain no approximations. They are extremely useful as an aid in picturing the performance of a line, as a check on the analytical solution of the various quantities involved, and, where a semigraphical solution is desired, as a means of greatly simplifying the work of predicting line performance over any desired range of load. All of the quantities usually desired, such as current, voltage, power, reactive power and power factor at both ends of the line, as well as line regulation, may readily be obtained. Since each diagram presented is a graphical interpretation of the exact line equations, the limit of accuracy is set only by the accuracy of the drawing. Experience shows that accuracy to within 0.5 per cent is generally easily attainable for lines of any length.

Diagrams are presented covering

1. Operation with voltage control by generator excitation.
2. Operation of a straight transmission line by phase control; *i.e.*, the synchronous reactor capacity is all connected to the line at the receiver end, and the line has no intermediate taps.

[1] Loew, E. A., "Vector Diagrams for Long Lines," *Elec. World*, March 8, 1924

Basic Equations.—Throughout the following discussion the equations upon which the diagrams are based are the equations of current and voltage, namely,

$$E_s = E_rA + I_rB$$
$$I_s = I_rA + E_rC.$$

While, in these equations, the constants A, B and C are used, referring to the line only, it is apparent that, if it should be desirable to introduce the impedances of the raising and lowering transformers and to combine these with the line constants to form new constants of the composite line, this may be done in the manner suggested in Chap. VIII. One may go a step further, and, instead of considering as constant, the terminal voltage at the generator busses, one may assume a constant induced voltage in the generators themselves. The synchronous reactance of the generators would then be added to the reactance of the transformers at the supply end, and their sum would be combined with the line constants to produce the equivalent constants A_0, B_0, C_0 and D_0.

Voltage Control by Generator Excitation.—This method of control assumes constant receiver voltage with supply-voltage variable, as pointed out in the discussion of the preceding chapter.

a. Voltage Diagram.—If the supply-voltage vector be expressed in terms of its two components, its equation is

$$E_s = E_1 + E_2 \tag{636}$$

where

$$E_1 = E_rA \text{ vector volts}$$

and

$$E_2 = I_rB \text{ vector volts.}$$

If the receiver-voltage vector be taken as the zero or reference vector, and since, consistent with the above assumption, its length or size is constant, the component E_1 is a vector of constant length and of fixed angle for all loads. Its length is the constant value

$$E_1 = E_r\sqrt{a_1^2 + a_2^2} \text{ volts} \tag{637}$$

and its angle is the angle of A, that is,

$$_E\theta_1 = \tan^{-1}\frac{a_2}{a_1}. \tag{638}$$

The component E_2 is a variable vector, for it is a function of the variable receiver current I_r; it varies both in length and in angular position. Its length depends upon the amount of the receiver current, and its angle upon the receiver power factor.

The length of E_2 is the product of the lengths of I_r and B, and its angle is the algebraic sum of the receiver power-factor angle and the angle of B. Since

$$\begin{aligned} E_2 &= I_r B \\ &= ({}_rI_1 - j{}_rI_2)(b_1 + jb_2) \text{ vector volts} \end{aligned}$$

the size of the vector is

$$E_2 = \sqrt{({}_rI_1^2 + {}_rI_2^2)(b_1^2 + b_2^2)} \tag{639}$$

and its angle is

$$\begin{aligned} {}_E\theta_2 &= \tan^{-1}\frac{b_2}{b_1} - \tan^{-1}\frac{{}_rI_2}{{}_rI_1} \\ &= \tan^{-1}\frac{b_2}{b_1} - \cos^{-1}\text{(receiver power factor)}. \end{aligned} \tag{640}$$

These vectors are shown in the diagram of Fig. 86. In this figure $E_1 = OM$, $E_2 = MN$, and $E_s = ON$. If the receiver current remains constant, the vector E_2 swings about M as a center with changing receiver power factor. The positions which it would occupy at various power factors are indicated by the radial power-factor lines. On the other hand, if the power factor remains constant and the current only changes, the locus of the vector E_2 is a straight line such as MN. Its angular position remains fixed, but its length changes. With constant receiver voltage, the kilovolt-amperes received varies directly as the current, and, by the addition of a suitable kilovolt-ampere scale as indicated, the magnitude and angular positions of all voltages, for any assumed load and power factor, may readily be obtained from the diagram. Figure 86, therefore, yields the generator voltage and the regulation.

b. Current Diagram.—The vector diagram of currents results from the current equation for the line in much the same way as does the voltage diagram from the voltage equation. The supply-end current vector consists of the component current vectors

$$I_1 = I_r A \text{ vector amp.} \tag{641}$$

$$I_2 = E_r C \text{ vector amp.} \tag{642}$$

Since the receiver voltage is a constant amount, the component vector I_2 is constant in length and fixed in position for

$$C = c_1 + jc_2$$

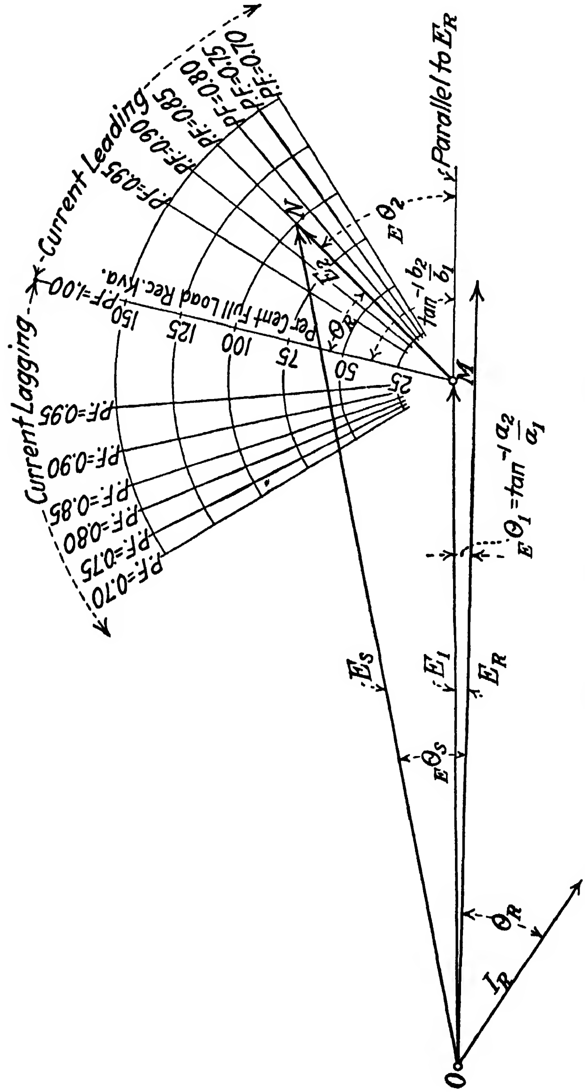


FIG. 86.—Vector diagram of voltages. Control by generator excitation.

and hence the length is

$$I_2 = E_r\sqrt{c_1^2 + c_2^2} \text{ amp.} \tag{643}$$

and its angle is

$$
\begin{aligned}
{}_I\theta_2 &= 0 + \tan^{-1}\frac{c_2}{c_1} \\
&= \tan^{-1}\frac{c_2}{c_1}.
\end{aligned} \tag{644}
$$

The component vector I_1 is a vector of varying length and varying angular position, depending upon the receiver load and power factor. At a fixed power factor, it elongates when the load increases and contracts when the load diminishes. Under these conditions its locus is a straight line. On the other hand, with constant kilovolt-amperes in the receiver circuit, the length of I_1 remains constant, but it swings through an angle with varying power factor. Its locus is now the arc of a circle.

The length of I_1 is the product of the receiver amperes and the numerical value of the constant A; its angle, with respect to the receiver voltage, is the algebraic sum of the angle of A and the receiver power-factor angle. For, since

$$
\begin{aligned}
I_1 &= I_rA \\
&= ({}_rI_1 - j{}_rI_2)(a_1 + ja_2) \text{ vector amp.}
\end{aligned}
$$

its length is

$$
I_1 = \sqrt{({}_rI_1^2 + {}_rI_2^2)(a_1^2 + a_2^2)} \text{ amp.} \tag{645}
$$

and its angle is

$$
\begin{aligned}
&= \tan^{-1}\frac{a_2}{a_1} - \tan^{-1}\frac{{}_rI_2}{{}_rI_1} \\
&= \tan^{-1}\frac{a_2}{a_1} - \cos^{-1}\text{(receiver power factor)}.
\end{aligned} \tag{646}
$$

After the derived line constants have been computed it is a simple matter to construct the vector diagram of currents from the above equations. From it the supply current may be found graphically for any load and power factor. Such a diagram is shown in Fig. 87.

In this figure, for full load at the receiver and 0.85 power factor, current lagging, the supply current is OQ', having the two components OB and BQ. At the same power factor but for varying receiver loads, the locus of this current is the straight line BQS, while, for full-load kilovolt-amperes and varying power factors, its locus is the arc of a circle, TQU. The angle ${}_I\theta_s$ of the current I_s, referred to the receiver voltage E_r may be measured on the diagram.

Thus, from the current diagram the vector currents at the supply end may be found for any receiver load and power factor. From the voltage diagram of Fig. 86, the corresponding angle $_E\theta_s$ of the supply voltage is obtained. The difference between these two angles is the generator power-factor angle. From the known supply voltage, current and power factor, the generator output may be calculated. The line loss, equal to the generator output less the receiver input, readily follows.

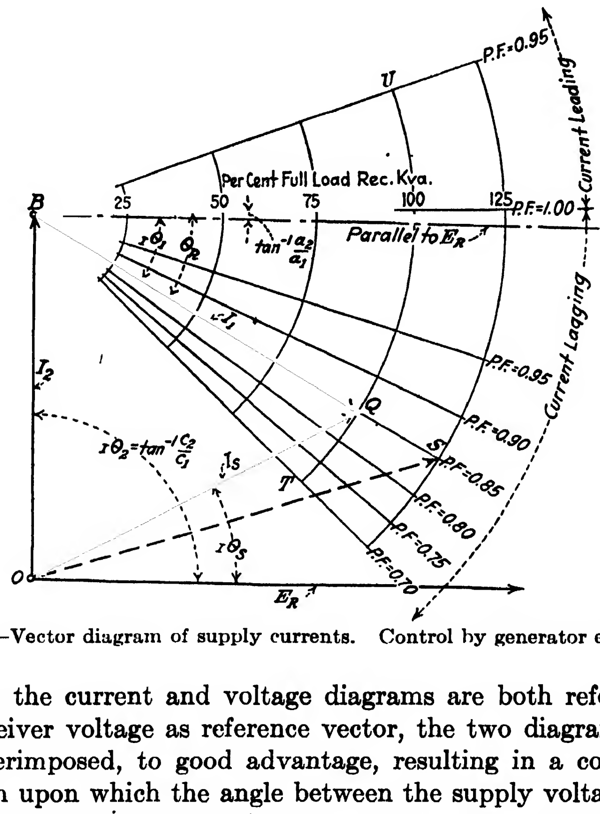


FIG. 87.—Vector diagram of supply currents. Control by generator excitation.

Since the current and voltage diagrams are both referred to the receiver voltage as reference vector, the two diagrams may be superimposed, to good advantage, resulting in a composite diagram upon which the angle between the supply voltages and current may be measured directly. Such a diagram is shown in Fig. 91, for the case of a line operating with phase control.

Voltage Control by Synchronous Reactors at Receiver Only.—By this method of control, constant voltages are maintained at both ends of the line, but both receiving-end and supply-end power factors vary with the loads, as shown in the discussion of the previous chapter. The vector diagrams of current and volt-

age for this case are not as simple as for the case just considered. They may be readily drawn on the basis of the theory considered below, however. These diagrams are exceedingly helpful as an aid to the better understanding of line performance under these conditions, besides offering a method by which accurate graphical solutions may be made.

a. Receiver-current Diagram.—When a line is operated with synchronous reactors at its receiving end to maintain constant voltages at both ends of the line, the locus of the receiver current is the circle defined by Eq. (461). It is

$$({}_rI_1 + E_rl)^2 + ({}_rI_2 + E_rm)^2 = E_s^2n^2$$

or

$${}_rI_2 = -E_rm + \sqrt{E^2n^2 - ({}_rI_1 + E_rl)^2}.$$

If the X-axis is used as the axis of the active currents ${}_rI_1$ and the Y-axis as the axis of reactive currents ${}_rI_2$, and if lagging currents (+ values of ${}_rI_2$) be represented as negative, and leading currents (− values of ${}_rI_2$) as positive in the diagram, the center of the current circle has the coordinates

$$x = -E_rl$$
$$y = E_rm$$

and its radius is

$$R = E_sn.$$

The constants are defined by Eqs. (462), (463) and (464). They are

$$l = \frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}$$

$$m = \frac{a_1b_2 - a_2b_1}{b_1^2 + b_2^2}$$

$$n = \frac{1}{\sqrt{b_1^2 + b_2^2}}.$$

Such a diagram of receiver currents is illustrated in Fig. 88, for a particular ratio of $E_s \div E_r$. The receiver current at 100 per cent load is the vector OF; its angle with respect to the receiver voltage is

$$\theta_r = \tan^{-1}\frac{HF}{MH}$$

and the receiver power factor (the power factor of the load plus the synchronous reactors) is

$$\cos \theta_r = \frac{MH}{MF}.$$

b. Receiver-power Diagram.—It has already been explained that the receiver power diagram may be obtained from the current diagram by simply multiplying Eq. (460) through by the receiver voltage. A complete discussion of this diagram is found in Chap. IX.

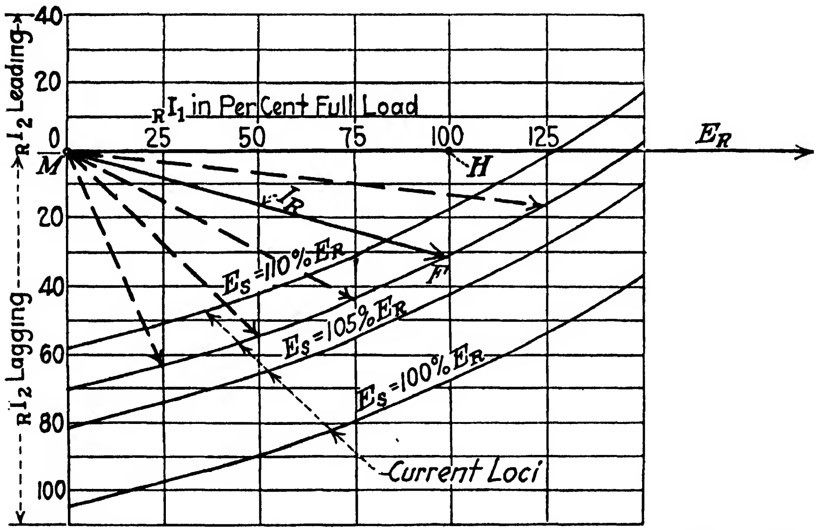


FIG. 88.—Receiver current loci for ratios of $E_s \div E_r$ shown. Operation with phase control.

Voltage Control by Synchronous Reactors at Receiving End.—*a. Supply-current Diagram.*—This diagram is obtained from an interpretation of the current equation

$$\begin{aligned} I_s &= I_rA + E_rC \\ &= I_1 + I_2 \text{ vector amp.} \end{aligned}$$

Considering the second component of current first, and using the usual notation

$$\begin{aligned} I_2 &= E_rC \\ &= E_r(c_1 + jc_2) \text{ vector amp.} \end{aligned} \tag{647}$$

Both E_r and C are constants in this equation. I_2 is therefore a vector of constant length and fixed angle, independent of the load. Its length is

$$I_2 = E_r\sqrt{(c_1^2 + c_2^2)} \tag{648}$$

and its angular position with respect to E_r is

$$_1\theta_2 = \tan^{-1}\frac{c_2}{c_1} \tag{649}$$

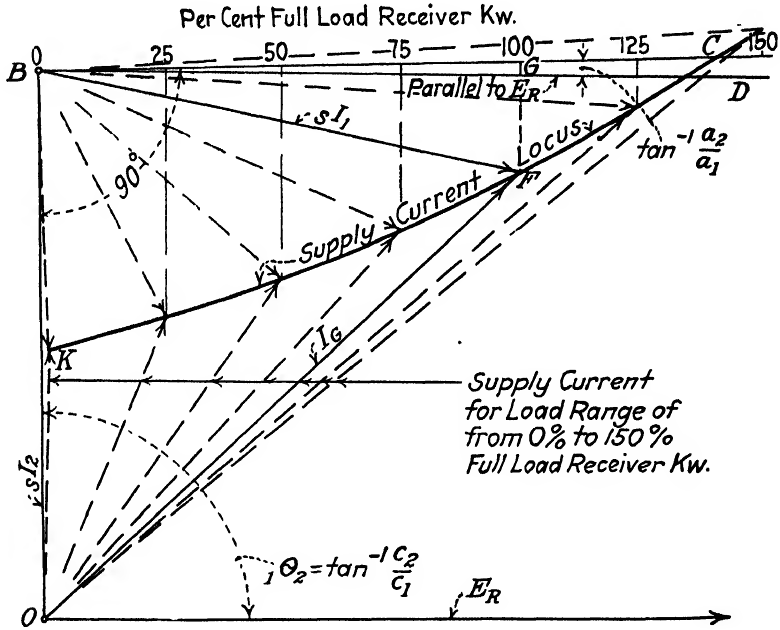


FIG. 89.—Supply current locus. Operation with phase control.

as shown in Fig. 89. As compared with c_1, c_2 is a large quantity and is positive, while c_1 is always negative. The angle $_1\theta_2$ is therefore an angle slightly over 90° and approaches 90° as a limit as the length of line approaches zero.

The supply current I_s is the sum of the constant vector I_2 above and the variable vector I_1. The latter may be resolved into two components proportional, respectively, to the in-phase and

quadrature components of the receiver load current. Using the complex notation,

$$
\begin{aligned}
{}_sI_1 &= I_r A \\
&= ({}_rI_1 - j\,{}_rI_2)(a_1 + ja_2) \\
&= {}_rI_1(a_1 + ja_2) + {}_rI_2(a_2 - ja_1) \\
&= I_{SP} + I_{SQ} \text{ vector amp.} \qquad (650)
\end{aligned}
$$

where the vector

$$I_{SP} = {}_rI_1(a_1 + ja_2) \text{ vector amp.} \qquad (651)$$

has the length

$$I_{SP} = {}_rI_1\sqrt{a_1^2 + a_2^2} \text{ amp.} \qquad (652)$$

and the angle with respect to the receiver voltage of

$$\theta_{IP} = + \tan^{-1}\frac{a_2}{a_1}. \qquad (653)$$

Similarly,

$$I_{SQ} = {}_rI_2(a_2 - ja_1) \text{ vector amp.} \qquad (654)$$

has the length

$$I_{SQ} = {}_rI_2\sqrt{a_1^2 + a_2^2} \text{ amp.} \qquad (655)$$

and the angular position with respect to the receiver voltage of

$$\theta_{IQ} = - \tan^{-1}\frac{a_1}{a_2}. \qquad (656)$$

From Eqs. (653) and (656) it is apparent that vector I_{SQ} lags vector I_{SP} by 90°.

The locus of vector I_{SP} is a straight line, since its angle $\tan^{-1}\frac{a_2}{a_1}$ is constant for all values of the vector; it is a line $\theta_a = \tan^{-1}\frac{a_2}{a_1}$ degrees ahead of parallelism with E_r. This vector is to be added to I_2. It is therefore drawn from B as an origin, making the angle $\theta_a = \tan^{-1}\frac{a_2}{a_1}$ with the line BD, the latter being parallel to E_r. The full-load value of ${}_rI_1$ is calculated from the known receiver load, or it may be taken from the receiver-current diagram. This value is multiplied by the numeric $A = \sqrt{a_1^2 + a_2^2}$ and is laid off as BG on the line OC. This represents the vector I_{SP} for 100 per cent receiver load. Corresponding vectors for other loads are found by dividing the line BC into sections proportional to the load.

It now remains to add the vector I_{sQ} to the two component vectors OB and BG. It has already been shown that the locus

of the current ${}_rI_2$ is a circle whose center has the coordinates $y = E_r m$, $x = -E_r l$. Since the length of A is constant, it is apparent, from Eq. (650), that the locus of the supply current is also a circle, its center having the coordinates $E_r m\sqrt{a_1^2 + a_2^2}$ and $-E_r l\sqrt{a_1^2 + a_2^2}$ with respect to the axes BC and BK, and its radius being $E_s n\sqrt{a_1^2 + a_2^2}$. With respect to the axes BC and BK, the supply current equation is obtained by multiplying the equation of the receiver current circle through by the constant $\sqrt{a_1^2 + a_2^2}$. This operation yields

$$\begin{aligned} I_{SQ} &= -E_r m\sqrt{a_1^2 + a_2^2} \\ &\qquad + \sqrt{E_2 n^2(a \quad + a\ ) - [{}_rI_1(a_1^2 + a_2^2)^{\frac{1}{2}} + E_r l(a_1^2 + a_2^2)^{\frac{1}{2}}]} \\ &= -E_r m_{si} + \sqrt{E^2 n_{si}^2 - (I_{SP} + E_r l_{si})^2} \end{aligned} \tag{657}$$

where

$$\left.\begin{aligned} l_{si} &= l\sqrt{a_1^2 + a_2^2} \\ m_{si} &= m\sqrt{a_1^2 + a_2^2} \\ n_{si} &= n\sqrt{a_1^2 + a_2^2} \end{aligned}\right\} \tag{658}$$

Having thus located the center and found the length of the radius, the circle may be drawn. At 100 per cent load the vector I_{SQ} is GF.

It will be observed that, with respect to the axes BK and BC, this diagram is identical in form with the receiver-current diagram already discussed, the only difference lying in the scale used. The supply-current locus may therefore be found from the receiver-current diagram by simply changing the scale of the latter in the ratio of $\sqrt{a_1^2 + a_2^2} \div 1$, and transferring it to the axes BC and BK. By drawing lines perpendicular to BC through the various points representing different percentages of receiver input, and by extending them until they meet the arc KFC, the ends of the corresponding supply-current vectors are found, and thus the vectors themselves are determined, both as to length and slope.

b. Voltage Diagram.—The supply voltage is

$$\begin{aligned} E_s &= E_1 + E_2 \\ &= E_r(a_1 + ja_2) + I_r(b_1 + jb_2) \text{ vector volts} \end{aligned} \tag{659}$$

The component E_1 (Fig. 90) is a vector of fixed length and slope. Its length is

$$E_1 = E_r\sqrt{a_1^2 + a^2} \tag{660}$$

and its angular position, referred to E_r is

$$\theta_a = \tan^{-1}\frac{a_2}{a_1}. \tag{661}$$

The second component E_2 is a variable vector, again consisting of two components, one E_{SP}, proportional to the active receiver current and the other E_{SQ}, proportional to the reactive component of the receiver current. That is,

$$\begin{aligned} E_2 &= E_{SP} + E_{SQ} \\ &= (b_1 + jb_2)({}_rI_1 - j{}_rI_2) \\ &= {}_rI_1(b_1 + jb_2) + {}_rI_2(b_2 - jb_1) \text{ vector volts} \end{aligned} \tag{662}$$

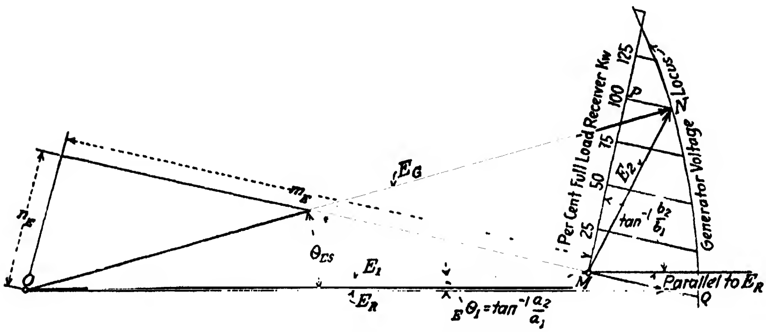


FIG. 90.—Supply voltage locus. Operation with phase control.

where

$$E_{SP} = {}_rI_1(b_1 + jb_2) \text{ vector volts} \tag{663}$$

and

$$E_{SQ} = {}_rI_2(b_2 - jb_1) \text{ vector volts.} \tag{664}$$

The length of E_{SP} is

$$E_{SP} = {}_rI_1\sqrt{b_1^2 + b_2^2} \text{ volts} \tag{665}$$

and its angular position is

$$\theta_{SP} = \tan^{-1}\frac{b_2}{b_1}. \tag{666}$$

For E_{SQ} the corresponding quantities are
The length is

$$E_{SQ} = {}_rI_2\sqrt{b_1^2 + b_2^2} \text{ volts} \tag{667}$$

and its angle is

$$\theta_{SQ} = -\tan^{-1}\frac{b_1}{b_2}. \tag{668}$$

From Eqs. (666) and (668) it is apparent that the vector E_{sQ} lags the vector E_{sP} by 90°.

The equation of the supply-voltage circle, with respect to the axes MP and MQ, is obtained by multiplying the equation for the receiver-current circle through by the constant $\sqrt{b_1^2 + b_2^2}$. this operation yields

$$
\begin{aligned}
E_{SQ} &= {}_rI_2\sqrt{b_1^2 + b_2^2} \\
&= -E_r m\sqrt{b_1 + b_2^2} \\
&\quad + \sqrt{E_s^2 n^2(b_1^2 + b_2^2) - [{}_rI_1(b_1^2 + b_2^2)^{\frac{1}{2}} + E_r l(b_1^2 + b_2^2)^{\frac{1}{2}}]^2} \\
&= -E_r m_{se} + \sqrt{E_s^2 n^2{}_{e} - [E_{SP} + E_r l_{se}]^2}
\end{aligned}
\qquad (669)
$$

where

$$
\left.
\begin{aligned}
l_{se} &= l\sqrt{b_1^2 + b_2^2} \\
m_{se} &= m\sqrt{b_1^2 + b_2^2} \\
n_{se} &= n\sqrt{b_1^2 + b_2^2}
\end{aligned}
\right\}
\qquad (670)
$$

In the vector diagram E_P is laid off at the angle $\theta_{EP} = \tan^{-1}\dfrac{b_2}{b_1}$, or along the line MP, the length of MP being taken as the length of E_P for the value of ${}_rI_1$ corresponding to full-load receiver input. The value of ${}_rI_1$ has already been computed from known full-load conditions. Since the supply voltage is constant for all loads, its locus is the arc of a circle, of radius E_s. The vector E_{SQ} lags E_{SP} by 90°, and for full-load receiver input is the line PN. E_2 is accordingly the vector MN. By dividing the line MP into sections proportional to the receiver load and drawing at right angles thereto the lines terminating in the supply-voltage locus on the arc of the circle, the corresponding ends of the supply-voltage vectors are located. The angle by which the supply voltage leads or lags the receiver voltage is θ_{ES}.

From the vector diagrams of current and voltage the complete performance of the line may be clearly visualized, and, if desired, performance curves may be worked out by their use.

c. Composite Diagram.—For use in making calculations to determine the complete performance of a line, it will be found most convenient to combine the diagrams of current and voltage into a single composite vector diagram as shown in Fig. 91. Figure 91 is drawn to represent operation with synchronous reactors connected to the line at the receiver only, and with constant receiver and supply voltages maintained at all loads. By the use of a single composite diagram the work of making the necessary drawings and of computing the performance curves is

a simple matter, once the line constants have been computed and the receiver voltage has been decided upon. The choosing of the proper receiver voltage is a matter of transmission-line economics that has already been considered in detail in Chap. XIII.

For a line having load taps either with or without additional reactor capacity, the problem of line performance may be worked out step-fashion by the method here developed for a line without taps.

Supply Power Circles.—It is desirable to have equations from which one may compute the power at the supply end of a line, or at any other point in the line where the voltage is maintained constant, such as at a tap point in a sectionalized line. For constant supply and receiver voltages the equations of supply power are circles, derived as shown below.

Since the supply power is the dot product of the supply current and voltage vectors, the method consists in finding this product from the fundamental line equations. Thus,

$$\begin{aligned} E_s &= E_1 + jE_2 \\ &= E_r(a_1 + ja_2) + ({}_rI_1 - j{}_rI_2)(b_1 + jb_2) \\ &= E_ra_1 + {}_rI_1b_1 + {}_rI_2b_2 + j(E_ra_2 + {}_rI_1b_2 - {}_rI_2b_1) \end{aligned} \tag{671}$$

and

$$\begin{aligned} I_s &= I_1 + jI_2 \\ &= ({}_rI_1 - j{}_rI_2)(a_1 + ja_2) + E_r(c_1 + jc_2) \\ &= E_rc_1 + {}_rI_1a_1 + {}_rI_2a_2 + j(E_rc_2 + {}_rI_1a_2 - {}_rI_2a_1) \end{aligned} \tag{672}$$

where

$$\left.\begin{aligned} {}_sE_1 &= E_ra_1 + {}_rI_1b_1 + {}_rI_2b_2 \\ {}_sE_2 &= E_ra_2 + {}_rI_1b_2 - {}_rI_2b_1 \\ {}_sI_1 &= E_rc_1 + {}_rI_1a_1 + {}_rI_2a_2 \\ {}_sI_2 &= E_rc_2 + {}_rI_1a_2 - {}_rI_2a_1 \end{aligned}\right\} \tag{673}$$

The supply power is

$$P_s = E_s \cdot I_s = E_1I_1 + E_2I_2. \tag{674}$$

Upon carrying out the multiplication indicated by Eq. (674), simplifying, and substituting for ${}_rI_1$ and ${}_rI_2$ their respective equivalents $\frac{P_r}{E_r}$ and $\frac{Q_r}{E_r}$, there results

$$P_s = \left(\frac{P_r}{E_r}\right)^2(a_1b_1 + a_2b_2) + P_r(a_1^2 + a_2^2 + b_1c_1 + b_2c_2) +$$
$$\left(\frac{Q_r}{E_r}\right)^2(a_1b_1 + a_2b_2) + Q_r(b_2c_1 - b_1c_2) + E_r^2(a_1c_1 + a_2c_2). \tag{675}$$

In order to simplify the notation in Eq. (675), let

$$\left.\begin{aligned} s &= a_1c_1 + a_2c_2 \\ u &= a_1^2 + a_2^2 + b_1c_1 + b_2c_2 \\ v &= b_2c_1 - b_1c_2 \\ w &= a_1b_1 + a_2b_2 \end{aligned}\right\} \tag{676}$$

Dividing Eq. (675) through by w and using the simplified notation,

$$\frac{P_s}{w} = \frac{P_r^2}{E_r^2} + \frac{P_r u}{w} + \frac{Q_r^2}{E_r^2} + \frac{Q_r v}{w} + \frac{E_r^2 s}{w}. \tag{677}$$

Completing squares

$$\left(\frac{P_r}{E_r} + \frac{E_r u}{2w}\right)^2 + \left(\frac{Q_r}{E_r} + \frac{Ev}{2w}\right)^2 = \frac{P_s}{w} + \left[\frac{u^2 + v^2}{4w^2} - \frac{s}{w}\right]E^2{}_r. \tag{678}$$

Equation (678) is the equation of a family of concentric circles, each representing a constant amount of supply power. The square root of the right-hand member is the radius. Remembering that, in order to represent lagging reactive power in the fourth and leading reactive power in the first quadrant, the sign of the ordinate to the center must be reversed, and making this change to conform to the usual method of representation for leading and lagging currents, the coordinates of the center are:

$$\left.\begin{aligned} \frac{P_r}{E_r} &= -\frac{E_r u}{2w} \\ \frac{Q_r}{E_r} &= +\frac{E_r v}{2w} \end{aligned}\right\} \tag{679}$$

Circles of Constant Power Loss.—By subtracting from the right-hand member of Eq. (678) the power delivered to the receiver circuit, the following expression for the line loss P_L, is obtained.

$$\begin{aligned} P_L &= (P_s - P_r) \\ &= \frac{P_r^2}{E_r^2}w + P_r(u - 1) + \frac{Q_r^2}{E_r^2}w + Q_r v + E_r^2 s \end{aligned} \tag{680}$$

or

$$\left[\frac{P_r}{E_r} + \frac{E_r(u - 1)}{2w}\right]^2 + \left[\frac{Q_r}{E_r} + \frac{E_r v}{2w}\right]^2 = \frac{P_L}{w} + E_r^2\left[\frac{(u - 1)^2 + v^2}{4w^2} - \frac{s}{w}\right] \tag{681}$$

where the symbols u, v, w and s have the meanings assigned to them by (676).

The circles of constant loss, given by Eq. (681), are concentric circles with centers at

$$\left.\begin{aligned} \frac{P_r}{E_r} &= -\frac{E_r(u-1)}{2w} \\ \frac{Q_r}{E_r} &= +\frac{E_r v}{2w} \end{aligned}\right| \tag{682}$$

and of radii

$$R = \sqrt{\frac{P_L}{w} + E_r^2\left[\frac{(u-1)^2 + v^2}{4w^2} - \frac{s}{w}\right]}. \tag{683}$$

Figure 92 shows the circles of constant power loss for the 200-mile line of Chap. XVI.

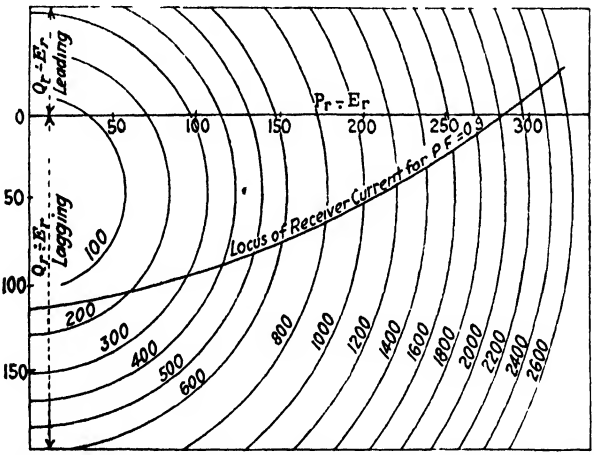


FIG. 92 — Loss circles for the 200-mile line of Chapter XVI. Indices on circles indicate kilowatts loss per phase

Supply Reactive Power Circles.—The reactive power at the supply end of a line, or at any other point in the line at which constant voltage is maintained, such as a tap point in a sectionalized line, may be expressed in terms of the receiver active and reactive powers, the receiver voltage and line constants. The expression is derived from the fundamental equations of current and voltage by a method exactly similar to that used in obtaining the active power equation for the supply end (Eqs. (671) and (672)). Since the reactive power is the cross product of current and voltage, the supply end reactive power Q_s is

$$\begin{aligned} Q_s &= E_s \times I_s \\ &= E_1 I_2 - E_2 I_1. \end{aligned} \tag{684}$$

From Eqs. (673) and (684),

$$Q_s = (E_r a_1 + {}_rI_1 b_1 + {}_rI_2 b_2)(E_r c_2 + {}_rI_1 a_2 - {}_rI_2 a_1) - (E_r a_2 + {}_rI_1 b_2 - {}_rI_2 b_1)(E_r c_1 + {}_rI_1 a_1 + {}_rI_2 a_2). \quad (685)$$

Upon completing the multiplication indicated in Eq. (685), simplifying and substituting for ${}_rI_1$ and ${}_rI_2$ their equivalents $\frac{P_r}{E_r}$ and $\frac{Q_r}{E_r}$, respectively, one obtains the expression

$$Q_s = \left(\frac{P_r}{E_r}\right)^2 (a_2b_1 - a_1b_2) - P_r(b_2c_1 - b_1c_2) + \left(\frac{Q_r}{E_r}\right)^2 (a_2b_1 - a_1b_2) - Q_r(a_1^2 + a_2^2 - b_1c_1 - b_2c_2) + E^2(a_1c_2 - a_2c_1). \quad (686)$$

In order to simplify the notation in Eq. (686), the following substitutions are made:

$$\left.\begin{aligned} s' &= a_1c_2 - a_2c_1 \\ u' &= a_1^2 + a_2^2 - b_1c_1 - b_2c_2 \\ v &= b_2c_1 - b_1c_2 \\ w' &= a_2b_1 - a_1b_2 \end{aligned}\right\}. \quad (687)$$

Equation (686) then becomes

$$\frac{Q_s}{w'} = \frac{P_r^2}{E_r^2} - \frac{P_r v}{w'} + \frac{Q_r^2}{E_r^2} - \frac{Q_r u'}{w'} + \frac{E_r^2 s'}{w'}. \quad (688)$$

Completing squares,

$$\left(\frac{P_r}{E_r} - \frac{E_r v}{2w'}\right)^2 + \left(\frac{Q_r}{E_r} - \frac{E_r u'}{2w'}\right)^2 = \frac{Q_s}{w'} + E_r^2\left[\frac{u'^2 + v^2}{4w'^2} - \frac{s'}{w'}\right]. \quad (689)$$

Here again, Eq. (689) is the equation of a family of concentric circles, each representing a constant value of reactive power Q_s. The radius of a circle of the family is

$$R = \sqrt{\frac{Q_s}{w'} + E_r^2\left[\frac{u_2' + v^2}{4w'^2} - \frac{s'}{w'}\right]}. \quad (690)$$

and the center is at

$$\left.\begin{aligned} \frac{P_r}{E_r} &= \frac{E_r v}{2w'} \\ \frac{Q_r}{E_r} &= -\frac{E_r u'}{2w'} \end{aligned}\right\}. \quad (691)$$

The circles of Eq. (689) are particularly helpful in finding the synchronous reactor capacity required at some tap point of a sectionalized line in order to maintain constant voltage at that point as well as at the supply and receiver ends of the line.

PROBLEMS

1. A three-phase, 60-cycle line is 250 miles long and delivers 84,000 kw. at full load. The line conductors are three 500,000-cir. mil, copper cables, and the generalized line constants are those found in Appendix D. The receiver voltage is constant and equal to 154 kv. at all loads. The supply voltage is also held constant at all loads by means of synchronous reactors installed at the receiver end of the line. The supply voltage is determined by the condition that its value shall be such as will require the minimum capacity in synchronous reactors at the receiver end. The full-load ratings of the reactors on the leading side are 1.5 times their ratings as generators of lagging kilovolt-amperes. If the *load* power factor may be considered constant,[1] and equal to 87 per cent lagging at all loads, draw a *composite diagram*,[2] and calculate the quantities called for below:

a. Draw the receiver-voltage vector diagram showing E_1, E_2 and E_r for full load and 87 per cent power factor.

b. Calculate the best ratio of $E_s \div E_r$, and check the calculated value graphically. Locate the corresponding E_s circle on the diagram. This will be the supply voltage used.

c. Calculate the synchronous reactor capacity required. Check graphically.

d. Calculate the scale for measuring active and reactive powers on the receiver power diagram, and locate the points representing 0, 25, 50, 75 and 100 per cent loads.

e. Draw the receiver-current vector for each load in (*d*).

f. Locate the center and draw the supply-current circle.

g. Draw the supply-current vectors.
h. Calculate the supply power factor.
i. Calculate the supply active power.
j. Calculate the supply reactive power.
k. Calculate the line percentage loss.
} For each load point mentioned in (*d*) above.

l. Draw curves of supply current, supply power factor and percentage line loss against receiver loads in kilowatts, using the latter as abscissas.

2. Draw the supply active power circles for the conditions of Problem 1, covering the range of loads from 0 to 100 per cent of full load. Read from

[1] Actually, the power factor at light loads will be considerably less than 87 per cent; but it is the maximum load that determines the synchronous condenser capacity required, and for this load the assumption of a power factor of 87 per cent is reasonable.

[2] Use a voltage scale of 10 kv. to the inch, and remember that graphical methods require that the drawings be carefully made if accurate results are to be expected.

the diagram the supply powers corresponding to each load mentioned in (*d*) of Problem 1, and compare with the values found in (*i*) of Problem 1.

3. Draw the supply reactive power circles for the conditions of Problem 1, covering the range of loads from 0 to 100 per cent of full load. Read from the diagram the reactive powers corresponding to the loads mentioned in (*d*) of Problem 1, and compare with the values found in (*j*) of Problem 1.

4. Draw the loss circles corresponding to the conditions of Problem 1, and covering the range of loads from 0 to 100 per cent of full load. Read from the diagram the loss corresponding to each load mentioned in (*d*) of Problem 1, and compare with the corresponding values computed in (*k*) of Problem 1.

CHAPTER XV

POWER LIMITS OF TRANSMISSION LINES

Power Limits. General.—The question of the power limits of transmission-line networks is a very complex one, and, therefore, one which, in the limited extent of the present volume, must necessarily be treated in a very incomplete manner. Perhaps, however, enough may be said to give a fair idea of the scope of the problem, to indicate what variables are involved, and to suggest how these variables affect the solution.

The power limit of a transmission line is not fixed by the line alone, but depends to some extent upon the nature and characteristics of every piece of apparatus connected to it. It is reached at the point where, for whatever reason, the synchronous apparatus at the two ends of the line break out of step. This may happen for either one of two reasons, namely, (*a*) because, due to *gradually applied loads*, the angle of displacement between the excitation voltages at the two ends of the line has reached its maximum possible value, or, (*b*) because the same angle has been reached due to changes in load resulting from a short circuit, the switching in of load, or other cause. These two conditions under which the limit of output of a line may be reached are designated as, (*a*) *steady-state stability;* and (*b*) *transient stability.* It is apparent that the limit of output which it is possible to reach is greater for (*a*) than for (*b*).

During the past several years the subject has received a great deal of attention at the hands of interested engineers, and considerable literature dealing therewith is now available in the technical journals. A short bibliography of important articles and discussions is given at the end of the chapter. In the discussion following, liberal use has been made of the material contained in these references.

Receiver Power Circle-diagram.—Because this diagram forms the basis of much of the discussion on power limits, it will be given further brief consideration. The equation upon which the diagram is based has been discussed in the chapter on voltage

control. It may be written in any one of several useful forms, one of which is

$$Q_r = E^2\left[-m + \sqrt{\left(\frac{E_s}{E_r}\right)^2 n^2 - \left(\frac{P_r'}{E_r^2} + l\right)^2}\right]. \tag{692}$$

If the constants l, m, and n are each multiplied by 1,000, that is, if the "kilo-constants" are used in the equation, then the voltages are in kilovolts, and the active and reactive powers are in kilowatts and kilovolt-amperes, respectively.

The active and reactive powers of Eq. (692) may be represented on the voltage diagram of Fig. 90, here redrawn and slightly modified in Fig. 93. The active power is laid off on the axis of reals MP while the reactive power appears on the MQ axis. Again, lagging reactive powers are negative, and leading reactive powers are positive in the diagram.

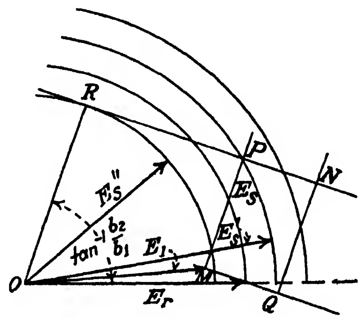


FIG. 93.—Theoretical limit of output.

That the component voltages E_{MQ} and E_{MP}, of Fig. 90, may be converted to power by simply changing the scale of the diagram is clear, for

$$E_{MP} = {}_rI_1\sqrt{b_1^2 + b_2^2} \text{ volts} \tag{693}$$

and

$$E_{MQ} = {}_rI_2\sqrt{b_1^2 + b_2^2} \text{ volts} \tag{694}$$

But

$$P_r = {}_rI_1E_r \text{ watts} \tag{695}$$

and

$$Q_r = {}_rI_2E_r \text{ volt amperes.} \tag{696}$$

Whence, substituting for ${}_rI_1$, and ${}_rI_2$ their values from Eqs. (693) and (694) in Eqs. (695) and (696),

$$P_r = \frac{E_rE_{sP}}{\sqrt{b_1^2 + b_2^2}} \text{ watts} \tag{697}$$

and

$$Q_r = \frac{E_rE_{sQ}}{\sqrt{b_1^2 + b_2^2}} \text{ volt-amp.} \tag{698}$$

Therefore, by multiplying the voltage scale by $\dfrac{E_r}{\sqrt{b_1^2 + b_2^2}}$, the

distances along MP and MQ become true and reactive powers, respectively.

Power Limit of Line and Transformers in Steady-state Operation. (*a*) *Graphical Method.*—Let us consider the steady-state load limit of the line and transformers, delivering power to a load having a fixed power factor, and let the synchronous condenser capacity available be as large as may be required. The voltage is assumed constant at the low-tension busses at both ends of the line. In other words, under this assumption the impedances of the generators are negligible with respect to the line and transformer impedances. This condition does not exist in practice, yet the case is of interest as illustrating the principle involved as well as preparing the way for later discussion. The load is assumed to be added in small increments so that all transient effects are of negligible magnitude.

A diagram similar to Fig. 90 may be drawn to include the influence of transformer impedances by using the equivalent constants in the calculation of l, m and n. In this diagram, Fig. 93, it is seen that for any constant receiver reactive power and constant receiver voltage, the supply voltage is variable, and its locus is a straight line parallel to MP, such as QN, for example. When the receiver power factor is unity, the locus is NP. On the other hand, for any constant receiver load, the locus of the supply voltage is a line parallel to MQ. At the load P it is RN. The minimum supply voltage that will suffice to deliver an assumed load MP, is the voltage $E_s'' = OR$, and if the supply voltage E_s'' is held constant at this value its locus is the arc of the circle shown. MP is the maximum power which can be delivered by the line under assumed conditions. Any further swing of the voltage vector away from the receiver voltage will bring about a reduction of delivered power, and the synchronous apparatus at the two ends of the line will fall out of step. The angle at which this occurs is $\theta_b = \tan^{-1} \dfrac{b_2}{b_1}$, since OR is parallel to MP. Thus the point of tangency of the normal to NP with the circle of radius E_s'' represents the limit of output of the line and transformers for the particular values chosen for E_s and E_r.

The delivered power could, of course, be further increased by increasing the supply voltage. The effect on the power limit of varying the supply voltage is clearly shown by drawing a series of power circles,[1] each representing a different receiver voltage,

[1] Evans and Sels, "Power Limitations," *Trans.*, A. I. E. E., p. 27, 1924.

the supply voltage remaining the same. Thus keeping the supply voltage constant and assuming various ratios of $E_r \div E_s = s$, such as 1.0, 0.9, 0.8, etc., one may draw a separate power circle for each ratio as in Fig. 94. Since

$$(P_r + E_r^2 l)^2 + (Q_r + E_r^2 m)^2 = E_s^2 E_r^2 n^2 = s^2 E_s^4 n^2$$

the coordinates of the centers are

$$P_r = s^2 E_s^2 l$$
$$Q_r = s^2 E_s^2 m$$

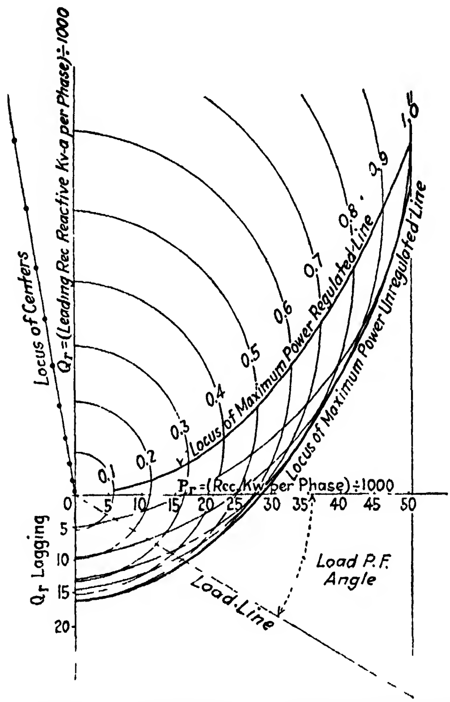


FIG. 94.—Line output circles for E_s constant and E_r variable.

and the radii are given by the equation

$$R = sE_s^2 n.$$

The locus of the centers is a straight line through the origin making the angle $\theta = \tan^{-1} \frac{m}{-l}$ with the power axis. The distance out on this line from the origin to the several centers is proportional to s^2. A curve through the points of tangency of the power circles with a line parallel to the OQ axis, is the locus of the maxi-

mum steady-state power limit of the regulated line, subject to the original assumptions. The capacity in synchronous condensers, required at this limit for the constant-load power factor represented by the load line, is given by the intercept of a line parallel to the OQ axis and lying between the load line and the point of tangency.

If no synchronous reactors were used at the end of the line to keep constant receiver voltage, the power limit would occur at the intersection of the load line with the envelope to the power circles. Since no power circles extend beyond this limit, the region beyond represents inoperative conditions.

b. Analytical Solution.—Obviously, the above conditions defining the power limits of the line (or of the line including transformers if equivalent constants are used) are mathematically deducible from the power circle equation for the regulated line, one form of which is

$$Q_r = -E_r^2 m + \sqrt{E_s^2 E_r^2 n^2 - (P_r + E_r^2 l)^2}.$$

In general, for all positive values of the radical, Q_r has two values, both representing intersections of the circle with a line parallel to the OQ axis. The lower intersection represents stable operation and the upper one represents unstable operation. For constant values of E_s the quantity Q_r is single-valved only (a) when E_r is zero, an impossible operating condition, or (b) when a line parallel to OQ is tangent to the circle, or at the maximum load limit. The value of P_r which satisfies condition (b) is found by equating the quantity under the radical to zero. For this condition,

$$Q_r = -mE_r^2$$

or, in terms of the ratio $E_r \div E_s = s$ and the constant supply voltage,

$$Q_r = -m\frac{E_s^2}{s^2}$$

$$= -\frac{E_s^2}{s^2}\,\frac{a_1b_2 - a_2b_1}{b_1^2 + b_2^2} \qquad (699)$$

where the negative sign means leading, reactive kilovolt-amperes.

The condition for corresponding maximum output is

$$P_r + E_r^2 l = E_r E_s n$$

or

$$P_r = E_s^2(sn - s^2 l). \qquad (700)$$

The above value of P_r may still be varied by varying the ratio, s. The ratio which yields the maximum output is found by differentiating with respect to s and solving for maximum power. Thus,

$$\frac{dP_r}{ds} = E_s^2 n - 2sE_s^2 l = 0$$

and

$$s = \frac{E_r}{E_s} = \frac{n}{2l}. \tag{701}$$

In terms of constant supply voltage E_s, the theoretically maximum power which can be transmitted is found by substituting this value of s in Eq. (700), whence,

$$\begin{aligned} {}_{\max.}P_r &= \frac{E_s^2 n^2}{4l} \\ &= \frac{E^2}{4(a_1b_1 + a_2b_2)}. \end{aligned} \tag{702}$$

Synchronous Motor Supplied from Constant Voltage Mains. Steady-state Stability.—In steady-state stability it is assumed that the load is added in increments so small that the transient conditions associated with the change in load are negligible. If conditions of operation are satisfactory at a given load, and if a slight increment of synchronous motor load is added, all excitations remaining the same, the limit of stability is reached when, under the added increment of load, the excitations are no longer sufficient to supply the power demanded. The assumption of constant excitations is valid, since a drop in voltage must occur before the voltage regulators can operate to bring it back to normal. When the load is the maximum which the given excitation makes it possible to carry, further load increase causes the motor to fall out of step.

A very good way to get a picture of the problem involved in steady-state power limits is to consider the case of a synchronous motor to which energy is being supplied from a constant voltage main. When the motor is unloaded and its excitation is such that the induced voltage E_m is equal to the induced voltage E_r of the equivalent generator representing the bus, then, neglecting losses, the two voltages are in phase. Considering the series circuit through the motor and the equivalent generator, these electromotive forces are in phase opposition as shown in Fig. 95, for E_m in the position 1. If an infinite bus be assumed, the imped-

ance of the equivalent generator is zero and E_r is also the terminal voltage. When the motor is loaded, its excitation remaining constant, its induced voltage drops behind to some new position as 2, for example, in order to permit the passage of the increased current required to carry the additional load. Assuming a constant synchronous impedance, the impedance drop in the motor armature is the potential difference, E_z, and the current through the motor is I_m, making a constant angle with E_z. Neglecting

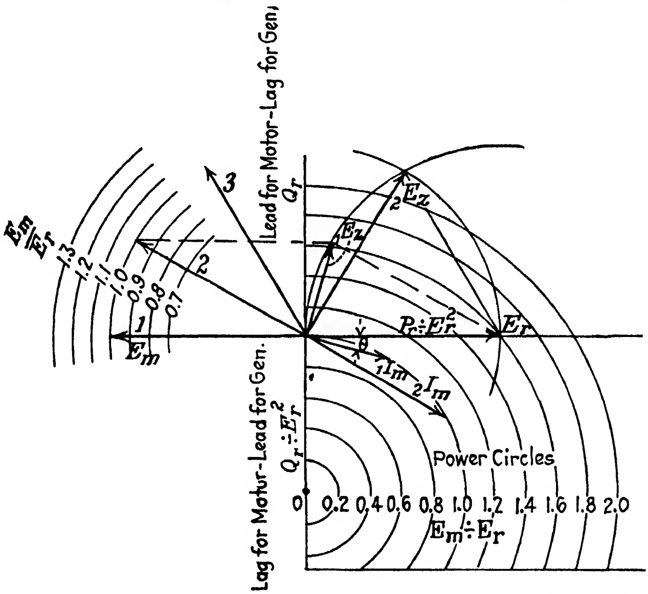


FIG. 95.—Synchronous motor operating on constant voltage bus.

motor resistance, this angle is 90°. The power delivered to the motor is

$$P_m = E_r I_m \cos \theta$$

where θ is the angle between E_r and I_m.

When the motor load is indefinitely increased, the vector E_m swings along the arc of a circle, occupying the successive positions 1, 2, 3, etc. The corresponding locus of E_z is likewise the arc of a circle, as is also that of the current I_m.

Since the impressed voltage is constant, the current circles are also circles of power, or of $P_r \div E_r^2$, to different scales. Distances parallel to the axis of $P_r \div E_r^2$ thus represent active power input to the motor, while those parallel to the quadrature axis or axis of $Q_r \div E_r^2$ represent reactive input power to the motor. The

equation of the circles is of the form of Eq. (470) in which $l = 0$ (since the resistance is assumed zero), $m = -\frac{1}{X_m}$ and the radius is $\frac{E_m}{E_r X_m}$, where X_m is the synchronous reactance of the motor. Thus, the equation is

$$\frac{P_r}{E_r^2} + \left(\frac{Q_r}{E_r^2} + m\right)^2 = \frac{E_m^2}{E_r^2 X_m^2}. \tag{703}$$

The circles in the figure indexed 0, 0.2, 0.4, 0.6, etc., are the power circles corresponding to these ratios of voltages $E_m \div E_r$.

As the motor load is increased, further and further, the angle between the voltages E_m and E_r increases accordingly, and with it the power input required to carry the load increases. An angle is finally reached, however, such that any further increase in angle will not result in a corresponding increase in power input to the motor. The resisting torque of the load then exceeds its propelling torque, and the motor falls out of step. In the circle diagram of Fig. 95, this condition occurs where a line parallel to the Q_r axis is tangent to the particular power circle representing the existing excitation of the motor. For a motor of zero resistance, the angle at pull-out is 90°. Actually, it is, $\theta = \tan^{-1}\frac{X_m}{r}$.

If, when the motor pulled out, the excitation was such that $E_m \div E_r = 1$, for example, the motor load could be further increased by increasing the excitation to a new value corresponding to some higher value of $E_m \div E_r$, such as 1.2. This increased excitation would increase the load at pull-out to 120 per cent of the former value. Thus it is seen that load conditions requiring a high excitation lead to a high pull-out load, and *vice versa*.

In the above discussion we have considered the impedance of the motor only. In practice the motors are connected in series with additional impedances. These include the impedance of the transmission circuits, and transformers at the supply and receiving ends of the line. Furthermore, the bus is not an "infinite bus," and the impedance of the equivalent generator, representing the generating system, further restricts the maximum load which can be carried under steady-state conditions; for it is the angle between the generator and motor excitation voltages which determines the point of pull-out.

Equivalent System.—When a network, consisting of one or more transmission lines having distributed resistance, inductance

and capacitance, is connected at the supply end through transformers to one or more generating stations in parallel, and at the receiving end through transformers to various kinds of load, the steady-state load limit of the line may still be calculated, provided the complex system can be replaced by an equivalent simple circuit.

The constants of two transmission lines in parallel may be combined to form the constants of an equivalent line. The resistances and the reactances of the transformer banks at the ends of the line may, in turn, be combined with the resultant line constants to form an equivalent line in which the transformer impedances are included. The synchronous impedance of the generating system may also be included, so that, finally, the entire network, including step-up and step-down transformers, the transmission line or lines, and the generators will be represented by four equivalent line constants, A_0, B_0, C_0 and D_0 as pointed out in Chap. VIII.

Equivalent Synchronous Impedance.[1]—In the foregoing, as well as in the following pages containing discussions on power limits, where the synchronous impedances of generators and synchronous condensers are involved, a constant equivalent impedance is used. The synchronous impedance, as ordinarily defined, is a variable quantity, depending upon operating conditions. An equivalent impedance, which may be calculated from the machine characteristics for any given condition of operation, is used instead. This impedance may be considered as a constant for the operating conditions for which it is computed.

Thus, in Fig. 97, are represented the characteristic curves of a synchronous condenser having the usual range of leading to lagging reactive kilovolt-amperes in the ratio of 1.5 to 1. Let X'_c and E'_c be the equivalent synchronous reactance and the equivalent excitation voltage of the synchronous condenser respectively at 100 per cent voltage and given excitation, and let

E_c = actual excitation voltage.
E_t = condenser terminal voltage.
X_c = synchronous impedance at 100 per cent voltage and given excitation as calculated from the condenser V-curves.

[1] Clarke, Edith, "Steady-state Stability in Transmission Systems," paper presented at the Midwinter Convention of the A. I. E. E., New York, Feb. 11, 1926.

From the characteristic curves of Fig. 97 and at 100 per cent voltage and for a given excitation such as that corresponding to the curve through P, for example, the slope of the characteristic curve is practically a straight line. Then, since kilovolt-amperes and current are proportional to each other at a given voltage, if the voltage is near normal, the current at any point on the characteristic curve such as P' is

$$I = I_0 + I_x.$$

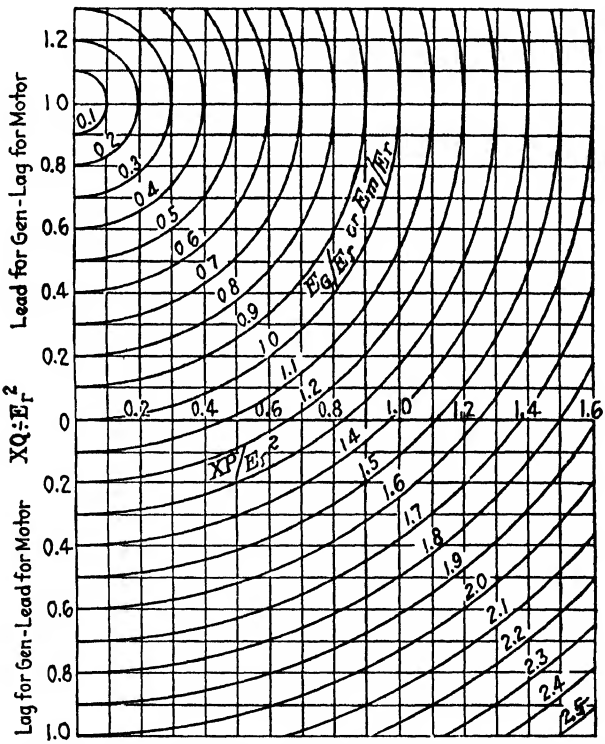


FIG. 96.—Circle diagram for synchronous motor or generator of constant synchronous reactance X, — 100,000 kv.-a. base.

Expressing terminal voltage in percentage of normal, and reactances in percentage,

$$I_0 = \frac{E_t - 1}{X}$$

and

$$\frac{dE_t}{dI} = \frac{1 - E_t}{I_x} = X', \text{ a constant}$$

whence

$$
\begin{aligned}
I &= \frac{E_c - 1}{X} + \frac{1 - E}{X'} \\
&= \left(\frac{E_cX'_c - X'_c + X_c}{X_c} - E_t\right)\frac{1}{X'_c} \\
&= \frac{E'_c - E_t}{X'_c}.
\end{aligned}
$$

Accordingly,

$$
E'_c = \frac{E_cX'_c - X'_c + X_c}{X_c} \tag{704}
$$

and

$$
X'_c = \frac{dE_t}{dI_c}. \tag{705}
$$

Calculation of Steady-state Power Limit for the 200-mile Line of Chap. XVI.—To illustrate somewhat in detail how a line operating under known conditions of load and voltage may be

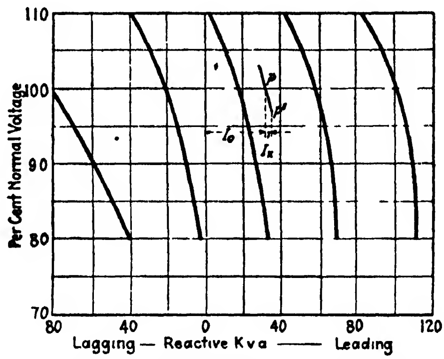


FIG. 97.—Synchronous condenser characteristic curves.

investigated for its steady-state power limit, the line whose design is worked out in Chap. XVI will be used as an example. The method[1] to be used is a graphical one. It is based on the power circle diagram, and fits in well with the previous discussion.

a. Given Data.—The following data will be assumed as applying to the 200-mile line to be investigated (Chap. XVI).*

[1] For further discussion of this method, see paper entitled, "Steady-state Stability in Transmission Systems," *Proc.*, A. I. E. E., April, 1926, by Miss Edith Clarke. By kind permission of the author Figs. 96, 97 and 98 are reproduced from her paper.

Generators:

Kilovolt-amperes installed = 100,000 or 33,330 per phase
Equivalent synchronous reactance (assumed) 85 per cent.
Frequency = 60 cycles

Length of line = 200 miles

Line constants (Chap. XVI) are

$$
\begin{aligned}
r &= 25.59 \text{ ohms} \\
x &= 162.57 \text{ ohms} \\
b &= 10.45 \times 10^{-4} \text{ mho} \\
g &= 0 \\
A &= a_1 + ja_2 = 0.9163 + j0.0130 \\
B &= b_1 + jb_2 = 24.16 + j158.07 \\
C &= c_1 + jc_2 = (-0.457 + j101.57)10^{-5} \\
l' &= 0\,9461 \\
m' &= 5.652 \\
n' &= 6.254
\end{aligned}
$$

Supply-end transformers:

Kilovolt-amperes installed = 100,000, (33,330 per phase)
Voltages 13,200 − 169,000
Resistance = 1 per cent
Reactance = 10 per cent

Receiver-end transformers:

Kilovolt-amperes installed = 90,000, (30,000 per phase)
Voltages 150,000 − 13,200
Resistance 1 per cent
Reactance 10 per cent.

b. Equivalent Line Constants.—The equivalent line constants, including transformers and generator impedances may be found from the given data by the procedure following All constants will be referred to the line voltages, and equivalent single-phase values will be used.

$$I_s = \text{supply current at rated transformer capacity} = \frac{100,000}{169} = 591.7 \text{ amp.}$$

$$I_r = \text{receiver current at rated transformer capacity} = \frac{90,000}{150} = 600.0 \text{ amp.}$$

X_G = generator synchronous reactance $= \dfrac{0.85 \times 169{,}000}{591.7} =$ 242.8 ohms.

r_{ST} = resistance of supply-end transformers $= \dfrac{0.01 \times 169{,}000}{591.7}$ = 2.86 ohms.

x_{ST} = reactance of supply-end transformers $= \dfrac{0.10 \times 169{,}000}{591.7}$ = 28.56 ohms.

r_{RT} = resistance of receiver-end transformers $= \dfrac{0.01 \times 150{,}000}{600}$ = 2.50 ohms.

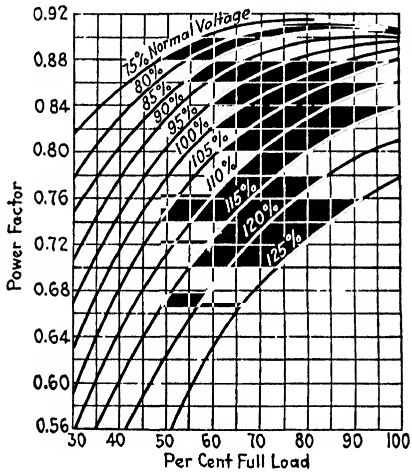


FIG. 98.—Variation of power factor of an average induction motor with load and voltage.

x_{RT} = reactance of receiver-end transformers $= \dfrac{0.10 \times 150{,}000}{600}$ = 25.00 ohms.

The complex impedances of the various elements to be considered in evaluating the equivalent circuit constants may now be tabulated. They are

Z_G = generator impedance $= + j242.8$
Z_L = line impedance $= 25.59 + j162.57$
Z_{ST} = supply-end transformer impedance $= 2.86 + j\,28.56$
Z_{RT} = receiver-end transformer impedance $= 2.50 + j\,25.00$

The equivalent constants for the line, including transformers, by the method of Chap. VIII, are

$$
\begin{aligned}
A_0 &= A + CZ_{ST} \\
&= 0.9163 + j0.0130 + 10^{-5}(-0.457 + j101.57)(2.86 + j28.56) \\
&= 0.8873 + j0.0158 \\
B_0 &= B + A(Z_{ST} + Z_{RT}) + CZ_{ST}Z_{RT} \\
&= 24.16 + j158.07 + (0.9163 + j0.0130)(5.36 + j53.56) \\
&\quad + 10^{-5}(0.457 + j101.57)(-706.86 + j142.90) \\
&= 28.23 + j206.50 \\
C_0 &= C = 10^{-5}(-0.457 + j101.57) \\
D_0 &= A + CZ_{RT} \\
&= 0.9163 + j0.0130 + 10^{-5}(-0.457 + j101.57)(2.50 + j25.00) \\
&= 0.8909 + j0.0154.
\end{aligned}
$$

The equivalent constants of the line, including transformers and the synchronous impedance of the generators, are

$$
\begin{aligned}
A_{00} &= A_0 + CZ_G \\
&= 0.08873 + j0.0158 + j242.8 \times 10^{-5}(-0.457 + j101.57) \\
&= 0.6407 + j0.0147 \\
B_{00} &= D_0Z_G + B_0 \\
&= j242.8(0.8909 + j0.0154) + 28.23 + j206.50 \\
&= 24.49 + j422.81 \\
C_{00} &= C_0 = 10^{-5}(-0.457 + j101.57) \\
D_{00} &= A_0 = 0.8873 + j0.0158.
\end{aligned}
$$

For line and transformers alone, the values l_0', m_0' and n_0' (for $E_s \div E_r = 1$) are

$$l_0' = \frac{1{,}000(0.9163 \times 28.23 + 0.0158 \times 206.50)}{\overline{28.23}^2 + \overline{206.50}^2} = 0.67$$

$$m_0' = \frac{1{,}000(0.9163 \times 206.50 - 0.0158 \times 28.23)}{\overline{28.23}^2 + \overline{206.50}^2} = 4.35$$

$$n_0' = \frac{1{,}000}{(\overline{28.23}^2 + \overline{206.50}^2)^{\frac{1}{2}}} = 4.80.$$

Similarly, for line, transformers and generators, the equivalent power circle constants are

$$l'_{00} = 0.12$$
$$m'_{00} = 1.51$$
$$n'_{00} = 2.36, \text{ (for } E_1 \div E_r = 1)$$

where E_1 is the excitation voltage of the generator.

In this case the line is designed for a constant supply voltage of 169 kv. and a constant receiver voltage of 150 kv. Hence $E_s \div E_r = 1.127$ and, for operating conditions,

$$n'_0 = 1.127 \times 4.80 = 5.41.$$

The line of Chap. XVI was designed for a full-load of 80,000 kw. approximately. Let the load be assumed to consist of the following:

1. P_m, a synchronous motor load of 25,000 kva. The motors are 80 per cent loaded, have 100 per cent synchronous impedance, and operate at unity power factor.
2. P_I induction motor load of 30,000 kw., operating at 80 per cent power factor under normal voltage.
3. P_R resistance load of 30,000 kw.
4. A synchronous condenser of 45,000 kva. capacity is connected to the line at the receiver end (Chap. XVI).

For normal operating conditions of the line, $E_r = 150$ kv. and $E_s = 169$ kv. It is required to investigate the line for steady-state stability under the given load.

Using the constants $l'_0 = 0.67$, $m'_0 = 4.35$, and $n'_0 = 5.41$, the circle ① of Fig. 99 is drawn. This is the power circle for line and transformers, and represents the steady-state operating conditions, assuming fixed voltages of 150 and 169 kv. at the two ends of the line. Using the constant $l'_{00} = 0.12$, $m'_{00} = 1.51$ and $n'_{00} = 2.36$, the power circle is drawn for line transformers and generator numbered 1.0 of family ②. The other circles of the family may be drawn by using the same center, but various radii bearing the ratios 1.1, 1.2, 1.3, etc., to the radius $n'_{00} = 2.36$. These are the power circles of line, transformers and generators. They represent the relation between active and reactive powers for the ratios given, the ratio being the quotient of generator excitation voltage and receiver voltage, referred to the high side. Each circle represents a particular excitation voltage.

Given the power circles of Fig. 99, the conditions of receiver loading and the characteristic curves of the various kinds of apparatus comprising the receiver equipment, the circuit may be tested for steady-state stability. As already pointed out, steady-state stability assumes additions of load to be made in increments so small that the transients associated with the change are negligible.

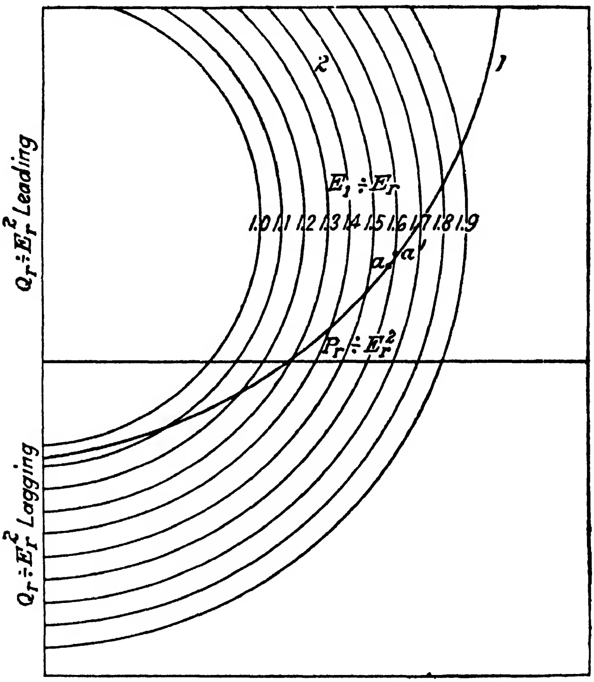


FIG. 99.—Power circles of line for, (1) line and transformers; (2) line, transformers and generators.

To test for stability at a given load the procedure is as follows: Calculate $P_r \div E_r^2$ for the assumed load and the normal voltage, and locate on curve 1 the point corresponding thereto. The particular curve of the family 2 that passes through this point determines the ratio of $E_1 \div E_r$ which is required, and the corresponding value of E_1 may be computed. Next assume that the receiver voltage has dropped slightly, say to 98 per cent of its original value, but that all excitations have remained unchanged. Calculate the active and reactive powers for this new voltage,

and, likewise, the values of $P_r \div E_r^2$ and $Q_r \div E_r^2$. Locate on the circle diagram the points corresponding to the new values of $P_r \div E_r^2$ and $Q_r \div E_r^2$, and note the value of $E_1 \div E_r$ corresponding to it. Compute the new value of E_1 from this ratio. If it is less than the original excitation voltage calculated for normal receiver voltage, the system is stable; if greater, it is unstable. The limit of stability can only be determined by the method of trial and error, that is, by assuming various loads and testing for each. When that load is found for which the excitation voltages at the reduced voltage and at normal voltage are just equal, the limit of stable operation has been found.

At normal voltage,

$$E_r = 150 \text{ kv.} = 100 \text{ per cent receiver voltage.}$$
$$P_r = 80{,}000 \text{ kw. total load over line.}$$
$$\frac{P_r}{E_r^2} = 80{,}000 \div 22{,}5000$$
$$= 3.56 \text{ (locates point } a \text{ on circle 1).}$$
$$\frac{Q_r}{E_r^2} = 0.99 \text{ corresponding to point } a \text{ on diagram.}$$
$$+jQ_r = j0.99 \times 22{,}500$$
$$= +j22{,}300 \text{ reactive kva. over line.}$$
$$\frac{E_1}{E_r} = 1.57 \text{ from circles (2) at point } a.$$
$$E_1 = 1.57 \times 150$$
$$= 235.5 \text{ kv. excitation voltage at generator end.}$$
$$P_m + jQ_m = 200{,}000 + j0 = \text{synchronous motor load.}$$
$$P_I - jQ_I = 30{,}000 - j22{,}500 = \text{induction motor load.}$$
$$P_R + jQ_R = 30{,}000 + j0 = \text{resistance load.}$$

Adding,

$$80{,}000 - j22{,}500 = \text{total motor and resistance load.}$$
$$P_r + jQ_r = 80{,}000 + j22{,}300 = \text{total receiver load over line.}$$

$Q_c = +j22{,}300 - (-j22{,}500) = +j44{,}800$ kva. (synchronous condenser).

X_m = synchronous motor impedance in per cent on 100,000 kva. base,

$$= \frac{100{,}000 \times 100}{25{,}000}$$
$$= 400 \text{ per cent.}$$
$$\frac{X_m(P_m + jQ_m)}{E_r^2} = \frac{400(20 + j0)}{100^2}$$
$$= 0.8 + j0.0.$$

Locating these values of Q_m and P_m on the circle diagrams for the synchronous motor (Fig. 96), $E_m \div E_r$ is found to be

$$\frac{E_m}{E_r} = 1.285$$

$$E_m = 1.285 \times 150$$
$$= 193 \text{ kv.} = \text{motor excitation voltage.}$$

When the receiver voltage drops to 98 per cent of its normal value,

$$E_r = 0.98 \times 150$$
$$= 147 \text{ kv.}$$

P_m = 28,800 kv. (synchronous motor load remains constant).

P_I = 30,000 kw. (induction motor load remains approximately constant).

P_R = 28,800 kw. (resistance load varies as E_r^2).

Adding,

P_r = 78,800 kw. total receiver load.

$$\frac{P_r}{E^2} = \frac{78{,}800}{147^2}$$
$$= 3.64.$$

E_m = 193 as before, since the excitation is assumed constant.

$$\frac{E_m}{E_r} = \frac{193}{147}$$
$$= 1.31$$

$$\frac{X_m P_m}{E_r^2} = \frac{400 \times 20}{98^2}$$
$$= 0.834$$

Using the bracketed values in connection with the circle diagram for the synchronous motor (Fig. 96), we find

$$\frac{X_m Q_m}{E_r^2} = 0.01$$

and from the characteristic curves of the synchronous condensers (Fig. 97), it is seen that the leading reactive kilovolt-amperes has increased about 3 per cent, or

$$
\begin{aligned}
Q_c &= +j46{,}400 \\
Q_m &= +j\frac{E_r^2}{x_m} \times 0.01 \\
&= \frac{(0.96 \times 0.01) \times 10^5}{4.00} \\
&= j240 \text{ kva.}
\end{aligned}
$$

The power factor on the induction motors has increased to about 81 per cent (see Fig. 98 for induction motor characteristics) and hence, under the reduced voltage,

$$Q_I = -j21{,}700.$$

Adding,

$$
\begin{aligned}
Q_r &= +j24{,}940 \text{ kva.} \\
\frac{Q_r}{E_r^2} &= \frac{24{,}940}{147^2} \\
&= 1.15.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{P_r}{E_r^2} + j\frac{Q_r}{E_r^2} &= 3.64 + j1\ 15, \text{ located at } a' \text{ (Fig. 99).} \\
\frac{E_1}{E_r} &= 1.60 \\
E_1 &= 1\ 60 \times 147 \\
&= 235.2 \text{ kv.}
\end{aligned}
$$

Since the required excitation voltage is less than the original 235.5 kv., the system is stable at 80,000-kw. load. A similar test, made with a load of 85,000 kw., shows the line to be unstable at that load. Hence the limit of steady-state ability for the line under assumed conditions of load lies between 80,000 and 85,000 kw.

Transient Stability.—Thus far the discussion has dealt almost entirely with the power limits of lines under steady-state operation. As a matter of fact, the limits of output are more likely to be reached on account of transient conditions. Especially, when a line is already operating under heavy load, a sudden application of additional load, due to switching, short circuit or other condition of operation, may set up transient conditions which will

cause the synchronous load to fall out of step with the generators, even though, if gradually applied, the added load could have been carried without reaching the load limit of the line.

When a load is suddenly applied the impedance of the receiver circuit is suddenly reduced and a sudden increase of generator current results. Due initially to the leakage reactance of the generator and later to its synchronous reactance, the increased current causes a drop in terminal voltage, which, in turn, permits the regulators to function, tending to bring the voltage back to normal. While the above series of events take place, the generator and motor armatures are swinging apart in phase; that is, the angle between their excitation voltages is increasing in order to permit of the increased interchange of power demanded by the new situation. This demands that the rotating masses momentarily slow down and later again accelerate when the correct new phase position has been reached. During the slowing down interval, some of the increased energy demanded is supplied from the energy of rotation of the generator rotor. Another element which assists in supplying the increased demand is the fact that the suddenly increasing armature current induces additional current in the field circuit, thereby momentarily increasing the excitation voltage. The readjustment of the rotor positions and the dropping of the terminal voltage at the generators go on simultaneously. The farther the voltage drops, the greater is the required phase displacement between the motor and generator excitation voltages. It is thus a contest between the exciters and regulators on the one hand, and the decelerating rotors on the other. If, during the initial swing, the angle between the motor and generator excitation voltages reaches the critical value before the regulators and exciters can operate to raise the voltage again, and thus limit the angle to a lesser value, the system will fall out of step. The inertia of the negatively accelerating rotors tend to cause an over swing, thus further endangering the stability.

Methods of Increasing Output Limits of Lines.—The steady-state power limits may obviously be increased by any measures which will reduce the impedance of the circuit involved. These include the use of the so-called "split conductor" suggested by Percy Thomas, as a means of reducing line reactances; two or more parallel transmission lines instead of a single line; transformers having lower reactances; generators designed for lower reactances; lower frequencies; shunt reactors at the generators,

thus requiring higher excitations; intermediate synchronous condenser stations in long lines.

Transient stability conditions are improved by using exciters whose voltages respond rapidly to changes in field resistance, and at the same time using quick acting voltage regulators to control the exciters.

Bibliography

Trans., A. I. E. E., pp. 1 to 104, 1924. A group of articles and discussions by Messrs. Thomas, Evans and Sels; Evans and Bergval; Fortescue and Wagner; E. B. Shand; together with important discussions by the above and Edward L. Moreland, R. D. Booth, V. Bush, R. E. Doherty, C. A. Nickle and others.

PHILLIP, R. A., "Economic Limitations," *Trans.*, A. I. E. E., 1911.

DOHERTY, R. E. and H. H. DEWEY, "Fundamental Considerations of Power Limits of Transmission Systems," *Trans.*, A. I. E. E., pp. 972 to 983, 1925.

FORTESCUE, C. L., "Transmission Stability," *Trans.*, A. I. E. E., pp. 984 to 994, 1925.

Discussions of Doherty, Dewey and Fortescue papers, *Trans.*, A. I. E. E., pp. 994 to 1001, 1925.

CLARKE, EDITH, "Steady State Stability in Transmission Systems," *Jour.*, A. I. E. E., April, 1926.

EVANS and WAGNER, "Studies in Transmission Stability," *Jour.*, A. I. E. E., April, 1926.

CHAPTER XVI

EXAMPLE OF LINE DESIGN AND PERFORMANCE CALCULATIONS

In the preceding chapters the theory underlying transmission-line design, as well as line performance, has been considered. This, the last chapter, will be devoted to the application of this theory to the design of a hypothetical problem, by way of illustration. In this problem, as with any problem, certain assumptions must be made as to the conditions prevailing. These assumptions will largely determine the design which follows. For example, an examination of Eq. (635), and a reading of the discussion following, shows that the conductor diameter and the voltage of a proposed line are chiefly dependent upon the r.m.s. load to be transmitted. Accordingly, assumptions affecting the amount of installed generating equipment, available power and load factor greatly influence the subsequent design. The proper selection of these factors often depends upon circumstances which can scarcely be discussed to advantage here. In choosing an illustrative example, therefore, a simple case is chosen in which the load transmitted is assumed to be determined entirely by the load curve of the distribution center.

TABLE 26.—ASSUMED DATA

Location of project	Eastern Washington
Number of lines	One
Length of line	200 miles
Continuously available power at generator terminals (average demand)	50,000 kw.
Power factor at peak load	90 per cent
Average annual load factor	55 per cent
Frequency	60 cycles per second
Maximum clearance of conductor to ground	30 ft.
Maximum elevation of line above sea level	2,000 ft.
Conductor materials considered	Copper and aluminum

The transmission voltage chosen for the problem is the one which fits the conductor diameter demanded by Eq. (635). It is realized, of course, that the final choice of voltage may be

affected by pre-existing standards, by the necessity for linking the new line with other systems, and other considerations. These will not be considered here.

Root-mean-square Kilowatts per Line.—The curve of Fig. 100, plotted to any convenient scale, is assumed to represent the average-day load curve of the community served. The curve is obtained by averaging a sufficient number of day load curves at regular intervals throughout the year, to secure a fair average for the community. This curve was obtained by averaging the load curves for every Tuesday of the year.

The mean load of the average day is readily found by integrating the area under the average-day load curve in terms of the

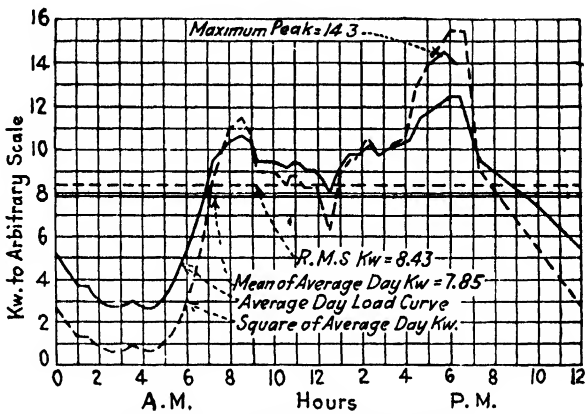


FIG. 100—Average-day load curve

arbitrary unit chosen and dividing this area by 24, or by scaling the area under the curve with the planimeter set to read the square inches, and dividing by the length of the base in inches. The result must again be translated into the units of the arbitrary scale. The ordinates of the horizontal full lines in the figure indicate the mean of the average-day load. As a check, the area under the horizontal full lines should equal the area under the average-day load curve.

The dotted curves are obtained by squaring the ordinates of the average-day load curve. The mean ordinate of the dotted curve is also found by integrating the area under this curve and dividing the result by the length of the base. The r.m.s. value is the square root of the mean square and is shown by the horizontal broken line.

The half-hour average annual load factor is defined as the ratio of the average to the half-hour peak demand of the year. This load factor may vary from 40 to 65 per cent or more. Here it is assumed to be 55 per cent. The storage provided by the basin behind the dam will be at least sufficient to take care of the variation in demand from the average, during any time of day or year. Thus, with 50,000 kw. of power continuously available at the generator terminals, if the power factor at peak load is 90 per cent, the installed machine capacity required is

$$\text{Installed capacity} = \frac{50{,}000}{0.55 \times 0.90}$$
$$= 101{,}000 \text{ kva., say } 100{,}000 \text{ kva.}$$

To the scale of the curve the average kilowatt is 7.85, while the r.m.s. value is 8.43. Therefore, the r.m.s. load to be supplied to the line is

$$\text{R.m.s. kilowatts supplied} = \frac{50{,}000 \times 8.43}{7.85}$$
$$= 53{,}700 \text{ kw.}$$

Factor J. Value of Wasted Energy.—From Eq. (635) the factor pertaining to the value of the energy wasted as heat in the line conductors is

$$J = \frac{U^2 \cos^2 \theta}{\rho A}.$$

Since only one line is contemplated, a single circuit tower of type A is the only type considered. The maximum altitude of any point on the line is assumed to be 2,000 ft. For this elevation the altitude factor is $\delta = 0.92$, from Table 24. The roughness factor for stranded cable may be assumed as $m_0 = 0.82$. Then, if we design the line so that, at the point of maximum elevation, the actual line voltage will be about 10 per cent under the critical disruptive value, the constant γ of Eq. (587) is 0.90. Thus,

$$m_0\delta\gamma = 0.82 \times 0.92 \times 0.90$$
$$= 0.678$$

and, by interpolation from the values of Table 23,

$$U_A = 116{,}900.$$

The resistance, per mil-foot of conductor and for copper and aluminum, may be estimated for any assumed average operating temperature from the equations

$$\rho_{Al} = 17.01[1 + 0.0039(t - 20)] \text{ for aluminum}$$
$$\rho_{Cu} = 10.37[1 + 0.00382(t - 20)] \text{ for copper}$$

where

t = assumed average operating temperature of line conductors in degrees centigrade.

The mean temperature for the year and locality may be obtained from the records of the U. S. Weather Bureau. For the project here considered, it is 48° F. The line conductors, in order to radiate the heat developed in them, must operate at a temperature somewhat higher than that of the surrounding air. If this difference is taken as an average of 15° F., the average temperature of the line conductors is 63° F., or 17.2° C., and the resistivities of aluminum and copper under average operating conditions are

$$\rho_{Al} = 17.01[1 + 0.0039(17.2\text{–}20)] = 16.8 \text{ ohms}$$
$$\rho_{Cu} = 10.37[1 + 0.00382(17.2\text{–}20)] = 10.25 \text{ ohms.}$$

The weighted average selling value of a kilowatt-hour of energy, represented by the constant A, is difficult to estimate. The value of a kilowatt-hour will depend upon the locality of the plant, market conditions, the distance of transmission, the time of day and year (if generated from water power with incomplete storage), whether or not steam generation is required at peak loads and other factors. A study should be made of the revenues derived from several years of past operation together with a forecast of future revenues with the new project added to the existing system. The value of A may vary from as low as 0.002 for some western hydroelectric plant which was constructed at a very low unit cost, where the transmission line is short, and where an ample supply of power exists, to perhaps 0.007 where conditions of generation and transmission are less favorable. For the purpose of the problem to be considered here, 0.006 will be assumed. The power factor at maximum load is assumed to be 0.90.

The factor J and the constants used in deriving it are summarized in Table 27.

TABLE 27

Item	Symbol	Value for aluminum	Value for copper
Constant U of equation......	U	116,900	116,900
Constant U^2................	U^2	136.7×10^8	136.7×10^8
Resistivity at 20° C..........	ρ_{20}	17.01	10.37
Selling price per kilowatt-hour (dollars)..................	A	0.006	0.006
Power factor of r.m.s. load (assumed)................	$\cos\theta$	0.90	0.90
$J = U^2 \cos^2\theta \div \rho A$ (Eq. (64))	J	10.8×10^{10}	17.8×10^{10}

Factor F. Annual Fixed Charges Due to Line Conductors.—By Eq. (634) the annual fixed charges due to the line conductors and ground cables of one line are

$$31.05(p_1BW + gp_{Fe}B_{Fe}W_{Fe}).$$

The several factors involved in the constant F are listed in Table 27 for the two kinds of conductor materials.

The values of p_1 and p_{Fe}, representing interest and depreciation on the conductors and ground cables, will vary somewhat for different projects, depending upon the condition of the money market and the credit of the company undertaking the construction. It is a matter of judgment how much should be allowed for the depreciation of line conductors. The depreciation of copper conductors is relatively little. The values allowed in this problem are illustrative rather than exact.

TABLE 28

Item	Symbol	Aluminum	Copper
Per cent interest and depreciation on line conductor...	p_1	10	9
Cost of conductor in dollars per pound	B	0.38	0.19
Weight per cubic foot of conductor material........	W	168.5	555
Guard cable constant (Eq. 625).......	g	0.167	0.333
Per cent interest and depreciation on guard cables......................	p_{Fe}	12	12
Cost of guard cables in dollars per pound	B	0.12	0.12
Weight per cubic foot of guard cable material..........................	W_{Fe}	490	490
Constant F of Eq. (64).........	F	23,540	36,760

The costs, per pound of stranded conductor and ground wires in the sizes and amounts required for the project, are represented by B and B_{Fe}. The values given in the table are approximate prices for these materials at the time and place of writing.

The values of W and W_{Fe} are the weights per cubic foot of the materials composing the line conductors and ground cables, respectively, corresponding to the densities of the materials, as given in Chap. II. The constants F, together with the quantities used in their calculation, are shown in Table 28.

Factor *G*, Tower Cost.—By Eq. (635) that part of the cost of transmission line towers, which is a function of conductor diameter, is the constant

$$G = 2{,}525p_2M.$$

The value of the factor M is to be estimated from data furnished by the manufacturers of towers as outlined in detail on pages 340 to 341. As there stated, the manufacturer is requested to furnish bids on two series of anchor and suspension towers of the type or types considered. (Only type A tower is being considered in this problem.) In the first series on which bids are requested, the towers are all of a given height (45 ft. in this instance) and all are designed for the same tension (6,000 lb.), but are to be built for three different voltages, namely, 100, 150 and 200 kv., the range in voltages being so chosen as to embrace the voltage of the contemplated line. The towers of the second series are all to be built for the same height as the first (45 ft.) and for one of the voltages specified in the first series (150 kv., say), but for three different tensions, one of which is also the tension specified in the first series (3,000, 6,000 and 9,000 lb). It is assumed that the suspension towers will be designed for one-half the conductor tension specified for the anchor towers, and that the total ground wire tension per tower for all towers is equal to the conductor tension specified for the anchor tower.

Table 29, together with a sketch of the type of tower as in Fig. 75 and a set of specifications, will furnish the manufacturer with the information required to make up cost estimates.

Specifications for Towers.—In order that the manufacturer of towers may know precisely for what loads, conductor clearances, tensions, etc., the structures are to be designed, it is necessary to define these quantities through the medium of written specifications. Such specifications will vary with the

locality in which the towers are to be used, the type of construction to be adopted, the judgment of the engineer responsible for the design and other factors. The items enumerated below will generally be covered in more or less detail in such specifications.

1. Conductor horizontal and vertical spacings. These will usually be determined upon in some such manner as indicated in the design of the towers illustrated in Fig. 75. The spacings given by Eq. (580) and as shown in Fig. 76 are representative of good practice.
2. Clearances of conductors to nearest tower members.
3. Location of guard cables and clearances to line conductors.
4. Materials to be used.
5. Galvanizing.
6. Foundations.
7. Types of towers, anchor and suspension towers.
8. Loads assumed:
   a. Wind on towers and cables
   b. Maximum tension in cable
   c. Weight of towers, insulators, ice and cables.
9. Unit stresses to be used in calculations:
   a. Tension
   b. Compression
   c. Shear on bolts and rivets
   d. Bearing on bolts and rivets.
10. Total number of anchor towers required.
11. Total number of suspension towers required.

The voltages, tensions, conductor clearances and tower heights, pertaining to the present problem and in accordance with which the manufacturer is requested to furnish bids on both anchor and suspension towers, are given in Table 29. These data together with the detailed specifications covering such items as those already enumerated under specifications, will furnish all the information the manufacturer may need to make up his cost estimates. It is assumed that all such data have been supplied.

The number of towers of each kind required for the present project is estimated by assuming the length of the average span to be 700 ft., and that there is one anchor tower for each seven suspension towers.

Then,

$$T_s + T_a = 7.$$

But,

$$T_s + T_a = 8T_a = \text{total number of towers}$$

or

$$8T_a = \frac{5{,}280 + 200}{700}$$
$$= 1{,}509$$

whence

$$T_a = 189$$

and

$$T_s = 1{,}320.$$

After prices have been secured from the manufacturer for the ten different towers of Table 29, these prices are itemized as in Table 30. In this table items 1, 2 and part of 4 are those which the tower manufacturer furnishes. To the part of item 4, the cost of foundation as furnished, must be added the estimated cost of excavation and back filling to secure the final item in the table. Item 3 is estimated from the total cost of the right-of-way divided by the number of tower locations. Item 6 is estimated from prices of insulator discs and hardware, as furnished by manufacturers' representative or jobbers. By allowing about 17,000 volts per unit of a string, the number of insulators per string (and the total number required) for any assumed line

TABLE 29

Index	Line voltage, assumed, (kilovolts)	Conductor tension, assumed, pounds	Ground cable tension, assumed, pounds	Spacing D (Fig. 76)	Clearance $a = D \div 3.31$ (Eq. 577)	Length of insulator string $= 2a \div \sqrt{3}$	Tower height, assumed	Class of tower
1	100	6,000	3,000	10′4″	3.13	3′8″	45′0″	Anchor
2	150	6,000	3,000	15′6″	4.68	5′4″	45′0″	Anchor
3	200	6,000	3,000	20′8″	6.25	7′2″	45′0″	Anchor
4	100	3,000	3,000	10′4″	3.13	3′8″	48′8″	Suspension
5	150	3,000	3,000	15′6″	4.68	5′4″	50′4″	Suspension
6	200	3,000	3,000	20′8″	6.25	7′2″	52′2″	Suspension
7	150	3,000	1,500	15′6″	4.68	5′4″	45′0″	Anchor
8	150	9,000	4,500	15′6″	4.68	5′4″	45′0″	Anchor
9	150	1,500	1,500	15′6″	4.68	5′4″	50′4″	Suspension
10	150	4,500	4,500	15′6″	4.68	5′4″	50′4″	Suspension

TABLE 30.—ITEMS OF TOWER COST

Index	Line voltage, kilovolts	Conductor tension, pounds	Item 1 tower at place of erection	Item 2 erection of tower	Item 3 tower site	Item 4 foundation	Item 5 location and inspection	Item 6 insulators at tower sites	Item 7 placing insulators and cable	Class of tower
1	100	6,000	$379	$104	$30	$205	$40	$110.00	$33.00	Anchor
2	150	6,000	386	106	30	205	40	168.00	33.00	
3	200	6,000	408	113	30	205	40	231.00	33.00	
4	100	6,000	251	68	30	116	40	52.50	33.00	Suspension
5	150	6,000	256	70	30	116	40	79.50	33.00	
6	200	6,000	269	75	30	116	40	109.50	33.00	
7	150	3,000	288	79	30	152	40	168.00	33.00	Anchor
8	150	9,000	495	136	30	250	40	168.00	33.00	
9	150	3,000	203	56	30	96	40	79.50	33.00	Suspension
10	150	9,000	308	85	30	132	40	79.50	33.00	

voltage, may readily be estimated. It is here assumed that the unchained strength of the suspension insulators, the strength of a single string, has been increased by the use of high-strength steel so that it will not be necessary to parallel insulator strings in order to sustain the loads due to the use of large cable diameters. A link is used to connect the top insulator of a string to the crossarm. Its length is such that the overall length of the insulator string is $2a \div \sqrt{3}$, (Fig. 75). The cost of this link is estimated as $0.50 per string.

In Table 31, the estimated prices of item 6 are shown in tabular form.

TABLE 31

	Anchor tower			Suspension tower		
Line voltage	100,000	150,000	200,000	100,000	150,000	200,000
Number of disks required	6	9	12	6	9	12
Insulators per string	$ 15.00	$ 22.50	$ 30.00	$ 15.00	$ 22.50	$ 30.00
Conductor clamps per string	2 30	3.50	5.00	1.50	2.00	3 00
Arc rings per string	0.50	1.50	3.00	0.50	1.50	3.00
Link per string	0.50	0.50	0.50	0.50	0.50	0.50
Total per string	$ 18.30	$ 28.00	$ 38.50	$ 17.50	$ 26.50	$ 36.50
Total per type *A* tower	110.00	168.00	231.00	52.70	79.50	109.50

Items 5 and 7, like item 3, are relatively unimportant, and have practically no influence on the choice of the most economical voltage. They are estimated and entered in Table 30 for the sake of completeness.

By Eq. (602) a given cost item for the *average* line support is related to the corresponding values of the item as applied to anchor and suspension towers as follows:

Cost item for average support =

$$\frac{\text{cost item for anchor tower} + \frac{T_s}{T_a}\text{ (cost item for suspension towers)}}{1 + \frac{T_s}{T_a}}$$

By the use of this equation the items of cost 1, 2, 3, etc., for the average line support, are calculated, as shown in Table 32.

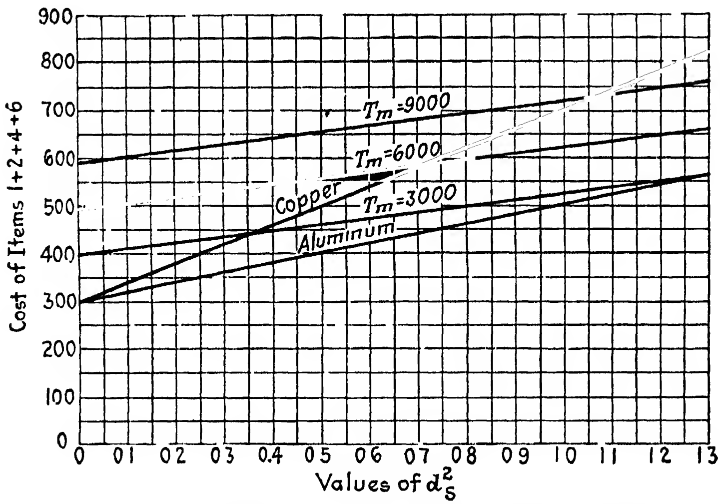


FIG. 101 — Cost items 1 + 2 + 4 + 6 for average line support

Finding the Values k_v, k_T, m and n.—After the cost items 1, 2, 3, etc., for the average support have been calculated and listed as in Table 32 [items (*a*)], the voltage and tension constants k_v and k_t and the constants m and n, of the same table are computed.

In Fig. 101 are plotted the cost items 1 + 2 + 4 + 6 of the average line support for columns I, II and III [item (*d*)], against the corresponding values of d_s^2 [item (*c*)]. This gives the middle

TABLE 32.—COST OF AVERAGE LINE SUPPORT
Computations for Constants k_v, k_T, m and n

Item	Index	I	II	III	IV	V
	Assumed line voltage (kilovolts)...	100	150	200	150	150
	Assumed conductor tension (pounds)..	6,000	6,000	6,000	3,000	9,000
	From Table 29 (Eq. (602)) — Average line support: Item 1...	267.0	272.20	286.40	213.60	331.40
	Combine indices: Item 2........	72.50	74.50	79.80	58.90	91.40
	1 and 4 for I — Item 3........	30.00	30.00	30.00	30.00	30.00
(a)	2 and 5 for II — Item 4........	127.10	127.10	127.10	103.00	146.80
	3 and 6 for III — Item 5........	40.00	40.00	40.00	40.00	40.00
	7 and 9 for IV — Item 6........	59.70	90.60	124.70	90.60	90.60
	8 and 10 for V — Item 7........	33.00	33.00	33.00	33.00	33.00
(b)	Equivalent conductor diameter, $d_s = (E \div \sqrt{3}U)$; $U = 116{,}900$	0.494 in.	0.741 in.	0.988 in.	0.741 in.	0.741 in.
(c)	Equivalent $d_s^2 = E^2 \div 3U^2$........	0.244	0.549	0.976	0.549	0.549
(d)	Average line support items 1 + 2 + 4 + 6........	526.30	564.40	618.00	466.10	660.20
(e)	From Fig. 101 and by Eq. (603) .... Cost items $(1 + 2 + 4 + 6) = k_v + md_s^2$........	←—— $k_v = 496$,		$m = 125$ ——→	$k_v = 397$ $m = 125$	$k_v = 590$ $m = 125$
(f)	d^2 corresponding to T_m (Eq. (605)).. $d_s^2 = 16.1 \times 10^{-5}$	....	0.966	......	0.483	1.449
	$d_s^2 = 8.04 \times 10^{-5}$	......	0.482	......	0.241	0.724
(g)	From Fig. 101 k_T for both aluminum and copper........	......	300			
	n for aluminum........	......	200			
	n for copper........	......	403			
(h)	$(m + n)$ for aluminum $= 125 + 200$........	......	325			
	$(m + n)$ for copper $= 125 + 403$........	......	528			
(i)	Cost items $(1 + 2 + 4 + 6) = k_T + (m + n)d_s^2$ for aluminum =........	......	$300 + 325d_s^2$			
	for copper =........	......	$300 + 528d_s^2$			

one of the three parallel straight lines of Fig. 101, marked $T_1 = 6{,}000$ whose equation is item (*e*), columns I, II, III. The corresponding cost curves for the tensions of 3,000 and 9,000 lb. are parallel to the curve already found and pass through the points located by items (*c*) and (*d*) of columns IV and V. The equations of the curve thus located are items (*e*) columns IV and V. Item (*f*), derived from Eq. (605), shows the values of d_s^2 corresponding to the three different tensions and the two kinds of conductor materials. By plotting the three values of k_v (item (*e*)), against the corresponding values of d_s^2 from item (*f*), corresponding to the aluminum and copper conductors, the two curves of Fig. 101, which intersect at cost = 300, $d_s^2 = 0$, are obtained. From these curves it is apparent that $k_{T'} = 300$ for both curves, and that $n = 200$ for aluminum and 403 for the copper conductors (item (*g*)). The methods by which items (*h*) and (*i*) are derived are apparent from the table.

The assumed height of the point of suspension of the conductor above ground, in all towers of Table 29, and upon which bids have been obtained, is 45 ft.

By definition (Eq. (557), Chap. XII),

$$k_3 = \text{items } (4 + 5 + 6 + 7)$$

and

$$k_2 = \text{items } (1 + 2 + 3)$$

hence

$$\frac{k_3}{k_2} = \frac{h^2 \text{ (items } 4 + 5 + 6 + 7)}{\text{(items } 1 + 2 + 3)}.$$

Thus, if we choose the tower of column II as representing approximately the cost of tower for the present project, its most economical height may be found by the method described in Chap. XII, as follows:

$$\frac{k_3}{k_2} = \frac{(45)^2(127.10 + 40.00 + 90.60 + 33.00)}{272.20 \div 74.50 + 30}$$

$$= 1{,}563.$$

By original assumption, the minimum permissible clearance to ground is 30 ft., or

$$k_1 = 30$$

From Plate I, for the above values of $k_3 \div k_2$ and k_1, the conductor sag for the most economical tower spacings or span is found

to be 20.3 ft. Accordingly, the most economical tower height is

$$h_e = 30 + 20.3 = 50.3 \text{ ft.}$$

Before the total cost of the towers of height h_e can be found, three items 1, 2, and 3, whose costs vary with the tower height, must be corrected in accordance with Eq. (612). Thus, the correction factor is

$$\begin{aligned} k_4 &= (\text{items } 1 + 2 + 3)\left(\frac{50.3}{45} - 1\right) \\ &= (272.20 + 74.50 + 30)(0.25) \\ &= 94.0. \end{aligned}$$

By Eq. (613):

$$\begin{aligned} k_5 &= \text{items } 5 + 7 \\ &= 73.0. \end{aligned}$$

From Eq. (615):

$$\begin{aligned} k &= k_T + k_4 + k_5 \\ &= 300 + 94 + 73 \\ &= 467. \end{aligned}$$

The total cost of line supports may now be expressed by Eq. (617) as soon as the constants k_6, k_7 and k_8 are found. These are obtained from Eq. (567) and from the charts of Plates 2 to 5, inclusive, by entering the charts with the values $k_3 \div k_2 = 1{,}563$ and $k_1 = 30$. The constants as found from the charts are given below:

$$\left.\begin{aligned} k_6 &= +0.20 \\ k_7 &= 978 \\ k_8 &= 428 \end{aligned}\right\} \text{aluminum} \qquad \left.\begin{aligned} k_6 &= -0.15 \\ k_7 &= 840 \\ k_8 &= 114 \end{aligned}\right\} \text{copper.}$$

By substituting in Eq. (619) the values of the above constants, the cost of towers, per foot of line, is expressed in terms of conductor diameter, as shown below.

$$\frac{[(m + n)d_s^2 + k](d_s + k_6)}{k_7(d_s + k_6) - k_8} = \frac{[325d_s^2 + 467](d_s + 0.20)}{978(d_s + 0.20) - 428}, \quad \text{aluminum}$$

or

$$= \frac{[528d_s^2 + 467](d_s - 0.15)}{840(d_s - 0.15) - 114}, \quad \text{copper.}$$

The cost of towers, per foot of line, is found by evaluating the above equations for various assumed values of d_s and plotting the resultant costs as a function of d_s^2. A series of such computations has been made, the results of which are listed in Table 33.

TABLE 33

d_s	k_4d^2 Aluminum	k_4d^2 Copper	$k_4d_s^2 + k$ Aluminum	$k_4d_s^2 + k$ Copper	$d_s + k_6$ Aluminum	$d_s + k_6$ Copper	$k_7(d_s + k_6)$ Aluminum	$k_7(d_s + k_6)$ Copper
0.5	81.3	132.0	548.3	599.0	0.7	0.35	684.6	294.0
0.6	117.0	190.1	584.0	657.1	0.8	0.45	782.4	378.0
0.7	159.3	258.7	626.3	725.7	0.9	0.55	880.2	462.0
0.8	208.0	337.9	675.0	804.9	1.0	0.65	978.0	546.0
0.9	263.3	427.7	730.3	894.7	1.1	0.75	1,075.8	630.0
1.0	325.0	528.0	792.0	995.0	1.2	0.85	1,173.6	714.0
1.1	393.3	638.9	860.3	1,105.9	1.3	0.95	1,271.4	798.0
1.2	468.0	760.3	935.0	1,227.3	1.4	1.05	1,369.2	882.0
1.3	549.3	892.3	1,016.3	1,359.3	1.5	1.15	1,467.0	966.0
1.4	637.0	1,034 0	1,104.0	1,501.9	1.6	1.25	1,564.8	1,050.0
1.5	731.3	1,188.0	1,198.3	1,655.0	1.7	1.35	1,662.6	1,134.0

TABLE 33 (*Continued*)

d_s	$[k_4d_s^2 + k](d_s + k_6)$ Aluminum	$[k_4d_s^2 + k](d_s + k_6)$ Copper	$k_7(d_s + k_6) - k_8$ Aluminum	$k_7(d_s + k_6) - k_8$ Copper	Cost per foot of line Aluminum	Cost per foot of line Copper	d_s^2
0.5	383.8	209.7	256.6	180.0	$1.50	$1.16	0.25
0.6	467.2	295.7	354.4	264.0	1.32	1.12	0.36
0.7	563.7	399.1	452.2	348.0	1.25	1.15	0.49
0.8	675.0	523.2	550.0	432.0	1.23	1.21	0.64
0.9	803.3	671.0	647.8	516.0	1.24	1.30	0.81
1.0	950.4	845.7	745.6	600.0	1.27	1.41	1.00
1.1	1,118.4	1,050.6	843.4	684.0	1.33	1.54	1.21
1.2	1,309.0	1,288.7	941.2	768.0	1 39	1.68	1.44
1.3	1,524.4	1,563.2	1,039.0	852.0	1.47	1.83	1.69
1.4	1,766.4	1,877.4	1,136.8	936.0	1.55	2.01	1.96
1.5	2,037 1	2,234.2	1,234.6	1,020.0	1.65	2.19	2 25

Plotting the costs of tower per foot of line against values of d_s^2, as shown in Table 33, yields the curves of cost (Fig. 102). In this figure, straight lines are now drawn tangent to the two curves. These tangents are practically coincident with the curves themselves over considerable distances, and, within these ranges, they express the cost of tower per foot of line correctly in terms of the parabolic law, whose equation is Eq. (620), namely,

Cost of towers per foot of line = $Md_s^2 + N$

From the curves the values of M and N are found to be

For aluminum

$$M = 0.275$$
$$N = 1.00$$

For copper

$$M = 0.612$$
$$N = 0.80$$

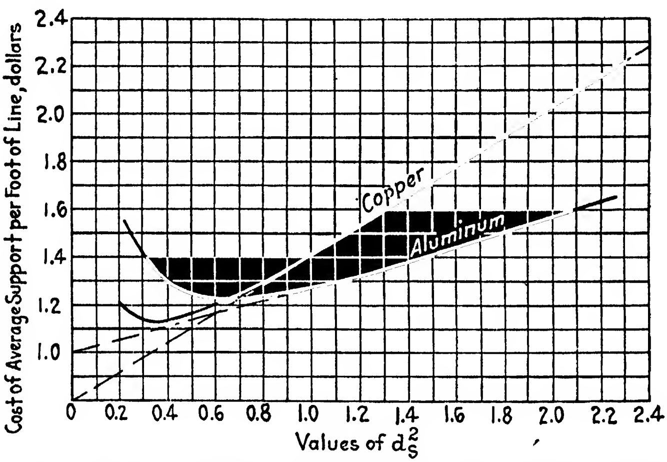


FIG. 102.—Cost of average line support as a function of cable diameter.

Assuming the depreciation and interest on tower investment at 12 per cent, the item G of Eq. (635) becomes

$$G = 2{,}525 p_2 M$$
$$= 2{,}525 \times 12 \times M$$

whence

$$G_{Al} = 8{,}330, \text{ for aluminum}$$

and

$$G_{Cu} = 18{,}540, \text{ for copper.}$$

Evaluation of Constant H**, Eq. (635).**—The example here considered is for a substation of 100,000-kva. capacity. It is assumed that the substation layout is comparable to type (Fig. 82), as regards the degree of complexity. Experience indicates that in all probability the economical transmission voltage for a 200-mile line having 100,000 kva. of connected generating capac-

ity will exceed 132 kv. The constants are therefore chosen from Table 25 from the columns representing a range in voltage from 132 to 220 kv. The constants are

$$k'_{11} = 11.1 \times 10^{-6}$$
$$k_{12} = 220{,}000$$

By Eq. (628),

$$k_{11} = 3k'_{11}\, U^2.$$

Table 27,

$$U^2 = 136.7 \times 10^8.$$

Hence

$$\begin{aligned} k_{11} &= 3 \times 11.1 \times 10^{-6} \times 136.7 \times 10^8 \\ &= 455{,}200. \end{aligned}$$

By Eq. (635), the factor pertaining to terminal equipment is

$$H = \frac{2{,}525 p_3 k_{11}}{L}.$$

The factor p_3, covering the percentage of interest and depreciation on terminal apparatus and housing, is here assumed as 12. The length of line is

$$\begin{aligned} L &= 200 \times 5{,}280 \\ &= 1{,}056 \times 10^3 \text{ ft.} \end{aligned}$$

Solving for the constant H,

$$\begin{aligned} H &= \frac{2.525 \times 12 \times 455{,}200}{1{,}056 \times 10^3} \\ &= 13{,}060 \text{ for both aluminum and copper conductors.} \end{aligned}$$

Solution of Eq. (635). Calculation of E and d_s.—All of the required terms of Eq. (635) have now been evaluated, and the conductor diameter may be found by substitution therein. For convenience, the constants already calculated, together with the solution of Eq. (635) for d_s and the corresponding value of the most economical voltage for the line, are collected below.

The total cost of the line and terminal equipment, for each of the two kinds of conductor materials, may now be estimated, by substituting in the appropriate equations the constants already computed. The results of these calculations and the equations from which the values were obtained are clearly indicated in Table 34. The estimates show that the total costs of the aluminum and copper lines are substantially equal.

TABLE 34

Item	Conductor material	
	Aluminum	Copper
R.m.s. kilowatts per line	53,700	53,700
J, loss factor (Table 27)	12.0×10^{10}	19.8×10^{10}
F, line conductors (Table 28)	23,540	36,760
G, line towers	8,330	18,540
H, terminal appliances and housing	13,060	13,060
$J(F + G + H)$	53.92×10^{14}	135.35×10^{14}
$[J(F + G + H)]^{\frac{1}{4}}$	418.8	488.2
$10 \times$ (r.m.s. kilowatts per line)$^{\frac{1}{2}}$	377.3	377.3
d_s, [Solution of Eq. (635)]	0.901	0.773
$E = \sqrt{3}d_sU$ = line voltage	182,400	156,500

Choice of Type of Conductor.—There are certain objections to the use of an all-aluminum cable, however, and before making a final decision as to the kind of conductor material to use, the comparative cost estimates should be extended to include steel-reinforced aluminum cable and possibly other types such as tubular conductors. From such a complete estimate the engineer may readily select the most suitable type of conductor. If there is no advantage in one kind of conductor over another, the conductor and line voltage which require the minimum total expenditure would naturally be chosen.

In the illustration here used the copper conductor fits the standard voltage, 154 kv., very well. If a steel-reinforced aluminum cable had been included as a part of the estimate, it would probably have been found to call for a voltage close to 220 kv., while the all-aluminum cable falls in between these extremes. Thus the influence of type of conductor upon the most economical voltage of transmission is well illustrated.

Line Specifications and Performance.—For the purpose of illustrating the remaining calculations, the copper line will be chosen. The diameter of the cable will be assumed as computed, namely $d_s = 0.773$ in.

Some of the important required data pertaining to the construction of this line and its performance are listed in Table 35 which follows. The methods used in making the calculations are well illustrated in columns 1 and 2 of the table.

TABLE 35

Item	Equation	Estimated cost	
		Aluminum	Copper
Line conductors [Eq. (625) and Table 27]	100 × Eq. (625) ÷ p	$ 779,000	$ 971,000
Tower cost (Eqs. (621) and (622))	LMd_s^2	236,000	386,000
	LN	1,056,000	845,000
Cost of terminal equipment, housing,[1]	$k_{11}'E^2$	388,000	260,000
Wiring, etc. (Eqs. (626) and (628))	k_{12}	220,000	220,000
Total cost of terminal equipment and line,		2,679,000	2,682,000

[1] It is assumed that the equipment will be purchased for a standard voltage. For the aluminum conductor the voltage, 187 kv., will be assumed as the nearest standard. For copper the nearest standard is 154 kv.

TABLE 36.—LINE SPECIFICATIONS AND PERFORMANCE DATA

Item	Reference, equations, etc	Quantity
Given data		
R m s kilowatts per line	Page 335	53 700
R m s kilowatts per phase		17,900
Generator kilovolt-amperes installed	Page 335	100 000
Assumed generator full-load power factor (approximate value)	Page 336	0 90
Kind of conductor material		Copper
Type of towers used (Fig 75)		*A*
d_s = diameter of stranded cable, in inches	Table 33	0 773 in
E_{av} = approximate average line voltage	Table 33	157 000
$_nE_{av}$ = approximate average e m f to neutral = 156 500 ÷ $\sqrt{3}$		90,360
L = length of line in feet	200 × 5 280	10 56 × 10⁶
Conductors and guard cables		
Circular mils metallic section	$10^6 d_s^2 \div (1\ 151)^2$	450 000
Number of strands (standard stranding)		37
Circular mils per strand	450 000 ÷ 37	12 162
Diameter of strand in mils	$(12\ 162)^{\frac{1}{2}}$	110 3
Maximum allowable tension pounds per square inch = 0 75 × 27,500	0 75 × elastic limit	21 000
Maximum conductor tension = 0 9 × combined tension		6 680
Diameter of strands of galvanized steel ground cable (inches)		•
D = actual conductor spacing (inches)	Eq (580)	194 in
D in feet and inches	195 ÷ 12	16 ft 2 in
D' = equivalent delta spacing in inches	1 26 × 195 in	244 in
Towers and insulators		
k_1 = minimum clearance to ground (feet)	Page 333	30
$k_3 \div k_2$ ratio	Page 344	1 563
Maximum cable deflection in feet	Page 345	20 3
h_e = the most economical tower height, anchor tower	Page 345	50 3
k_6		−0 15
k_7	Page 345	840
k_8		114
S = tower spacing in feet	Eq (566)	660
Approximate number of towers for line	$L \div S$	1 600
Number of anchor towers	1,600 ÷ 8	200
Number of suspension towers	1,600 ÷ 200	1 400
a = clearance conductor to tower = $E_n \div$ 1,538	Fig 75 and Eq (579)	59 in
Length of suspension insulator string	$2a \div \sqrt{3}$	68 in
Number of insulator discs per string	Page 340	9
Number of discs for anchor towers	2 × 3 × 9 × 200	10 800
Number of discs for suspension towers	3 × 9 × 1 400	37 800
Total number of discs		48,600

TABLE 36 (*Continued*)

Item	Reference, equations, etc.	Quantity
Electrical line constants:		
ρ = resistivity of medium hard drawn copper at 68° F. (ohms per mil-foot)	$10.37 \div 0.97$	10.69
r = resistance of line conductor (ohms)	$1.02\rho L \div$ C.M.	25.59
x = reactance of line conductor at 60 cycles (ohms)	....	162.57
b = susceptance one conductor to neutral (mho)	....	10.45×10^{-4}
z = impedance one conductor (ohms)	$(r^2 + x^2)^{\frac{1}{2}}$	164.53
$\sqrt{ZY}$ = complex line angle	$[jb(r + jx)]^{\frac{1}{2}}$	$0.41465/\underline{85° 31.63'}$
$= \alpha + j\beta$	....	$0.41465/\underline{\delta}$
α = attenuation constant	$0.41465 (\cos \delta)$	0.032347
β = phase constant	$0.41465 (\sin \delta)$	0.413384
Z_0 = complex surge impedance (vector ohms)	$\left[\frac{z}{b}/\underline{-90°+81° 3.27'}\right]^{\frac{1}{2}}$	$395.58 - j30.954$
Y_0 = complex surge admittance (vector mho)	$1 \div Z_0$	$(25.125 + j1.9660) \times 10^{-4}$
$A = \cosh \alpha \cos \beta + j \sinh \alpha \sin \beta$	$a_1 + ja_2$	$0.916246 + j0.012998$
$a_1 = \cosh \alpha \cos \beta$	1.00052×0.91577	0.916338
$a_2 = \sinh \alpha \sin \beta$	0.032357×0.40171	0.012998
$B = Z_0(\sinh \alpha \cos \beta + j \cosh \alpha \sin \beta)$	$b_1 + jb_2$	$24.1627 + j158.074$
$b_1 = 395.58(\sinh \alpha \cos \beta) + 30.954 (\cosh \alpha \sin \beta)$	....	24.1627
$b_2 = 395.58(\cosh \alpha \sin \beta) - 30.954 (\sinh \alpha \cos \beta)$	....	158.074
$C = Y_0(\sinh \alpha \cos \beta + j \cosh \alpha \sin \beta)$	$c_1 + jc_2$	$(-0.4569 + j101.565)10^{-5}$
$c_1 = 10^{-4}[25.125(\sinh \alpha \cos \beta) - 1.9660 (\cosh \alpha \sin \beta)]$	....	-0.4569×10^{-5}
$c_2 = 10^{-4}[25.125(\cosh \alpha \sin \beta) + 1.9660 (\sinh \alpha \cos \beta)]$	....	101.565×10^{-5}
A (length of vector)	$(a_1^2 + a_2^2)^{\frac{1}{2}}$	0.916338
B (length of vector)	$(b_1^2 + b_2^2)^{\frac{1}{2}}$	159.910
C (length of vector)	$(c_1^2 + c_2^2)^{\frac{1}{2}}$	101.566×10^{-5}
l' = abscissa of power circle center	$\dfrac{10^3(a_1b_1 + a_2b_2)}{b_1^2 + b_2^2}$	0.94613
m' = ordinate of power circle center	$\dfrac{10^3(a_1b_2 - a_2b_1)}{b_1^2 + b_2^2}$	5.6517
n' = radius of power circle for $E_s \div E_r = 1$	$10^3 \div (b_1^2 + b_2^2)^{\frac{1}{2}}$	6.2535
Preliminary performance analysis:		
Approximate generator full-load kilowatt output	$100{,}000 \times 0.90$	90,000
Approximate generator full-load kilowatts per phase	$90{,}000 \div 3$	30,000
Approximate generator full-load current (amperes)	$\dfrac{100{,}000{,}000}{\sqrt{3} \times 160{,}000}$	360

TABLE 36 (*Continued*)

Item	Reference, equations, etc.	Quantity
Estimated full-load line loss (kilowatts = $3rI^2 + 1{,}000$)	$\frac{3 \times 25.3 \times \overline{360}^2}{1{,}000}$	9,800
Approximate receiver full-load kilowatt	90,000 − 9,800	80,200
Hence, assume full-load receiver kilowatts per phase	....	27,000
Assume receiver power factor at full load	....	0.90
Assume receiver e.m.f. = approximate 0.95 × 157,000	....	150,000
Then E_r, receiver e.m.f. to neutral, is	$150{,}000 \div \sqrt{3}$	86,700
$10^{-3}Q_r$ = receiver reactance kilovolt-amperes required per phase to maintain any assumed ratio of supply to receiver voltages	$10^{-3}Q_r = 10^{-6}E_r^2\left[-m' + \sqrt{\frac{n_1'E_s^2}{E_r^2} - \left(\frac{10^3P_r}{E_r^2} + l'\right)^2}\right]$	

Using 86,700 as the receiver voltage to neutral, calculations are made to find the ratio of supply to receiver voltage to neutral, for which the synchronous reactor required is of minimum capacity. If, in the above equation, the reactive power (Q_r) be expressed in kilovolt-amperes, the receiver voltage (E_r) in kilovolts and the receiver power (P_r) in kilowatts, the reactive power required per phase in the receiver circuit is

$$Q_r = E_r^2\left[-m' + \sqrt{n'^2x^2 - \left(\frac{P_r}{E_r^2} + l'\right)^2}\right]$$

where x is the value of the ratio $E_s \div E_r$, which is now sought. The lagging reactive power required per phase in the receiver circuit at no load is

$$\begin{aligned} Q_{r0} &= E_r^2[-m' + \sqrt{n'^2x^2 - (l')^2}] \\ &= 7{,}516.9[-5.6517 + \sqrt{(6.2535)^2x^2 - (0.94613)^2}] \\ &= -42{,}483.2 + 7{,}516.9\sqrt{39.1063x^2 - 0.89516} \\ &= -42{,}483.2 + 47{,}006.1\sqrt{x^2 - 0.022890}. \end{aligned}$$

The reactive power required, per phase in the receiver circuit at full load, is

$$Q_{rL} = E_r^2\left[-m' + \sqrt{n'^2x^2 - \left(\frac{27{,}000}{E_r^2} + l'\right)^2}\right]$$

$$= 7{,}516.9\left[-5.6517 + \sqrt{(6.2535)^2x^2 - \left(\frac{27{,}000}{7516.9} + 0.94613\right)^2}\right]$$

$$= -42{,}483.2 + 7{,}516.9\sqrt{39.1063x^2 - 20.5938}$$
$$= -42{,}483.2 + 47{,}006.1\sqrt{x^2 - 0.52661}.$$

The load power factor has been assumed as constant and equal to 0.90, current lagging. Therefore, at full load, the receiver load supplies lagging reactive kilovolt-amperes to the amount of

$$Q_L = 27{,}000 \times \tan\cos^{-1} 0.90$$
$$= 0.4841 \times 27{,}000$$
$$= 13{,}070 \text{ kva.}$$

Accordingly, at full load the leading reactive power supplied by the synchronous reactor must be equal to Q_{rL} plus an additional amount sufficient to balance out the lagging reactive kilovolt-amperes of the load, or, at full load,

$$Q_{sr} = Q_{rL} - Q_L$$
$$= -13{,}070 - 42{,}483.2 + 47{,}006.1\sqrt{x^2 - 0.52661}$$
$$= -55{,}553.2 + 47{,}006.1\sqrt{x^2 - 0.52661}.$$

If we assume a reactor whose rating as a generator of leading, reactive kilovolt-amperes is equal to 1.5 times its corresponding rating on the lagging side, then it is evident that

$$1.5Q_{r0} + Q_{sr} = 0$$

or

$$1.5[-42{,}483.2 + 47{,}006.1\sqrt{x^2 - 0.022890}] - 55{,}553.2 + 47{,}006.1\sqrt{x^2 - 0.52661} = 0.$$

This equation, solved for x, yields

$$x = 1.126+$$

the theoretically correct ratio of $E_s \div E_r$.

The supply voltage to neutral is then

$$E_s = 1.126 \times 86{,}700$$
$$= 97{,}625 \text{ volts.}$$

The critical, disruptive voltage for this line at 2,000 ft. elevation is 175,000 volts for equilateral spacing of conductors, or approximately 169,000 volts for the mid-conductor of a flat spaced line. Thus, at the maximum elevation, the mid-conductor will operate approximately at the corona voltage.

Substituting the correct value of x in the appropriate equations for Q yields the values of Q_r for the corresponding assumed loads P_r. These values have been calculated for each 25 per cent of load up to 125 per cent, and are shown in the table on page 355. From this table the lagging kilovolt-amperes per phase furnished by the synchronous reactor is found to be 9,985 kva., while, at full load, the corresponding leading reactive kilovolt-amperes fur-

nished are 1,990 + 13,070 or 15,060. Thus, the rating of the synchronous reactor should be 45,000 kva.

The graphical solution for x, corresponding to the foregoing mathematical solution is shown in Fig. 103.

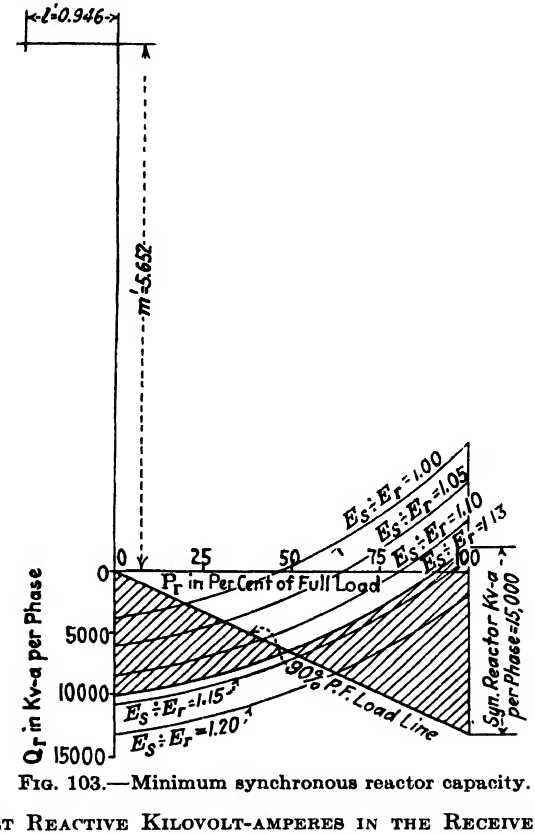


FIG. 103.—Minimum synchronous reactor capacity.

Q_r = NET REACTIVE KILOVOLT-AMPERES IN THE RECEIVER CIRCUIT

	(1)	(2)	(3)[1]	(4)	(5)[2]	(6)[3]
Load, per cent	$\frac{P_r}{E_r^2}$	$\left(\frac{P_r}{E_r^2} + l'\right)^2$	49.6180 − (2)	$\sqrt{(3)}$	−5.6517 + (4)	Q_r = 7,516.9 × (5)
0	0.00000	0.89516	48.72286	6.980	1.3283	9,985 (lagging)
25	0.89797	3.40070	46.21732	6.798	1.1466	8,619 (lagging)
50	1.79594	7.51895	42.09907	6.488	0.8366	6,289 (lagging)
75	2.69392	13.24993	36.36809	6.031	0.3793	2,850 (lagging)
100	3.59189	20.59363	29.02439	5.387	−0.2647	−1,990 (leading)
125	4.48986	29.54999	20.06803	4.480	−1.1717	−8,807 (leading)

[1] $n'^2 \times (1.126)^2 = 49.61802$.   [2] $-5.6517 = -m'$ [3] $7{,}516.9 = E_r^2$.

TABLE 37

Item		Reference, equations, etc.	Quantity
Data for exact performance diagram:			
E_r = receiver voltage to neutral (final value)		....	86,700
Receiver line voltage		$\sqrt{3} \times 86{,}700$	150,160
E_s = supply voltage to neutral		....	97,625
Supply line voltage		$\sqrt{3} \times 97{,}625$	169,090
${}_rI_1$ = full-load active component receiver amperes		$\dfrac{\text{Receiver full load kilowatts}}{3 \times 10^{-3} E_r}$	311.4
$E_1 = E_r\sqrt{a_1^2 + a_2^2}$	components of supply voltage	$86{,}700 \times 0.91634$	79.447
$E_2 = {}_rI_1\sqrt{b_1^2 + b_2^2}$	components of supply voltage	311.4×159.991	49,796
${}_sI_2 = E_r\sqrt{c_1^2 + c_2^2}$	components of supply current	$86{,}700 \times 101.566 \times 10^{-6}$	88.1
${}_sI_1 = {}_rI_1\sqrt{a_1^2 + a_2^2}$	components of supply current	311.4×0.91634	285.3
Line performance:			
Synchronous reactor kilovolt-amperes at full load (total)		3 × 15,000 kva.	45,000
Synchronous reactor kilovolt-amperes at no load (total)		3 × 10,000 kva.	30,000
Total receiver kilowatts for per cent load =	0	$= 3P_r$ Values assumed	0
	25		20,250
	50		40,500
	75		60,750
	100		81,000
	125		101,250
Total receiver reactive kilovolt-amperes for per cent load =	0	$= 3Q_r$	29,960
	25		25,860
	50		18,870
	75		8,550
	100		−5,970
	125		−26,420
Active receiver amperes for per cent load =	0	${}_rI_1 = \dfrac{P_r}{E_r}$	0.0
	25		77.8
	50		155.7
	75		233.5
	100		311.4
	125		389.3
Reactive receiver amperes for per cent load =	0	${}_rI_2 = \dfrac{Q_r}{E_r}$	115.2
	25		99.4
	50		72.5
	75		32.9
	100		−23.0
	125		−101.6
Supply vector voltage for per cent load =	0	From Eq. (372)	97,650 − j1,650
	25		97,020 + j11,030
	50		94,650 + j23,990
	75		90,270 + j37,240
	100		83,320 + j50,880
	125		72,770 + j 65,116

TABLE 37.—(*Continued*)

Item		Reference, equations, etc.	Quantity
Angle of supply voltage for per cent load =	0	$\theta_s = \tan^{-1} \frac{_sE_2}{_sE_1}$	$-0°\ 53.08'$
	25		$6°\ 29.3'$
	50		$14°\ 13.3'$
	75		$22°\ 25.0'$
	100		$31°\ 24.6'$
	125		$41°\ 49.4'$
Supply vector current for per cent load =	0	From Eq. (373)	$1.1 - j17.5$
	25		$72.2 - j2.0$
	50		$143.2 + j23.6$
	75		$214.0 + j60.9$
	100		$284.6 + j113.2$
	125		$355.1 + j186.2$
Angle of supply current for per cent load =	0	$\theta_i = \tan^{-1} \frac{I_2}{I_1}$	$-86°\ 24.0'$
	25		$-\ 1°\ 32.4'$
	50		$9°\ 25.5'$
	75		$15°\ 54.5'$
	100		$21°\ 41.4'$
	125		$27°\ 40.2'$
Supply current, amperes for per cent load =	0	$\sqrt{I_1^2 + I_2^2}$	17.5
	25		72.2
	50		145.1
	75		222.5
	100		306.3
	125		400.9
Supply power factor angle for per cent load =	0	$\theta = \theta_s - \theta_i$	$85°\ 25.9'$
	25		$8°\ 1.7'$
	50		$4°\ 47.8'$
	75		$6°\ 30.5'$
	100		$9°\ 43.2'$
	125		$14°\ 9.2'$
Supply power factor in per cent for per cent load =	0	$100 \cos \theta$	7.99
	25		99.00
	50		99.65
	75		99.36
	100		98.56
	125		96.96
Supply kilowatts for per cent load =	0	$\frac{3(_sE_{1s}I_1 + {_sE_{2s}}I_2)}{1,000}$	409
	25		20,920
	50		42,360
	75	$= 3P_s \div 1,000$	64,760
	100		88,420
	125		113,900
Line loss, (kilowatts) for per cent load =	0	$3P_L =$	409
	25		670
	50	$3(P_s - P_r)$	1,860
	75		4,010
	100		7,420
	125		12,650

TABLE 37.—(*Continued*)

Item	Reference, equations, etc.	Quantity
Per cent line loss for per cent load = 0	$\left[\frac{100 \times 3P_L}{3P_s}\right]$	100.0
25		3.2
50		4.4
75		6.2
100		8.4
125		11.1
Supply voltage to give rated receiver voltage (receiver end open)	$\sqrt{3}E_r\sqrt{a_1^2 + a_2^2}$	137,600
Supply current with rated receiver voltage (receiver end open)..................	$E_r\sqrt{c_1^2 + c_2^2}$	88.1
Receiver voltage with rated supply voltage impressed (receiver end open)....	$\frac{\sqrt{3}E_s}{\sqrt{a_1^2 + a_2^2}}$	184,550
Per cent voltage rise at receiver (receiver end open and rated supply e.m f. impressed)................... ..	$\frac{100(184,550 - 150,160)}{150,160}$	22.9

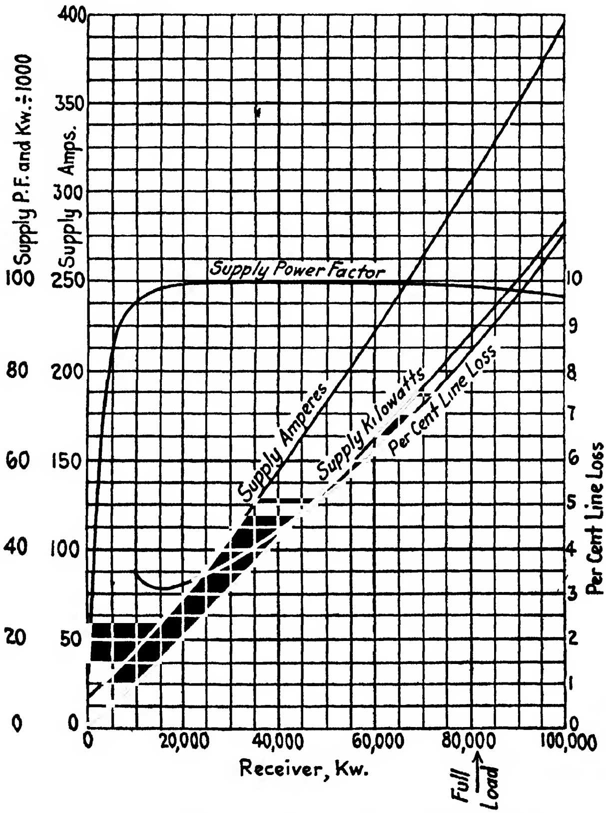


FIG. 105.—Performance curves for 200-mile line.

The performance diagram, representing a graphical solution of the line performance, is shown in Fig. 104, while the performance curves, plotted from the above data, are shown in Fig. 105.

Temperature-tension stringing charts are not worked out for this line, since the calculations are exactly similar to those of the illustrative problem at Chap. X, Tables 12 to 18, inclusive.

APPENDIX A

Important Relations of Circular and Hyperbolic Trigonometry

$$\operatorname{csch} u = \frac{1}{\sinh u}$$

$$\operatorname{sech} u = \frac{1}{\cosh u}$$

$$\coth u = \frac{1}{\tanh u}$$

$$\frac{d(\sinh u)}{du} = \cosh u$$

$$\frac{d(\cosh u)}{du} = \sinh u$$

$$\frac{d(\tanh u)}{du} = \operatorname{sech}^2 u$$

$$\frac{d(\coth u)}{du} = -\operatorname{csch}^2 u$$

$$\frac{d(\operatorname{sech} u)}{du} = -\operatorname{sech} u \tanh u$$

$$\frac{d(\operatorname{csch} u)}{du} = -\operatorname{csch} u \coth u$$

$$\operatorname{csch}^2 u = \coth^2 u - 1$$

$$\operatorname{sech}^2 u = 1 - \tanh^2 u$$

$$\cosh^2 u - \sinh^2 u = 1$$

$$\sinh (u \pm v) = \sinh u \cosh v \pm \cosh u \sinh v$$

$$\cosh (u \pm v) = \cosh u \cosh v \pm \sinh u \sinh v$$

$$\tanh (u \pm v) = (\tanh u \pm \tanh v) \div (1 \pm \tanh u \tanh v)$$

$$\coth (u \pm v) = (\coth u \coth v \pm 1) \div (\coth v \pm \coth u)$$

$$\sinh 2u = 2 \sinh u \cosh u$$

$$\cosh 2u = \cosh^2 u + \sinh^2 u = 2 \cosh^2 u - 1 = 1 + 2 \sinh^2 u$$

$$\tanh 2u = 2 \tanh u \div (1 + \tanh^2 u)$$

$$\tanh \frac{u}{2} = (\cosh u - 1) \div \sinh u = \sinh u \div (1 + \cosh u) = \sqrt{(\cosh u - 1) \div (\cosh u + 1)}$$

$$\csc \alpha = \frac{1}{\sin \alpha}$$

$$\sec \alpha = \frac{1}{\cos \alpha}$$

$$\cot \alpha = \frac{1}{\tan \alpha}$$

$$\frac{d(\sin \alpha)}{d\alpha} = \cos \alpha$$

$$\frac{d(\cos \alpha)}{d\alpha} = -\sin \alpha$$

$$\frac{d(\tan \alpha)}{d\alpha} = \sec^2 \alpha$$

$$\frac{d(\cot \alpha)}{d\alpha} = -\csc^2 \alpha$$

$$\frac{d(\sec \alpha)}{d\alpha} = \sec \alpha \tan \alpha$$

$$\frac{d(\csc \alpha)}{d\alpha} = -\csc \alpha \cot \alpha$$

$$\csc^2 \alpha = \cot^2 \alpha + 1$$

$$\sec^2 \alpha = \tan^2 \alpha + 1$$

$$\cos^2 \alpha + \sin^2 \alpha = 1$$

$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos (\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan (\alpha \pm \beta) = (\tan \alpha \pm \tan \beta) \div (1 \mp \tan \alpha \tan \beta)$$

$$\cot (\alpha \pm \beta) = (\cot \alpha \cot \beta \mp 1) \div (\cot \beta \pm \cot \alpha)$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha = 2 \cosh^2 \alpha - 1 = 1 - 2 \sin^2 \alpha$$

$$\tan 2\alpha = 2 \tan \alpha \div (1 - \tan^2 \alpha)$$

$$\tan \frac{\alpha}{2} = (1 - \cos \alpha) \div \sin \alpha = \sin \alpha \div (1 + \cos \alpha) = \sqrt{(1 - \cos \alpha) \div (1 + \cos \alpha)}$$

APPENDIX B

Natural Hyperbolic Functions

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0.000	1.000 000 000 0		0.000 000 000 0	
.001	.000 000 500 0	0.000 000 500 0	.001 000 000 2	0.001 000 000 2
.002	.000 002 000 0	.000 001 500 0	.002 000 001 3	.001 000 001 1
.003	.000 004 500 0	.000 002 500 0	.003 000 004 5	.001 000 003 2
.004	.000 008 000 0	.000 003 500 0	.004 000 010 7	.001 000 006 2
0.005	1.000 012 500 0	0.000 004 500 0	0.005 000 020 8	0.001 000 010 1
.006	.000 018 000 1	.000 005 500 1	.006 000 036 0	.001 000 015 2
.007	.000 024 500 1	.000 006 500 1	.007 000 057 2	.001 000 021 2
.008	.000 032 000 2	.000 007 500 1	.008 000 085 3	.001 000 028 1
.009	.000 040 500 3	.000 008 500 1	.009 000 121 5	.001 000 036 2
0.010	1.000 050 000 4	0.000 009 500 1	0.010 000 166 7	0.001 000 045 2
.011	.000 060 500 6	.000 010 500 2	.011 000 221 8	.001 000 055 1
.012	.000 072 000 9	.000 011 500 3	.012 000 288 0	.001 000 066 2
.013	.000 084 501 2	.000 012 500 3	.013 000 366 2	.001 000 078 2
.014	.000 098 001 6	.000 013 500 4	.014 000 457 3	.001 000 091 1
0 015	1.000 112 502 1	0 000 014 500 5	0.015 000 562 5	0.001 000 105 2
.016	.000 128 002 7	.000 015 500 6	.016 000 682 7	.001 000 120 2
.017	.000 144 503 5	.000 016 500 8	.017 000 818 8	.001 000 136 1
.018	.000 162 004 4	.000 017 500 9	.018 000 972 0	.001 000 153 2
.019	.000 180 505 4	.000 018 501 0	.019 001 143 2	.001 000 171 2
0.020	1.000 200 006 6	0.000 019 501 2	0.020 001 331 3	0.001 000 188 1
.021	.000 220 508 1	.000 020 501 5	.021 001 543 5	.001 000 212 2
.022	.000 242 009 8	.000 021 501 7	.022 001 774 7	.001 000 231 2
.023	.000 264 511 7	.000 022 501 9	.023 002 027 8	.001 000 253 1
.024	.000 288 013 8	.000 023 502 1	.024 002 304 1	.001 000 276 3
0.025	1.000 312 516 3	0 000 024 502 5	0.025 002 604 3	0.001 000 300 2
.026	.000 338 019 0	.000 025 502 7	.026 002 929 4	.001 000 325 1
.027	.000 364 522 1	.000 026 503 1	.027 003 280 6	.001 000 351 2
.028	.000 392 025 6	.000 027 503 5	.028 003 658 8	.001 000 378 2
.029	.000 420 529 5	.000 028 503 9	.029 004 065 0	.001 000 416 2
0.030	1.000 450 033 8	0.000 029 504 3	0.030 004 500 2	0.001 000 435 2
.031	.000 480 538 5	.000 030 504 7	.031 004 965 4	.001 000 465 2
.032	.000 512 043 7	.000 031 505 2	.032 005 461 6	.001 000 496 2
.033	.000 544 549 4	.000 032 505 7	.033 005 989 8	.001 000 528 2
.034	.000 578 055 7	.000 033 506 3	.034 006 551 1	.001 000 561 3
0.035	1.000 612 562 5	0.000 034 506 8	0.035 007 146 3	0.001 000 595 2
.036	.000 648 070 0	.000 035 507 5	.036 007 776 5	.001 000 630 2
.037	.000 684 578 1	.000 036 508 1	.037 008 442 8	.001 000 666 3
.038	.000 722 086 9	.004 037 508 8	.038 009 146 0	.001 000 704 2
.039	.000 760 596 4	.000 038 509 5	.039 009 887 3	.001 000 741 3
0.040	1.000 800 106 7	0.000 039 510 3	0.040 010 667 5	0.001 000 780 2
.041	.000 840 617 7	.000 040 511 0	.041 011 487 8	.001 000 820 3
.042	.000 882 129 7	.000 041 512 0	.042 012 349 1	.001 000 861 3
.043	.000 924 642 4	.000 042 512 7	.043 013 252 4	.001 000 903 3
.044	.000 968 156 2	.000 043 513 8	.044 014 198 7	.001 000 946 3
0.045	1.001 012 670 9	0.000 044 514 7	0.045 015 189 0	0.001 000 991 3

NATURAL HYPERBOLIC FUNCTIONS

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0.045	1.001 012 670 9	0.000 044 514 7	0.045 015 189 0	0.001 000 991 3
.046	.001 058 186 6	.000 045 515 7	.046 016 224 4	.001 001 035 4
.047	.001 104 703 3	.000 046 516 7	.047 017 305 7	.001 001 081 3
.048	.001 152 221 2	.000 047 517 9	.048 018 434 1	.001 001 128 4
.049	.001 200 740 2	.000 048 519 0	.049 019 610 5	.001 001 176 4
0.050	1.001 250 260 4	0.000 049 520 2	0.050 020 835 9	0.001 001 225 4
.051	.001 300 781 9	.000 050 521 5	.051 022 111 4	.001 001 275 5
.052	.001 352 304 7	.000 051 522 8	.052 023 435 8	.001 001 324 4
.053	.001 404 828 8	.000 052 524 1	.053 024 816 3	.001 001 380 5
.054	.001 458 354 3	.000 053 525 5	.054 026 247 8	.001 001 431 5
0.055	1.001 512 881 3	0.000 054 527 0	0.055 027 733 4	0.001 001 485 6
.056	.001 568 409 8	.000 055 528 5	.056 029 273 9	.001 001 540 5
.057	.001 624 939 8	.000 056 530 0	.057 030 870 5	.001 001 596 6
.058	.001 682 471 5	.000 057 531 7	.058 032 524 1	.001 001 653 6
.059	.001 741 004 9	.000 058 533 4	.059 034 235 8	.001 001 711 7
0.060	1.001 800 540 1	0.000 059 535 2	0.060 036 006 5	0.001 001 770 7
.061	.001 861 076 9	.000 060 536 8	.061 037 837 2	.001 001 830 7
.062	.001 922 615 8	.000 061 538 9	.062 039 729 0	.001 001 892 8
.063	.001 985 156 4	.000 062 540 6	.063 041 682 8	.001 001 953 8
.064	.002 048 699 1	.000 063 542 7	.064 043 699 6	.001 002 016 8
0.065	1.002 113 243 9	0.000 064 544 8	0.065 045 779 5	0.001 002 079 9
.066	.002 178 790 7	.000 065 546 8	.066 047 926 4	.001 002 146 9
.067	.002 245 339 7	.000 066 549 0	.067 050 138 4	.001 002 212 0
.068	.002 312 890 9	.000 067 551 2	.068 052 417 4	.001 002 279 0
.069	.002 381 444 5	.000 068 553 6	.069 054 764 5	.001 002 347 1
0.070	1.002 451 000 4	0.000 069 555 9	0.070 057 180 7	0.001 002 416 2
.071	.002 521 558 8	.000 070 558 4	.071 059 666 8	.001 002 486 1
.072	.002 593 119 9	.000 071 561 1	.072 062 224 1	.001 002 557 3
.073	.002 665 683 5	.000 072 563 6	.073 064 853 4	.001 002 629 3
.074	.002 739 249 7	.000 073 566 2	.074 067 555 8	.001 002 702 4
0.075	1.002 813 818 6	0.000 074 568 9	0.075 070 332 3	0.001 002 776 5
.076	.002 889 390 4	.000 075 571 8	.076 073 183 8	.001 002 851 5
.077	.002 965 965 0	.000 076 574 6	.077 076 111 4	.001 002 927 6
.078	.003 043 542 6	.000 077 577 6	.078 079 116 1	.001 003 004 7
.079	.003 122 123 3	.000 078 580 7	.079 082 197 8	.001 003 081 7
0.080	1.003 201 706 4	0.000 079 583 8	0.080 085 360 6	0.001 003 162 8
.081	.003 282 294 0	.000 080 587 6	.081 088 602 6	.001 003 242 0
.082	.003 363 884 3	.000 081 590 3	.082 091 925 6	.001 003 323 0
.083	.003 446 477 9	.000 082 593 6	.083 095 330 7	.001 003 405 1
.084	.003 530 074 9	.000 083 597 0	.084 098 818 9	.001 003 488 2
0.085	1.003 614 675 5	0.000 084 600 6	0.085 102 391 1	0.001 003 572 2
.086	.003 700 279 8	.000 085 604 3	.086 106 048 5	.001 003 657 4
.087	.003 786 887 7	.000 086 607 9	.087 109 792 0	.001 003 743 5
.088	.003 874 499 4	.000 087 611 7	.088 113 622 6	.001 003 830 6
.089	.003 963 114 9	.000 088 615 5	.089 117 541 4	.001 003 918 8
0.090	1.004 052 734 5	0.000 089 619 6	0.090 121 549 2	0.001 004 007 8

Natural Hyperbolic Functions

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0.090	1.004 052 734 5	0.000 089 619 6	0.090 121 549 2	0.001 004 007 8
.091	.004 143 358 1	.000 090 623 6	.091 125 647 2	.001 004 098 0
.092	.004 234 984 8	.000 091 626 7	.092 129 836 3	.001 004 189 1
.093	.004 327 617 8	.000 092 633 0	.093 134 117 5	.001 004 281 2
.094	.004 421 254 1	.000 093 636 3	.094 138 491 8	.001 004 374 3
0.095	1.004 515 894 8	0.000 094 640 7	0.095 142 960 3	0.001 004 468 5
.096	.004 611 540 0	.000 095 645 2	.096 147 523 9	.001 004 563 6
.097	.004 708 189 9	.000 096 649 9	.097 152 183 7	.001 004 659 8
.098	.004 805 844 4	.000 097 654 5	.098 156 940 7	.001 004 757 0
.099	.004 904 503 8	.000 098 659 4	.099 161 795 7	.001 004 855 0
0.100	1.005 004 168 0	0.000 099 664 2	0.100 166 750 0	0.001 004 954 3
.101	.005 104 837 3	.000 100 669 3	.101 171 804 4	.001 005 054 4
.102	.005 206 511 7	.000 101 674 4	.102 176 960 0	.001 005 155 6
.103	.005 309 191 3	.000 102 679 6	.103 182 217 8	.001 005 257 8
.104	.005 412 876 2	.000 103 684 9	.104 187 578 7	.001 005 360 9
0.105	1.005 517 566 5	0.000 104 690 3	0.105 193 043 9	0.001 005 465 2
.106	.005 623 262 3	.000 105 695 8	.106 198 614 2	.001 005 570 3
.107	.005 729 963 7	.000 106 701 4	.107 204 290 7	.001 005 676 5
.108	.005 837 670 9	.000 107 707 2	.108 210 074 4	.001 005 783 7
.109	.005 946 383 9	.000 108 713 0	.109 215 966 3	.001 005 891 9
0.110	1.006 056 102 9	0.000 109 719 0	0.110 221 967 5	0.001 006 001 2
.111	.006 166 827 8	.000 110 724 9	.111 228 078 9	.001 006 111 4
.112	.006 278 559 1	.000 111 731 3	.112 234 301 5	.001 006 222 6
.113	.006 391 296 5	.000 112 737 4	.113 240 636 4	.001 006 334 9
.114	.006 505 040 4	.000 113 743 9	.114 247 084 5	.001 006 448 1
0.115	1.006 619 790 7	0.000 114 750 3	0.115 253 646 8	0.001 006 562 3
.116	.006 735 547 7	.000 115 757 0	.116 260 324 4	.001 006 677 6
.117	.006 852 311 4	.000 116 763 7	.117 267 118 2	.001 006 793 8
.118	.006 970 082 0	.000 117 770 6	.118 274 029 3	.001 006 911 1
.119	.007 088 859 5	.000 118 777 5	.119 281 058 7	.001 007 029 4
0.120	1.007 208 644 1	0.000 119 784 6	0.120 288 207 4	0.001 007 148 7
.121	.007 329 436 0	.000 120 791 9	.121 295 476 3	.001 007 268 9
.122	.007 451 235 1	.000 121 799 1	.122 302 866 6	.001 007 390 3
.123	.007 574 041 7	.000 122 806 6	.123 310 379 1	.001 007 512 5
.124	.007 697 855 9	.000 123 814 2	.124 318 015 0	.001 007 635 9
0.125	1.007 822 677 8	0.000 124 821 9	0.125 325 775 1	0.001 007 760 1
.126	.007 948 507 5	.000 125 829 7	.126 333 660 7	.001 007 885 6
.127	.008 075 345 2	.000 126 837 7	.127 341 672 6	.001 008 011 9
.128	.008 203 190 9	.000 127 845 7	.128 349 811 8	.001 008 139 2
.129	.008 332 044 9	.000 128 854 0	.129 358 079 3	.001 008 267 3
0.130	1.008 461 907 1	0.000 129 863 2	0.130 366 476 2	0.001 008 396 9
.131	.008 592 777 9	.000 130 870 8	.131 375 003 4	.001 008 527 2
.132	.008 724 657 1	.000 131 879 2	.132 383 662 1	.001 008 658 7
.133	.008 857 545 2	.000 132 888 1	.133 392 453 1	.001 008 791 0
.134	.008 991 442 1	.000 133 896 9	.134 401 377 5	.001 008 924 4
0.135	1.009 126 348 0	0.000 134 906 9	0.135 410 436 3	0.001 009 058 8

Natural Hyperbolic Functions

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0.135	1.009 126 348 0	0.000 134 906 9	0.135 410 436 3	0.001 009 058 8
.136	.009 262 263 0	.000 135 915 0	.136 419 630 5	.001 009 194 2
.137	.009 399 187 3	.000 136 924 3	.137 428 961 2	.001 009 330 7
.138	.009 537 121 0	.000 137 933 7	.138 438 429 3	.001 009 468 1
.139	.009 676 064 2	.000 138 943 2	.139 448 035 8	.001 009 606 5
0.140	1.009 816 017 1	0.000 139 952 9	0.140 457 781 7	0.001 009 745 9
.141	.009 956 979 8	.000 140 962 7	.141 467 668 1	.001 009 886 4
.142	.010 098 952 5	.000 141 972 7	.142 477 696 0	.001 010 027 9
.143	.010 241 935 3	.000 142 982 8	.143 487 866 4	.001 010 170 4
.144	.010 385 928 3	.000 143 993 0	.144 498 180 2	.001 010 313 8
0.145	1.010 530 931 7	0.000 145 003 4	0.145 508 638 4	0.001 010 458 2
.146	.010 676 945 6	.000 146 013 9	.146 519 242 4	.001 010 604 0
.147	.010 823 970 2	.000 147 024 6	.147 529 992 8	.001 010 750 4
.148	.010 972 005 6	.000 148 035 4	.148 540 890 7	.001 010 897 9
.149	.011 121 052 0	.000 149 046 4	.149 551 937 2	.001 011 046 5
0.150	1.011 271 109 6	0.000 150 057 6	0.150 563 133 1	0.001 011 195 9
.151	.011 422 178 4	.000 151 068 8	.151 574 479 7	.001 011 346 6
.152	.011 574 258 6	.000 152 080 2	.152 585 977 8	.001 011 498 1
.153	.011 727 350 4	.000 153 091 8	.153 597 628 6	.001 011 650 8
.154	.011 881 453 9	.000 154 103 5	.154 609 432 9	.001 011 804 3
0.155	1.012 036 569 3	0.000 155 115 4	0.155 621 391 8	0.001 011 958 9
.156	.012 192 696 7	.000 156 127 4	.156 633 506 4	.001 012 114 6
.157	.012 349 836 3	.000 157 139 6	.157 645 777 5	.001 012 271 1
.158	.012 507 988 3	.000 158 152 0	.158 658 206 4	.001 012 428 9
.159	.012 667 152 8	.000 159 164 5	.159 670 793 8	.001 012 587 4
0.160	1.012 827 330 0	0.000 160 177 2	0.160 683 541 0	0.001 012 747 2
.161	.012 988 519 9	.000 161 189 9	.161 696 448 8	.001 012 907 8
.162	.013 150 722 9	.000 162 203 0	.162 709 518 4	.001 013 069 6
.163	.013 313 939 0	.000 163 216 1	.163 722 750 6	.001 013 232 2
.164	.013 478 168 5	.000 164 229 5	.164 736 146 6	.001 013 396 0
0.165	1.013 643 411 4	0.000 165 242 9	0.165 749 707 2	0.001 013 560 6
.166	.013 809 667 9	.000 166 256 5	.166 763 433 8	.001 013 726 6
.167	.013 976 938 3	.000 167 270 4	.167 777 327 0	.001 013 893 2
.168	.014 145 222 6	.000 168 284 3	.168 791 387 9	.001 014 060 9
.169	.014 314 521 1	.000 169 298 5	.169 805 617 8	.001 014 229 9
0.170	1.014 484 833 9	0.000 170 312 8	0.170 820 017 3	0.001 014 399 5
.171	.014 656 161 2	.000 171 327 3	.171 834 587 8	.001 014 570 5
.172	.014 828 503 2	.000 172 342 0	.172 849 330 0	.001 014 742 2
.173	.015 001 859 9	.000 173 356 7	.173 864 245 1	.001 014 915 1
.174	.015 176 231 7	.000 174 371 8	.174 879 334 1	.001 015 089 0
0.175	1.015 351 618 7	0.000 175 387 0	0.175 894 597 9	0.001 015 263 8
.176	.015 528 021 0	.000 176 402 3	.176 910 037 7	.001 015 439 8
.177	.015 705 438 8	.000 177 417 8	.177 925 654 3	.001 015 616 6
.178	.015 883 872 3	.000 178 443 5	.178 941 448 9	.001 015 794 6
.179	.016 063 321 8	.000 179 449 5	.179 957 422 4	.001 015 973 5
0.180	1.016 243 787 2	0.000 180 465 4	0.180 973 575 9	0.001 016 153 5

NATURAL HYPERBOLIC FUNCTIONS

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0.180	1.016 243 787 2	0.000 180 465 4	0 180 973 575 9	0.001 016 153 5
.181	.016 425 269 0	.000 181 481 8	.181 989 910 3	.001 016 334 4
.182	.016 607 767 1	.000 182 498 1	.183 006 426 7	.001 016 516 4
.183	.016 791 281 9	.000 183 514 8	.184 023 126 2	.001 016 699 5
.184	.016 975 813 4	.000 184 531 5	.185 040 009 6	.001 016 883 4
0.185	1.017 161 361 9	0.000 185 548 5	0.186 057 078 2	0.001 017 068 6
.186	.017 347 927 6	.000 186 565 7	.187 074 332 7	.001 017 254 5
.187	.017 535 510 7	.000 187 583 1	.188 091 774 3	.001 017 441 6
.188	.017 724 111 3	.000 188 600 6	.189 109 401 1	.001 017 629 8
.189	.017 913 729 6	.000 189 618 3	.190 127 222 9	.001 017 818 8
0 190	1.018 104 365 8	0.000 190 636 2	0.191 145 231 9	0.001 018 009 0
.191	.018 296 020 1	.000 191 654 3	.192 163 432 0	.001 018 200 1
.192	.018 488 692 7	.000 192 672 6	.193 181 824 2	.001 018 392 2
.193	.018 682 383 8	.000 193 691 1	.194 200 409 7	.001 018 585 5
.194	.018 877 093 6	.000 194 709 8	.195 219 189 3	.001 018 779 6
0.195	1.019 072 822 3	0.000 195 728 7	0.196 238 164 2	0.001 018 974 9
196	.019 269 570 0	.000 196 747 7	197 257 335 3	.001 019 171 1
197	.019 467 337 0	.000 197 767 0	.198 276 703 7	.001 019 368 4
.198	.019 666 123 5	.000 198 786 5	.199 296 270 3	.001 019 566 6
.199	.019 865 929 6	.000 199 806 1	.200 316 036 3	.001 019 766 0
0.200	1.020 066 755 6	0 000 200 826 0	0 201 336 002 5	0.001 019 966 2
.201	.020 268 601 7	.000 201 846 1	.202 356 170 1	.001 020 167 6
.202	.020 471 468 0	.000 202 866 3	203 376 540 1	.001 020 370 0
.203	.020 675 354 8	.000 203 886 8	.204 397 113 4	.001 020 573 3
.204	.020 880 262 3	.000 204 907 5	.205 417 891 1	.001 020 777 7
0.205	1.021 086 190 7	0.000 205 928 4	0.206 438 874 3	0.001 020 983 2
.206	.021 293 140 2	.000 206 949 5	.207 460 063 8	.001 021 189 5
.207	.021 501 110 9	.000 207 970 7	.208 481 460 9	.001 021 397 1
.208	.021 710 103 2	.000 208 992 3	.209 503 066 4	.001 021 605 5
.209	.021 920 117 1	.000 210 013 8	.210 524 881 4	.001 021 815 0
0 210	1.022 131 153 0	0.000 211 035 9	0.211 546 907 0	0 001 022 025 6
.211	.022 343 211 0	.000 212 058 0	.212 569 144 1	.001 022 237 1
.212	.022 556 291 3	.000 213 080 3	.213 591 593 7	.001 022 449 6
.213	.022 770 394 2	.000 214 102 9	.214 614 257 0	.001 022 663 3
.214	.022 985 519 9	.000 215 125 7	.215 637 134 9	.001 022 877 9
0.215	1.023 201 668 6	0.000 216 148 7	0.216 660 228 4	0.001 023 093 5
.216	.023 418 840 4	.000 217 171 8	.217 683 538 6	.001 023 310 2
.217	.023 637 035 7	.000 218 195 3	.218 707 066 4	.001 023 527 8
.218	.023 856 254 6	.000 219 218 9	.219 730 813 0	.001 023 746 6
.219	.024 076 497 4	.000 220 242 8	.220 754 779 3	.001 023 966 3
0.220	1.024 297 764 3	0.000 221 266 9	0.221 778 966 3	0.001 024 187 0
.221	.024 520 055 4	.000 222 291 1	.222 803 375 1	.001 024 408 8
.222	.024 743 371 1	.000 223 315 7	.223 828 006 8	.001 024 631 7
.223	.024 967 711 5	.000 224 340 4	.224 852 862 2	.001 024 855 4
.224	.025 193 076 9	.000 225 365 4	.225 877 942 5	.001 025 080 3
0.225	1.025 419 467 2	0.000 226 390 3	0.226 903 248 7	0.001 025 306 2

NATURAL HYPERBOLIC FUNCTIONS

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0.225	1.025 419 467 2	0.000 226 390 3	0.226 903 248 7	0.001 025 306 2
.226	.025 646 883 2	.000 227 416 0	.227 928 781 8	.001 025 533 1
.227	.025 875 325 1	.000 228 441 9	.228 954 542 8	.001 025 761 0
.228	.026 104 792 6	.000 229 467 5	.229 980 532 8	.001 025 990 0
.229	.026 335 286 3	.000 230 493 7	.231 006 752 7	.001 026 219 9
0.230	1.026 566 806 2	0.000 231 519 9	0.232 033 203 7	0.001 026 451 0
.231	.026 799 352 7	.000 232 546 5	.233 059 886 7	.001 026 683 0
.232	.027 032 926 1	.000 233 573 4	.234 086 802 8	.001 026 916 1
.233	.027 267 526 4	.000 234 600 3	.235 113 952 9	.001 027 150 1
.234	.027 503 153 7	.000 235 627 3	.236 141 338 2	.001 027 385 3
0.235	1.027 739 809 2	0.000 236 655 5	0.237 168 959 5	0.001 027 621 3
.236	.027 977 492 1	.000 237 682 9	.238 196 818 1	.001 027 858 6
.237	.028 216 202 9	.000 238 710 8	.239 224 914 9	.001 028 096 8
.238	.028 455 942 0	.000 239 739 1	.240 253 250 9	.001 028 336 0
.239	.028 696 709 2	.000 240 767 2	.241 281 827 1	.001 028 576 2
0.240	1.028 938 505 7	0.000 241 796 5	0.242 310 644 6	0.001 028 817 5
.241	.029 181 330 6	.000 242 824 9	.243 339 704 5	.001 029 059 9
.242	.029 425 185 2	.000 243 854 6	.244 369 007 6	.001 029 303 1
.243	.029 670 068 9	.000 244 883 7	.245 398 555 2	.001 029 547 6
.244	.029 915 982 4	.000 245 913 5	.246 428 348 1	.001 029 792 9
0.245	1.030 162 925 7	0.000 246 943 3	0.247 458 387 5	0.001 030 039 4
.246	.030 410 899 2	.000 247 973 5	.248 488 674 3	.001 030 286 8
.247	.030 659 903 2	.000 249 004 0	.249 519 209 6	.001 030 535 3
.248	.030 909 937 7	.000 250 034 5	.250 549 994 5	.001 030 784 9
.249	.031 161 003 2	.000 251 065 5	.251 581 029 8	.001 031 035 3
0.250	1.031 413 099 9	0.000 252 096 7	0.252 612 316 8	0.001 031 287 0
.251	.031 666 227 9	.000 253 128 0	.253 643 856 4	.001 031 539 6
.252	.031 920 387 7	.000 254 159 8	.254 675 649 6	.001 031 793 2
253	.032 175 579 3	.000 255 191 6	.255 707 697 5	.001 032 047 9
254	.032 431 803 1	.000 256 223 8	.256 740 001 1	.001 032 303 6
0 255	1.032 689 059 4	0.000 257 256 3	0.257 772 561 5	0.001 032 560 4
256	.032 947 348 4	.000 258 289 0	.258 805 379 6	.001 032 818 1
257	.033 206 670 3	.000 259 321 9	.259 838 456 5	.001 033 076 9
.258	.033 467 025 4	.000 260 355 1	.260 871 793 3	.001 033 336 8
.259	.033 728 413 9	.000 261 388 5	.261 905 390 9	.001 033 597 6
0.260	1.033 990 836 2	0.000 262 422 3	0.262 939 250 4	0.001 033 859 5
.261	.034 254 292 5	.000 263 456 3	.263 973 372 9	.001 034 122 5
.262	.034 518 783 1	.000 264 490 6	.265 007 759 3	.001 034 386 4
.263	.034 784 308 1	.000 265 525 0	.266 042 410 8	.001 034 651 5
.264	.035 050 867 9	.000 266 559 8	.267 077 328 3	.001 034 917 5
0.265	1.035 318 462 9	0.000 267 595 0	0.268 112 512 9	0.001 035 184 6
.266	.035 587 093 1	.000 268 630 2	.269 147 965 6	.001 035 452 7
.267	.035 856 758 9	.000 269 665 8	.270 183 687 4	.001 035 721 8
.268	.036 127 460 6	.000 270 701 7	.271 219 679 4	.001 035 992 0
.269	.036 399 198 4	.000 271 737 8	.272 255 942 7	.001 036 263 3
0.270	1.036 671 972 6	0.000 272 774 2	0.273 292 478 2	0.001 036 535 5

$\frac{x}{c}$	Cosh $\frac{x}{c}$	Difference	Sinh $\frac{x}{c}$	Difference
0.270	1.036 671 972 6	0.000 272 774 2	0.273 292 478 2	0.001 036 535 5
.271	.036 945 783 4	.000 273 810 8	.274 329 287 0	.001 036 808 8
.272	.037 220 631 2	.000 274 847 8	.275 366 370 1	.001 037 083 1
.273	.037 496 516 3	.000 275 885 1	.276 403 728 6	.001 037 358 5
.274	.037 773 438 8	.000 276 922 5	.277 441 363 5	.001 037 634 9
0.275	1.038 051 399 0	0.000 277 960 2	0.278 479 275 8	0.001 037 912 3
.276	.038 330 397 4	.000 278 998 4	.279 517 466 6	.001 038 190 8
.277	.038 610 434 1	.000 280 036 7	.280 555 936 9	.001 038 470 3
.278	.038 891 509 4	.000 281 075 3	.281 594 687 8	.001 038 750 9
.279	.039 173 623 5	.000 282 114 1	.282 633 720 3	.001 039 032 5
0.280	1.039 456 776 0	0.000 283 153 4	0.283 673 035 4	0.001 039 315 1

APPENDIX C

$\frac{x}{c}$	$\frac{\operatorname{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\operatorname{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\operatorname{Sinh}\frac{x}{c}}$	$\frac{x}{c}$	$\frac{\operatorname{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\operatorname{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\operatorname{Sinh}\frac{x}{c}}$
0.000	0.000 000 000		0.040	0.000 266 688	50.013 337 9
.001	.000 000 200	2 000.000 300	.041	.000 280 190	48.794 158 5
.002	.000 000 650	1 000.000 675	.042	.000 294 026	47.633 052 2
.003	.000 001 500	666.667 666	.043	.000 308 195	46.525 966 3
.004	.000 002 675	500.001 331	.044	.000 322 698	45.469 217 2
0.005	0 000 004 160	400 001 668	.045	0.000 337 533	44.459 450 0
.006	.000 006 000	333 335 332	.046	.000 352 704	43.493 600 3
.007	.000 008 174	285 716 618	.047	.000 368 206	42.568 864 3
.008	.000 010 662	250.002 667	.048	.000 384 046	41.682 673 4
.009	.000 013 500	222.225 222	.049	.000 400 214	40.832 666 8
0.010	0.000 016 670	200 003 330	0.050	0.000 416 718	40.016 674 2
.011	.000 020 164	181 821 847 1	.051	.000 433 557	39.232 694 4
.012	.000 024 000	166.670 666 7	.052	.000 450 688	38.478 880 9
.013	.000 028 169	153.850 488 3	.053	.000 468 232	37.753 524 9
.014	.000 032 664	142.861 811 4	.054	.000 486 070	37.055 046 5
0.015	0.000 037 500	133.338 313 3	0.055	0.000 504 244	36.381 979 9
0.16	.000 042 669	125 005 332 3	.056	.000 522 748	35.732 963 1
0.17	.000 048 164	117 652 725 2	.057	.000 541 588	35.106 730 5
0 18	.000 054 000	111.117 111 5	.058	.000 560 760	34.502 103 9
0.19	.000 060 168	105.269 492 1	.059	.000 580 268	33.917 984 2
0.020	0.000 066 550	100.006 675 5	0.060	0.000 600 108	33.353 346 5
.021	.000 073 500	95.245 094 7	.061	.000 620 282	32.807 232 4
.022	.000 080 668	90.916 424 3	.062	.000 640 790	32.278 745 6
.023	000 088 165	86.964 188 9	.063	.000 661 632	31.767 047 0
.024	.000 096 004	83.341 334 7	.064	.000 682 806	31.271 349 2
.025	0.000 104 172	80.008 335 0	0.065	0.000 704 300	30.790 914 3
.026	.000 112 669	76.931 745 3	.066	.000 726 158	30.325 047 9
.027	.000 121 503	74.083 074 7	.067	.000 748 334	29.873 098 1
.028	.000 130 671	71.437 906 1	.068	.000 770 844	29.434 450 6
.029	.000 140 172	68.975 185 6	.069	.000 793 688	29.008 527 4
0 030	0.000 150 007	66.676 668 3	0.070	0.000 816 867	28.594 782 8
.031	.000 160 174	64.526 464 5	.071	.000 840 377	28.192 702 5
.032	.000 170 675	62.510 668 2	.072	.000 864 224	27.801 800 6
.033	.000 181 510	60.617 062 6	.073	.000 888 403	27.421 617 4
.034	.000 192 679	58.834 865 2	.074	.000 912 916	27.051 718 4
0.035	0 000 204 180	57.154 526 6	0.075	0.000 937 764	26.691 692 5
.036	.000 216 014	55.567 558 0	.076	.000 962 945	26.341 149 6
.037	.000 228 184	54.066 390 2	.077	.000 988 460	25.999 720 6
.038	.000 240 684	52.644 249 0	.078	.001 014 309	25.667 054 6
.039	.000 253 521	51.295 055 0	.079	.001 040 478	25.342 819 3
0.040	0.000 266 688	50.013 337 9	0.080	0.001 067 008	25.026 697 8

$\frac{x}{c}$	$\frac{\text{Sinh}\,\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\text{Cosh}\,\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\text{Sinh}\,\frac{x}{c}}$	$\frac{x}{c}$	$\frac{\text{Sinh}\,\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\text{Cosh}\,\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\text{Sinh}\,\frac{x}{c}}$
0.080	0.001 067 008	25.026 697 8	0.120	0.002 401 728	16.706 772 07
.081	.001 093 859	24.718 390 3	.121	.002 441 953	16.569 366 95
.082	.001 121 044	24.417 610 8	.122	.002 482 513	16.434 220 00
.083	.001 148 563	24.124 087 1	.123	.002 523 407	16.301 276 26
.084	.001 176 415	23.837 560 0	.124	.002 564 637	16.170 482 41
0.085	0.001 204 601	23.557 782 7	0.125	0.002 606 201	16.041 786 33
.086	.001 233 122	23.284 519 4	.126	.002 648 101	15.915 138 13
.087	.001 261 977	23.017 546 0	.127	.002 690 335	15.790 490 14
.088	.001 291 166	22.756 647 7	.128	.002 732 905	15.667 794 91
.089	.001 320 689	22.501 619 9	.129	.002 775 809	15.547 007 09
0.090	0.001 350 546	22.252 266 8	0.130	0.002 819 048	15.428 082 73
.091	.001 380 738	22.008 401 3	.131	.002 862 621	15.310 979 80
.092	.001 411 264	21.769 844 6	.132	.002 906 531	15.195 655 79
.093	.001 442 123	21.536 425 5	.133	.002 950 775	15.082 071 24
.094	.001 473 316	21.307 979 8	.134	.002 995 354	14.970 186 67
0.095	0.001 504 845	21.084 350 6	0.135	0.003 040 269	14.859 964 94
.096	.001 536 707	20.865 387 3	.136	.003 085 518	14.751 369 41
.097	.001 568 904	20.650 945 8	.137	.003 131 104	14.644 363 98
.098	.001 601 436	20.440 887 4	.138	.003 177 024	14.538 914 16
.099	.001 634 300	20.235 097 4	.139	.003 223 279	14.434 986 42
0.100	0.001 667 500	20.033 393 7	0.140	0.003 269 869	14.332 548 69
.101	.001 701 034	19.835 710 2	.141	.003 316 795	14.231 568 34
.102	.001 734 902	19.641 907 9	.142	.003 364 056	14.132 015 11
103	.001 769 105	19.451 875 6	.143	.003 411 653	14.033 859 13
104	.001 803 641	19.265 504 5	.144	.003 459 585	13.937 071 33
0.105	0.001 838 513	19.082 690 13	0.145	0.003 507 851	13.841 622 98
.106	.001 873 719	18.903 331 00	.146	.003 556 455	13.747 487 05
107	.001 909 259	18.727 330 36	.147	.003 605 393	13.654 636 38
.108	.001 945 133	18.554 595 87	.148	.003 654 667	13.563 044 94
.109	.001 981 342	18.385 036 01	.149	.003 704 277	13.472 687 43
0.110	0.002 017 886	18.218 565 97	0.150	0.003 754 221	13.383 539 67
.111	.002 054 765	18.055 101 50	.151	.003 804 501	13.295 576 78
.112	.002 091 978	17.894 562 16	.152	.003 855 117	13.208 775 86
.113	.002 129 526	17.736 869 58	.153	.003 906 069	13.123 114 18
114	.002 167 407	17.581 950 51	.154	.003 957 356	13.038 569 58
.115	0.002 205 624	17.429 731 12	0.155	0.004 008 979	12.955 120 06
.116	.002 244 176	17.280 141 67	.156	.004 060 938	12.872 744 61
.117	.002 283 062	17.133 114 83	.157	.004 113 232	12.791 423 17
.118	.002 322 282	16.988 586 23	.158	.004 165 863	12.711 135 34
.119	.002 361 838	16.846 492 23	.159	.004 218 829	12.631 862 08
0.120	0.002 401 728	16.706 772 07	0.160	0.004 272 131	12.553 583 61

$\frac{x}{c}$	$\dfrac{\sinh \frac{x}{c}}{\frac{x}{c}} - 1$	$\dfrac{\cosh \frac{x}{c}}{\frac{x}{c}} + \dfrac{1}{\sinh \frac{x}{c}}$	$\frac{x}{c}$	$\dfrac{\sinh \frac{x}{c}}{\frac{x}{c}} - 1$	$\dfrac{\cosh \frac{x}{c}}{\frac{x}{c}} + \dfrac{1}{\sinh \frac{x}{c}}$
0.160	0.004 272 131	12.553 583 61	0.200	0.006 680 013	10.067 155 40
.161	.004 325 769	12.476 281 83	.201	.006 747 115	10.017 744 80
.162	.004 379 743	12.399 938 70	.202	.006 814 555	9.968 826 91
.163	.004 434 053	12.324 536 59	.203	.006 882 332	9.920 394 49
.164	.004 488 690	12.250 057 97	.204	.006 950 447	9.872 440 17
0.165	0.004 543 680	12.176 486 46	0.205	0.007 018 899	9.824 957 23
.166	.004 598 999	12.103 805 70	.206	.007 087 688	9.777 988 60
.167	.004 654 653	12.031 999 00	.207	.007 156 816	9.731 377 56
.168	.004 710 642	11.961 051 51	.208	.007 226 281	9.685 267 54
.169	.004 766 969	11.890 947 66	.209	.007 296 083	9.639 602 33
0.170	0.004 823 631	11.821 672 61	0.210	0.007 366 224	9.594 375 13
.171	.004 880 630	11.753 211 83	.211	.007 436 701	9.549 580 21
.172	.004 937 965	11.685 551 17	.212	.007 507 517	9.505 211 01
.173	.004 995 636	11.618 676 46	.213	.007 578 624	9.461 261 78
.174	.005 053 644	11.552 574 84	.214	.007 650 163	9.417 726 45
.175	0.005 111 989	11.487 232 14	0.215	0.007 721 992	9.374 599 26
.176	.005 170 669	11.422 636 01	.216	.007 794 160	9.331 874 78
.177	.005 229 685	11.358 773 92	.217	.007 866 665	9.289 547 25
.178	.005 289 039	11.295 632 88	.218	.007 939 509	9.247 611 45
.179	.005 348 728	11.233 201 47	.219	.008 012 691	9.206 061 63
0.180	0.005 408 755	11.171 467 26	0.220	0.008 086 210	9.164 892 72
.181	.005 469 118	11.110 419 34	.221	.008 160 068	9.124 099 61
.182	.005 529 817	11.050 045 84	.222	.008 234 264	9.083 677 21
.183	.005 590 854	10.990 336 01	.223	.008 308 799	9.043 620 57
.184	.005 652 226	10.931 279 12	.224	.008 383 672	9.003 924 62
0.185	0.005 713 936	10.872 864 21	0.225	0.008 458 883	8.964 584 77
.186	.005 775 982	10.815 081 35	.226	.008 534 433	8.925 595 93
.187	.005 838 365	10.757 920 12	.227	.008 610 321	8.886 953 84
.188	.005 901 086	10.701 370 56	.228	.008 686 547	8.848 653 79
.189	.005 964 142	10.645 423 08	.229	.008 763 112	8.810 691 27
0.190	0.006 027 536	10.590 068 17	0.230	0.008 840 017	8.773 062 03
.191	.006 091 267	10.535 296 55	.231	.008 917 258	8.735 761 44
.192	.006 155 334	10.481 099 15	.232	.008 994 840	8.698 785 70
.193	.006 219 739	10.427 466 79	.233	.009 072 759	8.662 130 27
.194	.006 284 481	10.374 391 01	.234	.009 151 018	8.625 791 07
0.195	0.006 349 560	10.321 863 31	0.235	0.009 229 615	8.589 764 25
.196	.006 414 976	10.269 879 80	.236	.009 308 551	8.554 045 69
.197	.006 480 729	10.218 418 04	.237	.009 387 826	8.518 631 68
.198	.006 546 820	10.167 484 28	.238	.009 467 441	8.483 518 05
.199	.006 613 248	10.117 065 88	.239	.009 547 394	8.448 701 22
0.200	0.006 680 013	10.067 155 40	0.240	0.009 627 685	8.414 177 61

$\frac{x}{c}$	$\frac{\text{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\text{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\text{Sinh}\frac{x}{c}}$	$\frac{x}{c}$	$\frac{\text{Sinh}\frac{x}{c}}{\frac{x}{c}} - 1$	$\frac{\text{Cosh}\frac{x}{c}}{\frac{x}{c}} + \frac{1}{\text{Sinh}\frac{x}{c}}$
0.240	0.009 627 685	8.414 177 61	0.260	0.011 304 809	7.780 047 680
.241	.009 708 317	8.379 943 39	.261	.011 392 233	7.750 920 996
.242	.009 789 287	8.345 995 04	.262	.011 479 997	7.722 019 390
.243	.009 870 597	8.312 329 01	.263	.011 568 102	7.693 340 291
.244	.009 952 246	8.278 941 83	.264	.011 656 546	7.664 881 176
0.245	0.010 034 234	8.245 830 108	0.265	0.011 745 331	7.636 639 556
.246	.010 116 562	8.212 990 475	.266	.011 834 457	7.608 612 977
.247	.010 199 299	8.180 419 627	.267	.011 923 922	7.580 799 028
.248	.010 282 236	8.148 114 315	.268	.012 013 729	7.553 195 327
.249	.010 365 581	8.116 071 343	.269	.012 103 876	7.525 799 531
0.250	0.010 449 267	8.084 287 562	0.270	0.012 194 363	7.498 609 332
.251	.010 533 292	8.052 759 877	.271	.012 285 191	7.471 622 456
.252	.010 617 657	8 021 485 240	.272	.012 376 360	7.444 836 662
.253	.010 702 361	7.990 460 652	.273	.012 467 870	7.418 249 741
.254	.010 787 406	7.959 683 163	.274	.012 559 720	7.391 859 517
0.255	0.010 872 790	7.929 149 863	0.275	0.012 651 912	7.365 663 847
.256	.010 958 514	7.898 857 896	.276	.012 744 444	7.339 660 616
.257	.011 044 578	7.868 804 445	.277	.012 837 317	7.313 847 742
.258	.011 130 981	7.838 986 738	.278	.012 930 531	7.288 223 172
.259	.011 217 725	7.809 402 045	.279	.013 024 087	7.262 784 882
0.260	0.011 304 809	7.780 047 680	0.280	0.013 117 983	7.237 530 880

APPENDIX D

Derived Constants—Type *A* Tower Construction

Line Constants a_1

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	0.99469	0.97883	0.95250	0.91605	0.86964	0.81375	0.67545	0.50593
	300,000	0.99468	0.97884	0.95256	0.91612	0.86981	0.81411	0.67712	0.51486
	400,000	0.99471	0.97884	0.95258	0.91619	0.87000	0.81449	0.67769	0.51152
	500,000	0.99470	0.97884	0.95260	0.91622	0.87008	0.81464	0.67821	0.51243
	750,000	0.99470	0.97885	0.95262	0.91625	0.87017	0.81482	0.67872	0.51362
	1,000,000	0.99470	0.97885	0.95260	0.91627	0.87019	0.81488	0.67892	0.51406
	1,250,000	0.99470	0.97884	0.95262	0.91626	0.87019	0.81489	0.67897	0.51418
	1,500,000	0.99470	0.97885	0.95262	0.91627	0.87022	0.81491	0.67904	0.51435
	1,750,000	0.99470	0.97885	0.95262	0.91627	0.87021	0.81492	0.67906	0.51442
	2,000,000	0.99469	0.97885	0.95261	0.91627	0.87022	0.81492	0.67906	0.51444
Copper	250,000	0.99470	0.97883	0.95257	0.91616	0.86990	0.81446	0.67768	0.51126
	300,000	0.99470	0.97883	0.95257	0.91616	0.87000	0.81453	0.67802	0.51210
	400,000	0.99470	0.97884	0.95258	0.91618	0.87012	0.81470	0.67843	0.51324
	500,000	0.99470	0.97883	0.95258	0.91620	0.87018	0.81472	0.67867	0.51351
	750,000	0.99470	0.97884	0.95260	0.91626	0.87019	0.81488	0.67896	0.51420
	1,000,000	0.99470	0.97885	0.95263	0.91628	0.87022	0.81492	0.67904	0.51437
	1,250,000	0.99470	0.97885	0.95263	0.91628	0.87027	0.81495	0.67912	0.51450
	1,500,000	0.99470	0.97886	0.95265	0.91631	0.87029	0.81503	0.67926	0.51475
	1,750,000	0.99470	0.97886	0.95263	0.91630	0.87026	0.81500	0.67920	0.51467
	2,000,000	0.99470	0.97886	0.95262	0.91628	0.87024	0.81497	0.67916	0.51460
Steel	250,000	0.99155	0.96458	0.91948	0.85397	0.76620	0.65381	0.34567	−0.08469
	300,000	0.99128	0.96497	0.92055	0.85806	0.77288	0.66881	0.39021	+0.01476
	400,000	0.99130	0.96514	0.92146	0.85621	0.78083	0.68360	0.43526	0.11725
	500,000	0.99128	0.96555	0.92183	0.86136	0.78405	0.69027	0.45600	0.16402
	750,000	0.99130	0.96560	0.92231	0.86283	0.78781	0.69725	0.47640	0.21179
	1,000,000	0.99131	0.96534	0.92246	0.86333	0.78871	0.69963	0.48394	0.22818
	1,250,000	0.99131	0.96535	0.92250	0.86354	0.78920	0.70182	0.48666	0.23496
	1,500,000	0.99131	0.96535	0.92255	0.86359	0.78940	0.70184	0.48861	0.23904
	1,750,000	0.99131	0.96540	0.92269	0.86381	0.78978	0.70186	0.48971	0.24181
	2,000,000	0.99131	0.96537	0.92264	0.86378	0.78976	0.70193	0.49048	0.24300

Derived Constants—Type *A* Tower Construction

Line Constants a_2

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250 000	0 0024019	0 0095584	0 021313	0 037422	0 05753	0 081213	0 13711	0 20012
	300 000	0 0020064	0 0079831	0 017802	0 031258	0 048056	0 067831	0 11454	0 16651
	400 000	0 0015050	0 0059875	0 013353	0 023446	0 036046	0 050886	0 085929	0 12542
	500 000	0 0012032	0 0047871	0 010678	0 018747	0 028819	0 040684	0 068698	0 10030
	750 000	0 0002025	0 0031929	0 0071201	0 012504	0 019221	0 027135	0 045820	0 06690
	1 000 000	0 0006018	0 0023945	0 0053399	0 0093746	0 014412	0 020349	0 034364	0 05017
	1 250 000	0 0004811	0 0019140	0 0042683	0 0074940	0 011521	0 016263	0 027467	0 04011
	1 500 000	0 0004008	0 0015944	0 0035556	0 0062426	0 0095976	0 013548	0 022888	0 03340
	1 750 000	0 0003435	0 0013668	0 0030483	0 0053513	0 0082272	0 011614	0 019612	0 02864
	2 000 000	0 0003006	0 0011958	0 0026668	0 0046818	0 0051283	0 010161	0 017159	0 02505
Copper	250 000	0 0014647	0 0058212	0 012984	0 022795	0 035045	0 049472	0 083531	0 12195
	300 000	0 0012240	0 0048691	0 010861	0 019068	0 029311	0 041379	0 069871	0 10201
	400 000	0 0009178	0 0036513	0 008143	0 014298	0 021979	0 031027	0 052398	0 07650
	500 000	0 0007341	0 0029210	0 006514	0 011432	0 017583	0 024816	0 041910	0 06119
	750 000	0 0004872	0 0019378	0 004324	0 0075902	0 011669	0 016471	0 027820	0 04062
	1 000 000	0 0003667	0 0014588	0 003254	0 0057110	0 0087798	0 012397	0 020929	0 03057
	1 250 000	0 0002922	0 0011660	0 002600	0 0045635	0 0070177	0 0098951	0 016732	0 02443
	1 500 000	0 0002437	0 0009694	0 0021617	0 0037958	0 0058354	0 0082380	0 013911	0 02031
	1 750 000	0 0002088	0 0008307	0 0018525	0 0032525	0 0050012	0 0070590	0 011921	0 01741
	2 000 000	0 0001820	0 0007258	0 0016183	0 0028417	0 0043702	0 0061667	0 010416	0 01520
Steel	250 000	0 015503	0 061162	0 13632	0 23728	0 36057	0 50136	0 81056	1 10510
	300 000	0 012950	0 051328	0 11381	0 19813	0 30117	0 41904	0 67934	0 93199
	400 000	0 0097067	0 038488	0 085340	0 13825	0 22595	0 31454	0 51120	0 70572
	500 000	0 0077673	0 030800	0 068264	0 11890	0 18087	0 25180	0 40962	0 56713
	750 000	0 0051754	0 020527	0 045526	0 079219	0 12059	0 16794	0 27354	0 37976
	1 000 000	0 0038816	0 015401	0 034157	0 059480	0 090461	0 12603	0 20536	0 28532
	1 250 000	0 0031049	0 012309	0 027290	0 047525	0 072302	0 10068	0 16419	0 22828
	1 500 000	0 0025884	0 010261	0 022749	0 039623	0 060273	0 083942	0 13682	0 19036
	1 750 000	0 0022119	0 008771	0 019447	0 033869	0 051523	0 071768	0 11707	0 16275
	2 000 000	0 0019398	0 007688	0 017046	0 029690	0 045173	0 062919	0 10259	0 14271

DERIVED CONSTANTS—TYPE *A* TOWER CONSTRUCTION

Line Constants b_1

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250 000	18 598	36 804	54 265	70 476	85 688	98 104	117 08	125 36
	300 000	15 535	30 740	45 112	58 874	71 185	81 946	97 857	104 47
	400 000	11 653	23 059	33 974	44 164	53 401	61 479	73 404	78 685
	500 000	9 3166	18 435	27 164	35 310	42 694	49 153	58 691	62 921
	750 000	6 2137	12 295	18 118	23 550	28 474	32 784	39 150	41 981
	1 000 000	4 6598	9 2206	13 686	17 659	21 353	24 576	29 361	31 487
	1 250 000	3 7247	7 3702	10 859	14 115	17 068	19 650	23 468	25 168
	1 500 000	3 1029	6 1398	9 0460	11 759	14 219	16 370	19 554	20 966
	1 750 000	2 6598	5 2631	7 7547	10 080	12 188	14 034	16 758	17 973
	2 000 000	2 3270	4 6024	6 7842	8 8189	10 664	12 277	14 662	15 725
Copper	250 000	11 331	22 416	33 029	42 932	51 911	59 763	71 352	76 478
	300 000	9 4710	18 740	27 614	35 893	43 398	49 962	59 654	63 935
	400 000	7 1045	14 058	20 712	26 926	32 544	37 481	44 755	47 989
	500 000	5 6786	11 231	16 546	21 505	26 007	30 012	35 707	38 337
	750 000	8 7706	7 4597	10 994	14 290	17 279	19 891	23 758	25 477
	1 000 000	2 8372	6 0053	8 2710	10 751	12 999	14 968	17 872	19 170
	1 250 000	2 2657	4 4899	6 6156	8 6178	10 398	11 971	14 298	15 335
	1 500 000	1 8909	3 7417	5 5130	7 1671	8 6665	9 9786	11 919	12 788
	1 750 000	1 6163	3 1983	4 7122	6 1255	7 4074	8 5278	10 185	10 924
	2 000 000	1 4079	2 7887	4 1084	5 3407	6 4588	7 4342	8 8785	9 5191
Steel	250 000	119 39	235 04	343 02	437 73	515 62	572 53	607 10	571 75
	300 000	100 15	196 76	286 44	365 89	431 06	479 84	514 16	447 46
	400 000	75 081	147 54	214 81	264 89	323 89	361 11	391 14	351 17
	500 000	60 056	118 04	171 80	219 53	259 26	289 29	314 65	286 43
	750 000	40 028	78 685	114 57	146 35	173 01	193 21	211 09	194 95
	1 000 000	30 033	59 041	85 979	109 88	129 35	145 09	158 81	147 30
	1 250 000	24 011	47 193	68 702	87 801	103 77	116 02	126 99	118 00
	1 500 000	20 011	39 322	57 254	73 177	86 481	96 639	105 88	98 487
	1 750 000	17 107	33 620	48 956	62 569	73 952	82 653	90 588	84 319
	2 000 000	15 725	30 617	43 880	54 748	62 538	66 640	61 966	38 278

Derived Constants—Type *A* Tower Construction
Line Constants b_2

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	41.110	81.870	121.93	160.97	198.64	234.61	300.19	355.30
	300,000	41.105	81.829	121.81	160.68	198.09	233.68	298.09	349.87
	400,000	41.098	81.797	121.69	160.41	197.54	232.76	295.99	347.51
	500,000	41.095	81.776	121.64	160.27	197.29	232.32	295.01	345.79
	750,000	41.094	81.760	121.63	160.15	197.05	231.90	294.06	344.02
	1,000,000	41.093	81.754	121.56	160.09	196.95	231.74	293.71	343.39
	1,250,000	41.094	81.756	121.56	160.08	196.92	231.69	293.56	343.12
	1,500,000	41.095	81.754	121.55	160.06	196.89	231.65	293.53	342.96
	1,750,000	41.093	81.752	121.55	160.06	196.88	231.62	293.43	342.86
	2,000,000	41.094	81.754	121.55	160.06	196.87	231.61	293.39	342.80
Copper	250,000	41.104	81.806	121.71	160.41	197.54	233.29	295.80	347.36
	300,000	41.110	81.785	121.65	160.29	197.33	232.36	295.08	345.78
	400,000	41.094	81.765	121.60	160.17	197.10	232.00	294.28	343.33
	500,000	41.083	81.676	121.46	160.07	196.93	231.61	293.62	343.41
	750,000	41.086	81.750	121.54	160.05	196.91	231.65	293.52	343.05
	1,000,000	41.098	81.741	121.56	160.06	196.89	231.65	293.45	342.90
	1,250,000	41.081	81.724	121.51	160.00	196.81	231.53	293.39	342.78
	1,500,000	41.078	81.715	121.49	160.00	196.77	231.49	293.24	342.61
	1,750,000	41.094	81.731	121.55	160.00	196.81	231.53	293.27	342.63
	2,000,000	41.077	81.722	121.50	160.00	196.79	231.49	293.21	342.56
Steel	250,000	67.870	138.67	213.73	295.94	386.56	486.50	713.31	963.92
	300,000	67.723	136.79	208.57	284.11	364.44	449.26	632.30	823.59
	400,000	67.534	135.37	203.68	275.11	342.17	412.04	550.51	680.98
	500,000	67.487	134.74	201.48	267.38	332.02	394.92	512.68	614.41
	750,000	67.344	133.97	199.10	261.90	321.64	377.66	475.17	548.28
	1,000,000	67.318	133.75	198.30	260.04	315.88	371.70	461.96	525.22
	1,250,000	67.350	133.62	197.87	259.15	316.47	368.89	456.05	514.60
	1,500,000	67.353	133.63	197.78	258.81	315.70	367.54	452.75	508.90
	1,750,000	67.327	133.61	197.52	258.37	314.99	366.43	450.70	505.22
	2,000,000	67.418	133.65	197.61	258.26	314.51	365.38	447.68	499.50

Derived Constants—Type A Tower Construction

Line Constants c_1 = Values in table $\times 10^{-5}$

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	−0.0217	−0.1640	−0.529	−1.306	−2.518	−4.298	−9.886	−18.565
	300,000	−0.0178	−0.1378	−0.462	−1.086	−2.203	−3.591	−8.238	−15.064
	400,000	−0.0129	−0.1043	−0.347	−0.816	−1.578	−2.694	−6.194	−11.616
	500,000	−0.0104	−0.0828	−0.276	−0.651	−1.262	−2.152	−4.954	−9.302
	750,000	−0.0070	−0.0551	−0.1875	−0.4336	−0.8410	−1.4357	−3.304	−6.2034
	1,000,000	−0.0053	−0.0414	−0.1388	−0.3267	−0.6319	−1.0766	−2.4768	−4.6529
	1,250,000	−0.0043	−0.0329	−0.1110	−0.2608	−0.5406	−0.8619	−1.9799	−3.7192
	1,500,000	−0.0035	−0.0275	−0.0923	−0.2174	−0.4202	−0.7175	−1.6497	−3.0990
	1,750,000	−0.0029	−0.0236	−0.0790	−0.1861	−0.3600	−0.6152	−1.4144	−2.6563
	2,000,000	−0.0026	−0.0221	−0.0692	−0.1631	−0.3150	−0.5381	−1.2375	−2.3241
Copper	250,000	−0.0113	−0.101	−0.336	−0.792	−1.534	−2.620	−6.02	−11.30
	300,000	−0.0104	−0.084	−0.279	−0.661	−1.283	−2.188	−5.04	−9.45
	400,000	−0.0079	−0.063	−0.212	−0.496	−0.963	−1.643	−3.78	−7.09
	500,000	−0.0059	−0.050	−0.169	−0.400	−0.771	−1.320	−3.05	−5.68
	750,000	−0.0042	−0.035	−0.111	−0.264	−0.512	−0.873	−2.00	−3.77
	1,000,000	−0.0032	−0.022	−0.0847	−0.209	−0.386	−0.656	−1.51	−2.83
	1,250,000	−0.0032	−0.020	−0.0676	−0.160	−0.308	−0.525	−1.21	−2.27
	1,500,000	−0.0030	−0.0175	−0.0645	−0.143	−0.269	−0.452	−1.02	−1.91
	1,750,000	−0.0030	−0.0144	−0.0482	−0.123	−0.219	−0.374	−0.860	−1.61
	2,000,000	−0.0024	−0.0125	−0.0421	−0.122	−0.191	−0.327	−0.751	−1.41
Steel	250,000	−0.129	−1.176	−3.565	−8.327	−15.989	−27.078	−60.852	−103.130
	300,000	−0.114	−0.885	−2.955	−6.900	−13.379	−22.590	−50.864	−92.544
	400,000	−0.0828	−0.665	−2.223	−7.286	−10.005	−16.950	−38.168	−69.622
	500,000	−0.0655	−0.525	−1.785	−4.165	−8.003	−13.568	−30.534	−55.826
	750,000	−0.0429	−0.353	−1.190	−2.792	−5.323	−9.048	−23.579	−41.549
	1,000,000	−0.0331	−0.267	−0.890	−2.081	−3.840	−6.784	−15.276	−27.953
	1,250,000	−0.0289	−0.2077	−0.709	−1.666	−3.203	−5.402	−12.246	−22.386
	1,500,000	−0.0222	−0.1780	−0.595	−1.390	−2.670	−4.529	−10.196	−18.653
	1,750,000	−0.0203	−0.1486	−0.5020	−1.183	−2.277	−3.860	−8.719	−15.955
	2,000,000	−0.0183	−0.1126	−0.4446	−1.006	−2.002	−3.391	−7.643	−13.997

DERIVED CONSTANTS—TYPE *A* TOWER CONSTRUCTION

Line Constants c_2 = Values in table $\times 10^{-5}$

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	25.741	51.208	76.135	100.24	123.28	145.01	183.55	214.13
	300,000	25.740	51.207	76.126	100.24	123.28	145.01	183.59	214.35
	400,000	25.738	51.206	76.129	100.24	123.29	145.03	183.64	214.37
	500,000	25.739	51.205	76.131	100.24	123.29	145.03	183.67	214.48
	750,000	25.740	51.205	76.130	100.24	123.30	145.04	183.69	214.55
	1,000,000	25.740	51.205	76.130	100.24	123.30	145.04	183.70	214.57
	1,250,000	25.740	51.206	76.129	100.24	123.30	145.04	183.70	214.58
	1,500,000	25.740	51.206	76.128	100.24	123.30	145.04	183.70	214.58
	1,750,000	25.740	51.206	76.130	100.24	123.30	145.04	183.70	214.58
	2,000,000	25.741	51.205	76.130	100.25	123.30	145.05	183.71	214.60
Copper	250,000	25.740	51.226	76.129	100.24	123.29	145.03	183.65	214.42
	300,000	25.742	51.227	76.130	100.29	123.35	145.09	183.73	214.48
	400,000	25.740	51.226	76.130	100.24	123.39	145.03	183.68	214.52
	500,000	25.745	51.224	76.130	100.29	123.38	145.05	183.74	214.64
	750,000	25.747	51.226	76.130	100.28	123.35	145.08	183.75	214.65
	1,000,000	25.745	51.222	76.130	100.26	123.32	145.06	183.73	214.61
	1,250,000	25.742	51.220	76.130	100.25	123.32	145.06	183.72	214.60
	1,500,000	25.736	51.224	76.130	100.24	123.29	145.04	183.70	214.60
	1,750,000	25.738	51.220	76.130	100.24	123.29	145.04	183.69	214.60
	2,000,000	25.739	51.223	76.130	100.28	123.30	145.09	183.70	214.67
Steel	250,000	25.704	50.889	75.300	98.203	119.13	137.50	163.77	171.10
	300,000	25.703	50.939	75 302	98.244	119.40	137.91	165.62	176.47
	400,000	25.700	50.954	75.306	98.350	119.49	138.40	167.55	182.08
	500,000	25.703	50.984	75.345	98.379	119.64	138.68	168.47	184.70
	750,000	25.702	50.972	75.353	98.389	119.70	138.88	169.35	186.24
	1,000,000	25.702	50.974	75.355	98.407	119.70	138.96	169.63	187.13
	1,250,000	25.710	50.974	75.352	98.414	119.72	138.94	169.78	188.52
	1,500,000	25.717	50.984	75.361	98.429	119.77	139.02	169.87	188.78
	1,750,000	25.715	50.962	75.329	98.428	119.77	139.00	169.95	188.94
	2,000,000	25.719	50.973	75.340	98.416	119.77	139.02	169.95	189.02

Derived Constants—Type *B* Tower Construction
Line Constants a_1

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	0 99469	0 97878	0 95245	0 91588	0 86936	0 81331	0 67455	0 50428
	300,000	0 99469	0 97880	0 95248	0 91595	0 86955	0 81372	0 67577	0 50715
	400 000	0 99469	0 97880	0 95249	0 91602	0 86974	0 81410	0 67700	0 51028
	500 000	0 99469	0 97880	0 95251	0 91610	0 86983	0 81429	0 67757	0 51139
	750,000	0 99469	0 97880	0 95252	0 91610	0 86991	0 81447	0 67813	0 51276
	1,000,000	0 99469	0 97881	0 95253	0 91610	0 86995	0 81453	0 67832	0 51321
	1,250 000	0 99469	0 97881	0 95253	0 91611	0 86998	0 81457	0 67843	0 51346
	1,500 000	0 99469	0 97881	0 95253	0 91612	0 86998	0 81458	0 67848	0 51353
	1,750 000	0 99469	0 97881	0 95253	0 91612	0 86998	0 81459	0 67850	0 51358
	2,000 000	0 99468	0 97881	0 95253	0 91611	0 86998	0 81461	0 67852	0 51368
Copper	250,000	0 99469	0 97879	0 952471	0 91598	0 86968	0 81401	0 67689	0 50995
	300 000	0 99469	0 97880	0 952508	0 91606	0 86983	0 81427	0 67751	0 51129
	400,000	0 99469	0 97880	0 952511	0 91608	0 86990	0 81443	0 67799	0 51238
	500,000	0 99469	0 97881	0 952512	0 91610	0 86993	0 81448	0 67818	0 51285
	750,000	0 99468	0 97881	0 952518	0 91611	0 86995	0 81454	0 67837	0 51332
	1,000,000	0 99468	0 97881	0 952522	0 91611	0 86996	0 81458	0 67847	0 51353
	1,250,000	0 99468	0 97881	0 952527	0 91612	0 86998	0 81459	0 67851	0 51367
	1,500,000	0 99468	0 97881	0 952530	0 91612	0 86998	0 81460	0 67853	0 51369
	1,750,000	0 99468	0 97881	0 952530	0 91612	0 86998	0 81460	0 67854	0 51370
	2,000,000	0 99468	0 97881	0 952532	0 91613	0 86998	0 81460	0 67855	0 51371
Steel	250,000	0 99106	0 96391	0 91742	0 84986	0 75901	0 64222	0 32014	0 01354
	300,000	0 99109	0 96413	0 91849	0 85323	0 76708	0 66700	0 36924	0 02342
	400,000	0 99109	0 96434	0 91958	0 85670	0 77521	0 67515	0 37740	0 07616
	500,000	0 99110	0 96443	0 92008	0 85818	0 77897	0 68280	0 44172	0 14145
	750,000	0 99110	0 96454	0 92057	0 85972	0 78267	0 69034	0 46436	0 19303
	1,000,000	0 99112	0 96458	0 92074	0 86028	0 78400	0 69300	0 47239	0 21125
	1,250,000	0 99111	0 96459	0 92083	0 86051	0 78458	0 69420	0 47603	0 21961
	1,500,000	0 99112	0 96460	0 92087	0 86066	0 78503	0 69488	0 47805	0 22420
	1,750,000	0 99112	0 96460	0 92091	0 86075	0 78513	0 69529	0 47925	0 22726
	2,000,000	0 99112	0 96461	0 92092	0 86080	0 78523	0 69554	0 47997	0 22866

DERIVED CONSTANTS—TYPE *B* TOWER CONSTRUCTION
Line Constants a_2

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250 000	0 0025273	0 010056	0 022425	0 039374	0 060526	0 085429	0 14422	0 21048
	300,000	0 0021107	0 0083980	0 018730	0 032883	0 050540	0 070866	0 12047	0 17583
	400 000	0 0015835	0 0063001	0 014052	0 024670	0 037925	0 053538	0 090390	0 13458
	500 000	0 0012661	0 0050366	0 011233	0 019724	0 030321	0 042795	0 072269	0 10549
	750,000	0 00084425	0 0033592	0 0074907	0 013152	0 020218	0 028544	0 048197	0 070363
	1,000 000	0 00063347	0 0025205	0 0056208	0 0098663	0 015166	0 021417	0 036159	0 052793
	1,250,000	0 00050608	0 0020136	0 0044891	0 0078815	0 012117	0 017109	0 028891	0 042177
	1,500 000	0 00042156	0 0016772	0 0037406	0 0065672	0 010095	0 014248	0 024070	0 035131
	1,750 000	0 00036112	0 0014368	0 0032040	0 0056250	0 0086477	0 012207	0 020612	0 030098
	2,000 000	0 00031610	0 0012577	0 0028046	0 0049248	0 0075712	0 010686	0 018046	0 026344
Copper	250,000	0 0015412	0 0061318	0 013677	0 0240085	0 036915	0 052107	0 097991	0 12840
	300,000	0 0012844	0 0051094	0 011396	0 0200073	0 030759	0 043419	0 073308	0 106935
	400,000	0 00096269	0 0038297	0 0085400	0 0149963	0 023056	0 032544	0 054951	0 080216
	500,000	0 00077180	0 0030710	0 0068480	0 0120220	0 018486	0 026091	0 044062	0 064324
	750,000	0 00051472	0 0020479	0 0045815	0 0080171	0 012325	0 017397	0 029380	0 042893
	1,000 000	0 00038598	0 0017408	0 0034250	0 0060128	0 009244	0 013048	0 022031	0 032171
	1,250,000	0 00030844	0 0012272	0 0027368	0 0048048	0 007387	0 0104271	0 017606	0 025702
	1,500,000	0 00025701	0 0010226	0 0023903	0 0040035	0 006155	0 0086884	0 014670	0 021418
	1,750 000	0 00021973	0 0008744	0 0019499	0 0034235	0 005263	0 0074296	0 012545	0 018315
	2,000,000	0 00019264	0 0007665	0 0017094	0 0030011	0 004614	0 0065127	0 010997	0 016055
Steel	250 000	0 016317	0 064686	0 14335	0 24940	0 37873	0 52606	0 84784	1 1498
	300,000	0 013626	0 054017	0 11972	0 20832	0 31645	0 43985	0 71097	0 97099
	400,000	0 010217	0 040507	0 089783	0 15625	0 23744	0 33024	0 53538	0 73626
	500,000	0 0081689	0 032341	0 071784	0 12493	0 18988	0 26418	0 42887	0 59165
	750,000	0 0054761	0 021597	0 047873	0 083330	0 12670	0 17629	0 28657	0 39657
	1,000,000	0 0040864	0 016204	0 035915	0 062516	0 095035	0 13228	0 21513	0 29804
	1,250,000	0 0032657	0 012949	0 028701	0 049960	0 075955	0 10572	0 17197	0 23836
	1,500,000	0 0027209	0 010788	0 023914	0 041625	0 063281	0 088082	0 14330	0 19868
	1,750,000	0 0023274	0 0092286	0 020452	0 035606	0 054128	0 075347	0 12257	0 17999
	2,000,000	0 0020402	0 0080867	0 017929	0 031209	0 047442	0 066043	0 10745	0 14902

DERIVED CONSTANTS—TYPE *B* TOWER CONSTRUCTION

Line Constants b_1

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	18 598	36 800	54 218	70 475	85 201	98 067	117 01	125 23
	300,000	15 534	30 738	45 288	58 866	71 163	81 922	97 767	104 69
	400,000	11 653	23 058	33 974	44 161	53 392	61 464	73 367	79 220
	500,000	9 3185	18 438	27 166	35 310	42 695	49 145	58 674	62 878
	750,000	6 2122	12 293	18 110	23 541	28 462	32 768	39 121	41 936
	1,000,000	4 6600	9 2211	13 585	17 657	21 348	24 580	29 346	31 460
	1,250,000	3 7244	7 3698	10 856	14 112	17 063	19 645	23 456	25 147
	1,500,000	3 1019	6 1378	9 0429	11 755	14 212	16 360	19 537	20 943
	1,750,000	2 6580	5 2595	7 7489	10 072	12 178	14 020	16 740	17 949
	2,000,000	2 3235	4 5975	6 7734	8 8050	10 646	11 255	14 632	15 685
Copper	250,000	11 3389	22 436	33 058	42 9659	51 9521	59 804	71 386	76 4706
	300,000	9 4499	18 6973	27 5491	35 8080	43 2962	49 841	59 497	63 7414
	400,000	7 0847	14 2500	20 6505	26 8448	32 4588	37 5534	44 610	47 8147
	500,000	5 6791	11 2381	16 5591	21 5201	26 0224	29 9546	35 766	38 3386
	750,000	3 7884	7 4951	11 0421	14 3529	17 3539	19 9777	23 854	25 5729
	1,000,000	2 8402	5 6203	8 2802	10 7627	13 0131	14 9908	17 886	19 1881
	1,250,000	2 2697	4 4912	6 6171	8 6010	10 3995	11 9719	14 294	15 3255
	1,500,000	1 8912	3 7422	5 6442	7 1666	8 6652	9 9753	11 911	12 7697
	1,750,000	1 6173	3 2006	4 7152	6 1294	7 4111	8 5318	10 187	10 9218
	2,000,000	1 4179	2 7600	4 1337	5 3732	6 4969	7 4793	8 930	9 5748
Steel	250,000	119 93	235 57	342 60	428 90	513 49	568 70	597 37	615 73
	300,000	100 15	196 73	286 15	364 93	429 62	476 90	506 82	451 79
	400,000	75 098	147 52	214 63	273 86	322 77	386 06	386 07	341 62
	500,000	60 040	117 86	171 61	219 01	258 27	287 65	310 89	271 18
	750,000	40 144	78 659	114 46	146 11	172 42	192 21	208 80	191 62
	1,000,000	30 038	59 015	85 872	109 63	129 38	144 30	157 04	144 13
	1,250,000	24 005	47 158	68 623	88 111	103 40	115 34	125 64	115 57
	1,500,000	19 999	39 289	57 175	73 012	86 162	96 927	104 93	96 479
	1,750,000	17 107	33 608	48 903	62 442	73 697	83 123	89 620	82 646
	2,000,000	14 996	29 456	42 869	55 634	64 600	72 076	78 576	72 473

Derived Constants—Type *B* Tower Construction

Line Constants b_2

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250,000	39.14	77.96	116.38	153.35	189.30	223.66	286.45	339.45
	300,000	39.14	77.93	116.01	153.06	188.70	222.70	284.27	335.42
	400,000	39.14	77.88	115.87	152.75	188.14	221.70	282.02	331.39
	500,000	39.14	77.87	115.84	152.64	187.90	221.26	281.05	329.48
	750,000	39.13	77.86	115.77	152.49	187.62	220.82	280.01	327.58
	1,000,000	39.13	77.86	115.73	152.44	187.51	220.63	279.62	326.89
	1,250,000	39.13	77.85	115.73	152.42	187.51	220.62	279.53	326.68
	1,500,000	39.13	77.84	115.73	152.40	187.46	220.54	279.39	326.46
	1,750,000	39.13	77.83	115.73	152.39	187.44	220.51	279.33	326.35
	2,000,000	39.12	77.82	115.71	152.36	187.40	220.47	279.26	326.23
Copper	250,000	39.135	77.890	115.91	152.69	188.13	221.67	281.90	331.05
	300,000	39.129	77.863	115.82	152.62	187.88	221.26	281.04	329.51
	400,000	39.128	77.858	115.81	152.52	187.69	220.93	280.26	328.05
	500,000	39.128	77.855	115.80	152.48	187.60	220.77	279.88	327.40
	750,000	39.128	77.850	115.75	152.43	187.50	220.61	279.52	326.67
	1,000,000	39.128	77.845	115.73	152.40	187.50	220.54	279.37	326.41
	1,250,000	39.127	77.834	115.73	152.38	187.43	220.50	279.29	326.33
	1,500,000	39.126	77.837	115.72	152.38	187.42	220.48	279.26	326.28
	1,750,000	39.127	77.839	115.72	152.38	187.42	220.48	279.24	326.18
	2,000,000	39.126	77.837	115.72	152.38	187.41	220.47	279.22	326.15
Steel	250,000	66.060	134.85	208.92	287.36	381.04	481.57	711.52	893.13
	300,000	65.857	133.28	203.68	276.50	357.79	442.67	626.86	810.49
	400,000	65.664	131.69	198.39	265.93	334.35	403.40	541.18	671.80
	500,000	65.569	130.98	195.93	260.23	323.48	385.18	501.41	580.69
	750,000	65.544	130.24	193.38	254.63	312.85	367.28	462.17	533.34
	1,000,000	65.442	129.97	192.66	252.64	309.03	360.97	448.38	509.13
	1,250,000	65.431	129.80	192.29	251.70	307.30	358.07	441.10	497.89
	1,500,000	65.423	129.79	192.07	251.24	306.39	356.48	438.53	491.81
	1,750,000	65.416	129.76	191.94	250.94	305.77	355.52	436.43	488.12
	2,000,000	65.422	129.73	191.86	250.76	305.42	354.91	435.11	485.78

DERIVED CONSTANTS—TYPE *B* TOWER CONSTRUCTION
Line Constants c_1 = Values in table $\times 10^{-5}$

	Circular mils	Length in miles: 50	100	150	200	250	300	400	500
Aluminum	250 000	−0 0232	−0 1800	−0 6100	−1 4357	− 2 7827	− 4 7557	−10 939	− 20 539
	300 000	−0 0195	−0 1526	−0 5104	−1 2026	− 2 3240	− 3 9750	− 9 1392	− 17.159
	400 000	−0 0142	−0 1150	−0 3813	−0 8974	− 1 7461	− 2 9804	− 6 8556	− 12.497
	500 000	−0 0120	−0 0918	−0 3077	−0 7231	− 1 3967	− 2 3835	− 5 4844	− 10.298
	750,000	−0 0076	−0 0609	−0 2044	−0 4807	− 0 9307	− 1 5885	− 3 6542	− 6.8572
	1,000,000	−0 0052	−0.0447	−0 1519	−0 3600	− 0 6979	− 1 1890	− 2 7405	− 5.1465
	1,250,000	−0 0046	−0.0367	−0 1235	−0 2901	− 0 5599	− 0 9534	− 2.1919	− 4.1153
	1,500,000	−0 0038	−0.0305	−0 1021	−0 2399	− 0 4650	− 0 7947	− 1 8241	− 3 4285
	1,750,000	−0 0035	−0.0265	−0 0883	−0 2070	− 0 3995	− 0 6816	− 1 5659	− 2 9383
	2,000,000	−0 0032	−0.0196	−0 0720	−0 1735	− 0 3403	− 0 5867	− 1.3572	− 2.5571
Copper	250,000	−0 01038	−0 11420	−0 3795	−0 8838	− 1.7050	− 2 9093	−10 3107	− 12 910
	300,000	−0 01179	−0 09345	−0 31089	−0 7324	− 1 4160	− 2 4184	− 5 5628	− 10 460
	400,000	−0 00892	−0 06977	−0 23445	−0 54836	− 1 0604	− 1 8126	− 4 1685	− 7 8283
	500,000	−0 00701	−0 05590	−0 18943	−0 44094	− 0 85072	− 1 4546	− 3 3416	− 6 2765
	750,000	−0 00504	−0 03716	−0 12502	−0 29365	− 0 56810	− 0 97018	− 2 2289	− 4 1859
	1,000,000	−0 00357	−0 02779	−0 09356	−0 22013	− 0 42590	− 0 72725	− 1 6725	− 3 1314
	1,250,000	−0 00273	−0.02238	−0 07490	−0 17585	− 0 32300	− 0 58120	− 1 3363	− 2 5087
	1,500,000	−0 00233	−0.01857	−0 06380	−0 15391	− 0 28343	− 0 48430	− 1 1133	− 2.0905
	1,750,000	−0 00214	−0 01592	−0 05286	−0 12527	− 0 24236	− 0 41383	− 0 9518	− 1.7873
	2,000,000	−0 00183	−0 01394	−0 04674	−0 10983	− 0 21247	− 0 36296	− 0 8344	− 1.5670
Steel	250,000	−0 148	−1 174	−3 932	−7 665	−17 676	−29 914	−67 117	−116.50
	300,000	−0 123	−0 982	−3 285	−6 687	−14 763	−24 994	−56 140	− 97.642
	400,000	−0 0930	−0 736	−2 462	−5 761	−11 073	−18 752	−42 169	− 74.50
	500,000	−0 0738	−0 612	−1 966	−4 607	− 8 855	−14 994	−33 746	− 61 60
	750,000	−0 0860	−0 393	−1 313	−3 074	− 5 912	−10 004	−22 519	− 41 104
	1,000,000	−0 0376	−0 294	−0 985	−2 307	− 4 431	− 7 504	−16 895	− 30 851
	1,250,000	−0 0302	−0 2354	−0 788	−2 037	− 3 540	− 5 999	−13 504	− 24 663
	1,500,000	−0 0250	−0 1965	−0 655	−1 535	− 2 953	− 4 998	−11 252	− 20 552
	1,750,000	−0 0210	−0 1671	−0 561	−1 312	− 2 522	− 4 274	− 9 624	− 17 571
	2,000,000	−0 0190	−0 1479	−0 491	−1 151	− 2 213	− 3 747	− 8 438	− 15 413

Derived Constants—Type *B* Tower Construction

Line Constants c_2 = Values in table $\times 10^{-5}$

	Circular mils	Length in miles							
		50	100	150	200	250	300	400	500
Aluminum	250 000	27 08	53 87	80 09	105 46	129 70	152 55	193 07	225 19
	300 000	27 08	53 87	80 09	105 46	129 70	152 55	193 11	225 33
	400 000	27 08	53 87	80 09	105 46	129 70	152 56	193 16	225 60
	500 000	27 08	53 87	80 09	105 46	129 70	152 56	193 20	225 62
	750,000	27 08	53 87	80 09	105 46	129 70	152 56	193 25	225 65
	1,000,000	27 08	53 88	80 10	105 46	129 71	152 57	193 25	225 67
	1,250,000	27 08	53 88	80 10	105 47	129 71	152 58	193 25	225 68
	1,500,000	27 08	53 88	80 10	105 47	129 72	152 59	193 26	225 70
	1 750,000	27 08	53 88	80 10	105 47	129 72	152 60	193 26	225 71
	2 000,000	27 08	53 88	80 11	105 48	129 73	152 61	193 27	225 72
Copper	250 000	27 093	53 897	80 143	105 51	129 77	152 64	193 25	225 57
	300 000	27 087	53 883	80 129	105 48	129 73	152 56	193 22	225 60
	400 000	27 083	53 873	80 106	105 46	129 71	152 58	193 22	225 62
	500 000	27 081	53 879	80 103	105 47	129 72	152 59	193 23	225 66
	750 000	27 082	53 879	80 102	105 47	129 72	152 59	193 24	225 69
	1 000 000	27 083	53 879	80 101	105 47	129 72	152 59	193 25	225 70
	1 250 000	27 084	53 881	80 104	105 47	129 73	152 60	193 26	225 71
	1 500 000	27 084	53 881	80 104	105 47	129 73	152 60	193 26	225 72
	1 750 000	27 080	53 883	80 105	105 47	129 71	152 60	193 26	225 72
	2 000 000	27 080	53 882	80 106	105 49	129 73	152 60	193 26	225 72
Steel	250 000	27 047	53 608	79 172	103 15	125 07	144 14	170 82	191 14
	300,000	27 049	53 613	79 194	103 27	125 30	144 69	173 03	192 75
	400,000	27 049	53 614	79 211	103 35	125 52	145 24	175 26	193 90
	500,000	27 051	53 614	79 222	103 39	125 63	145 50	176 30	194 36
	750,000	27 050	53 617	79 230	103 42	125 76	145 75	177 30	195 32
	1 000,000	27 047	53 616	79 226	103 42	125 76	145 83	177 66	196 35
	1 250,000	27 048	53 617	[illegible]	103 42	125 78	145 87	177 82	196 83
	1,500,000	27 050	53 620	79 2[illegible]3	103 44	125 81	145 89	177 92	197 10
	1,750,000	27 049	53 620	79 234	103 44	125 80	145 92	177 97	197 27
	2,000,000	27 052	53 617	79 232	103 44	125 80	145 92	178 01	197 36

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stant from the mid-point of the previous interval to the mid-point of the interval in question. In other words, this assumption is identical with the one used in Method II. No further assumption, however, is made in this case.

From equation (2) it follows that

$$\omega_n = \omega_{n-1} + \frac{\Delta t}{2M}(\Delta P_{n-1} + \Delta P_n) = \omega_{n-1} + \frac{k'}{2}(\Delta P_{n-1} + \Delta P_n)$$
$$= \omega_{n-1} + \Delta\omega_n \tag{17}$$

Using equation (3), it follows that

$$\delta_n = \delta_{n-1} + \Delta t\omega_{n-1} + \frac{\Delta t^2}{8M}(3\Delta P_{n-1} + \Delta P_n)$$
$$= \delta_{n-1} + \Delta t\omega_{n-1} + \frac{k''}{4}(3\Delta P_{n-1} + \Delta P_n)$$
$$= \delta_{n-1} + \Delta\delta_n \tag{18}$$

From these the increments in angular velocity and angle during the nth interval may be written

$$\Delta\omega_n = \frac{\Delta t}{2M}(\Delta P_{n-1} + \Delta P_n) = \frac{k'}{2}(\Delta P_{n-1} + \Delta P_n) \tag{19}$$

$$\Delta\delta_n = \Delta\delta_{n-1} + \frac{\Delta t^2}{8M}(\Delta P_{n-2} + 6\Delta P_{n-1} + \Delta P_n)$$
$$= \Delta\delta_{n-1} + \frac{k''}{4}(\Delta P_{n-2} + 6\Delta P_{n-1} + \Delta P_n) \tag{20}$$

The comments previously made in connection with equations (11) and (12) apply.

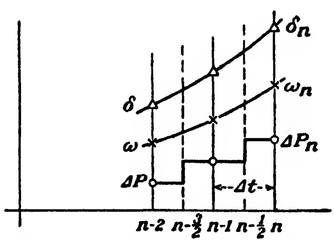


FIG. 173.—Point-by-point Method III.

At a discontinuity coincident with the beginning of the nth interval equations (17), (18), and (19) are not affected except that, of course, the power differential ΔP_{n-1} must be that existing *after* the discontinuity has occurred. In equation (20), on the other hand, the average value must be used, *i.e.*,

$$\Delta P_{n-1} = \tfrac{1}{2}[(\Delta P_{n-1})_F + (\Delta P_{n-1})_C]$$

assuming that the discontinuity is caused by the clearing of a

fault. For the first interval after a fault occurring at $t = 0$ evidently $\Delta P_{n-2} = 0$ and $\Delta P_{n-1} = \Delta P_0/2$.

Method IV.—This method assumes a linear variation of the power differential over the interval, but involves no further assumption. Graphically the situation is illustrated in Fig. 174. The power differential during the nth interval may be expressed as follows:

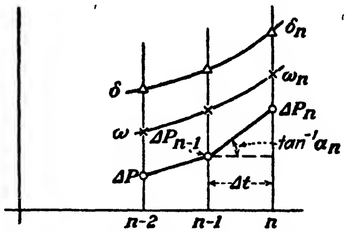


FIG. 174.—Point-by-point Method IV.

$$\Delta P_n = \Delta P_{n-1} + a_n t = \Delta P_{n-1} + \frac{\Delta P_n - \Delta P_{n-1}}{\Delta t} t \qquad (21)$$

Introducing this in equations (5) and (6) yields the following solutions:

$$\begin{aligned}
\omega_n &= \omega_{n-1} + \frac{\Delta t}{M}\Delta P_{n-1} + \frac{\Delta t}{2M}(\Delta P_n - \Delta P_{n-1}) \\
&= \omega_{n-1} + \frac{\Delta t}{2M}(\Delta P_{n-1} + \Delta P_n) \\
&= \omega_{n-1} + \frac{k'}{2}(\Delta P_{n-1} + \Delta P_n) = \omega_{n-1} + \Delta\omega_n \qquad (22)
\end{aligned}$$

$$\begin{aligned}
\delta_n &= \delta_{n-1} + \Delta t\omega_{n-1} + \frac{\Delta t^2}{2M}\Delta P_{n-1} + \frac{\Delta t^2}{6M}(\Delta P_n - \Delta P_{n-1}) \\
&= \delta_{n-1} + \Delta t\omega_{n-1} + \frac{\Delta t^2}{6M}(2\Delta P_{n-1} + \Delta P_n) \\
&= \delta_{n-1} + \Delta t\omega_{n-1} + \frac{k''}{3}(2\Delta P_{n-1} + \Delta P_n) = \delta_{n-1} + \Delta\delta_n \qquad (23)
\end{aligned}$$

From these equations the increments in angular velocity and angle during the nth interval become

$$\Delta\omega_n = \frac{\Delta t}{2M}(\Delta P_{n-1} + \Delta P_n) = \frac{k'}{2}(\Delta P_{n-1} + \Delta P_n) \qquad (24)$$

$$\begin{aligned}
\Delta\delta_n &= \Delta\delta_{n-1} + \frac{\Delta t^2}{6M}(\Delta P_{n-2} + 4\Delta P_{n-1} + \Delta P_n) \\
&= \Delta\delta_{n-1} + \frac{k''}{3}(\Delta P_{n-2} + 4\Delta P_{n-1} + \Delta P_n) \qquad (25)
\end{aligned}$$

It will be noted that the solution of any or all of the above equations necessitates the knowledge of the power differential

at the beginning as well as at the end of the interval. The latter, however, is not directly known, but there are three ways, in which this difficulty may be overcome:

1. A curve of ΔP may be plotted as the analysis proceeds, and extrapolated over the next interval.

2. The linear variation in ΔP may be based on the preceding interval. This merely involves the substitution in the above equations of ΔP_{n-1} for ΔP_n, and ΔP_{n-2} for ΔP_{n-1}.

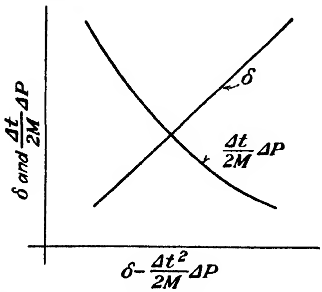


FIG 175.—Auxiliary curves for point-by-point Method IV.

3. A special scheme may be used involving the plotting and use of auxiliary curves, as described below.[1]

The procedures suggested in (1) and (2) are applicable in multi-machine systems. The procedure under (3), on the other hand, is limited to two-machine systems.

For the purpose of applying procedure 3, the terms in equations (22) and (23) are rearranged as follows:

$$\omega_n = \omega_{n-1} + \frac{\Delta t}{2M}\Delta P_{n-1} + \frac{\Delta t}{2M}\Delta P_n \tag{26}$$

$$\delta_n - \frac{\Delta t^2}{6M}\Delta P_n = \delta_{n-1} + \Delta t\omega_{n-1} + \frac{\Delta t^2}{3M}\Delta P_{n-1} \tag{27}$$

Since in the two-machine system, ΔP is determinable as a function of δ (being the difference between input and output), auxiliary curves of δ and $\Delta t\Delta P/(2M)$ may be plotted versus

$$\delta - \frac{\Delta t^2}{6M}\Delta P,$$

as suggested in Fig. 175. These auxiliary curves are then used in the solution of equations (26) and (27) as follows: The right-hand members of equation (27) are fixed by conditions at the beginning of the interval. Consequently the value of $\delta_n - \Delta t^2\Delta P/(6M)$ may be evaluated, the auxiliary curve entered, and the corre-

[1] Suggested (1935) by A. H. Howell, Graduate Student at the Massachusetts Institute of Technology.

sponding values of δ_n and $\Delta t \Delta P_n/(2M)$ selected. Knowing the latter equation (26) yields a solution for ω_n.

A schedule for this solution may be set up as suggested in Table 52. Details of computations based on this method will be found in Example 4.

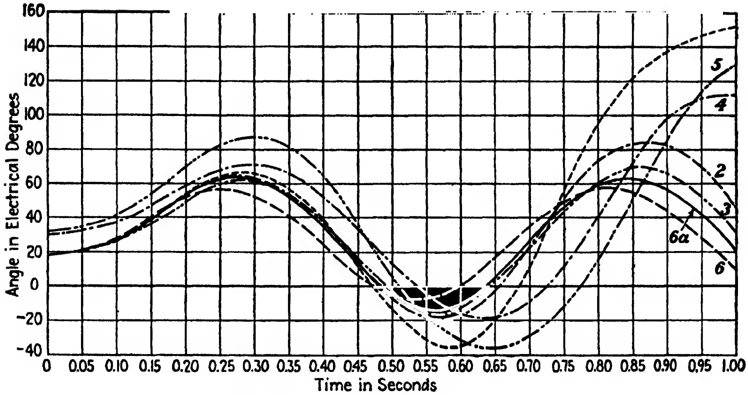
TABLE 52.—BASIC SCHEDULE FOR POINT-BY-POINT CALCULATION OF MACHINE SWING CURVES
(Method IV, 3)

(1) Int. no.	(2) t	(3) ΔP	(4) $\Delta t \cdot \omega$	(5) $\frac{\Delta t^2}{3M}\Delta P$	(6) $\delta - \frac{\Delta t^2}{6M}\Delta P$	(7) $\frac{\Delta t}{2M}\Delta P$	(8) ω	(9) δ
0	0							
1	Δt							
2	$2\Delta t$							
etc.	etc.							

Col. $(6)_n = (4)_{n-1} + (5)_{n-1} + (9)_{n-1}$
Col. $(7)_n$: From auxiliary curve, entering with $(6)_n$
δ_n: Col. $(9)_n$: From auxiliary curve entering with $(6)_n$
ω_n: Col. $(8)_n = (7)_{n-1} + (7)_n + (8)_{n-1}$
ΔP_n: Col. $(3)_n = \dfrac{(7)_n}{\Delta t/2M}$ (unnecessary)

Relative Merits of the Various Point-by-point Methods.—The accuracy of any point-by-point method depends upon the length of the time interval used. The shorter the time interval, the better the accuracy. On the other hand, a very short interval very materially increases the time and labor required for the analysis. It is desirable, therefore, to use a time interval as long as is commensurate with sufficient accuracy. As has previously been stated, 0.05 second is usually found suitable.

The errors introduced are cumulative and rapidly increase as time progresses, except with those methods where improved or additional assumptions tend to compensate for them. If computations are required over a relatively short time only (for instance, from the occurrence of a disturbance until it is cleared by the operation of automatic circuit breakers, in order to correlate switching time and switching angle so that the equal-area stability criterion may be applied), any one of the above-mentioned methods may yield sufficient accuracy. Inherently, however, Method II with its additional assumption and Method



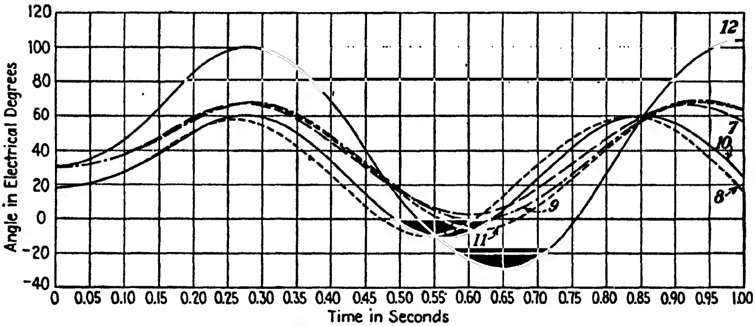


FIG. 176.—Angle-time curves for representative two-machine system when equilibrium is upset by the occurrence of a line-to-line fault which is cleared in 0.2 second. The curves illustrate the relative merits of the four methods of point-by-point analysis as well as the differences arising from the basic assumptions made. Time interval used is 0.05 second except where otherwise specifically stated. For details see Example 3.

(1) Point-by-point Method I. Assumption *A*.
(2) Point-by-point Method I. Assumption *A*. Time interval 0.02 second.
(3) Point-by-point Method I. Assumption *A*. Time interval 0.01 second.
(4) Point-by-point Method I. Assumption *B*.
(5) Point-by-point Method I. Assumption *A*; using actual rotor angle.
(6) Point-by-point Method II. Assumption *A*.
(7) Point-by-point Method II. Assumption *B*.
(8) Point-by-point Method III. Assumption *A*.
(9) Point-by-point Method III. Assumption *B*.
(10) Point-by-point Method IV. Assumption *A*.
(11) Point-by-point Method IV. Assumption *B*.
(12) Point-by-point Method IV. Assumption *A*; using actual rotor angle.

Assumption A considers the generator represented by its direct-axis transient reactance and the voltage behind this reactance constant and uses the angle to this voltage rather than the actual rotor angle (unless otherwise specifically stated).

Assumption B considers the direct-axis flux linkages (*i.e.*, the voltage e_d') constant.

IV with its improved representation of the variation of the power differential give better accuracy than Methods I and III and, in general therefore, are preferable. Method II excels by virtue of computational simplicity and is probably, therefore, the most advantageous one in usual problems. Method IV when properly executed is no doubt superior from the standpoint of accuracy, but it is questionable whether its additional complexity warrants its use except in special cases where a great degree of accuracy is aimed at.

In Example 3 will be found the solution of a representative two-machine stability problem by means of the four methods of point-by-point analysis outlined above. Comparison of the results will bear out the statement made with regard to their relative merits. Swing curves covering a time of one second, so as to include also a second swing apart of the machines, will be found in Fig. 176. It will be noted that in connection with Method I, shorter time intervals have also been used so that this effect may be observed. In Table 53 are listed the numerical quantities which are particularly pertinent in comparing the methods. It gives the initial angular displacement and also the angular displacement and the corresponding time at the first and second maxima and the first minimum of the oscillations.

Integraph Solution.—When, in general, not more than two machines are involved the swing equation can be solved by integraph methods. This can be done whether damping is included or omitted, provided the synchronous part as well as the damping part of the output can be represented graphically as functions of the angle and the rate of change of the angle with respect to time (*i.e.*, slip).

Simplified Criterion of Stability in Two-machine Systems.—When two machines oscillate with respect to each other as the result of the shock caused by a disturbance, the amplitudes of the oscillations will, when the system is stable, ordinarily decrease continually until new steady-state conditions prevail. In cases where the disturbance results in instability, loss of synchronism between the machines will usually occur during the first swing, *i.e.*, the angle between the machines will continually increase. If the system manages to swing by the first amplitude of the oscillation—the angle between the machines reaches its first maximum and begins to decrease—the remaining part of the

TABLE 53.—SUMMARY OF RESULTS FROM DIFFERENT METHODS OF POINT-BY-POINT ANALYSIS AND ASSUMPTIONS
(Based on Example 3)

Solution number	Method	Assumption	Time interval Δt sec.	Initial δ deg.	1st Maximum		2d Maximum		1st Minimum		Remarks
					δ deg.	t sec.	δ deg.	t sec.	δ deg.	t sec.	
1	I	*A*	0.05	18.2	66	0.28	152*	1.0	−36	0.59	
2	I	*A*	0.02	18.2	63	0.28	83.5	0.87	−19.5	0.57	
3	I	*A*	0.01	18.2	62.5	0.28	69	0.87	−17	0.56	
4	I	*B*	0.05	30	71	0.3	112.5†	1.0	−18.5	0.63	
5	I	*A*	0.05	32.5‡	87	0.3	130§	1.0	−36	0.65	Actual rotor angle
6	II	*A*	0.05	18.2	56	0.25	57.3	0.82	−8	0.53	
6*a*	II*a*	*A*	0.05	18.2	64	0.28	63.5	0.84	−13.5	0.56	Method II with modification at switching instant
7	II	*B*	0.05	30.5	67	0.28	66.5	0.92	3	0.59	
8	III	*A*	0.05	18.0	57.5	0.26	60	0.84	−10	0.54	
9	III	*B*	0.05	30.5	67.0	0.285	69	0.94	0	0.62	
10	IV	*A*	0.05	18.2	60.2	0.275	60.0	0.855	−10.8	0.565	
11	IV	*B*	0.05	30.5	67	0.28	68	0.95	−7	0.62	
12	IV	*A*	0.05	30.5	100	0.28	104†	1.0	−29.5	0.65	Actual rotor angle

* Value at 1.0 sec Analysis definitely indicates pull-out on second swing

† Value at 1.0 sec

‡ Slight inaccuracy in initial angle

§ Value at 1.0 sec Analysis would presumably indicate pull-out on second swing if continued beyond 1 sec

oscillations will usually be less severe and the system will be stable. This, however, may not *always* be true,[1] but the assumption is often, and correctly, made. Hence the fact as to whether or not the machines in a two-machine system *come to rest with respect to each other may be taken as the criterion of stability in a two-machine system.*[2]

FIG. 177.—General two-machine system.

The mathematical formulation of this theorem is readily established. Referring to the general two-machine system in Fig. 177, and considering input and output positive when the machines act as generators, the accelerations of the rotors are given by

$$M_1\frac{d^2\delta_1}{dt^2} = P_{1i} - P_1 = \Delta P_1 \tag{28}$$

$$M_2\frac{d^2\delta_2}{dt^2} = P_{2i} - P_2 = \Delta P_2 \tag{29}$$

The angle between the rotors is $\delta = \delta_1 - \delta_2$. Hence

$$\frac{d^2\delta}{dt^2} = \frac{d^2\delta_1}{dt^2} - \frac{d^2\delta_2}{dt^2} = \frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2} \tag{30}$$

which may also be written

$$d\left[\left(\frac{d\delta}{dt}\right)^2\right] = 2\left(\frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2}\right)d\delta \tag{31}$$

The angular velocity with which the machines swing apart is consequently

$$\frac{d\delta}{dt} = \sqrt{2\int_{\delta_0}^{\delta}\left(\frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2}\right)d\delta} \tag{32}$$

The machines will swing apart until this angular velocity becomes zero. This occurs when

$$\int_{\delta_0}^{\delta}\left(\frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2}\right)d\delta = 0 \tag{33}$$

which is also the desired stability criterion.

[1] Depending on such factors as time of switching, exciter action, damping, and power dissipation in the connecting network.

[2] See paper by Park and Bancker, *loc. cit.*

This criterion assumes that if the machines come to rest with respect to each other, the system is stable. Equation (19), as it stands, *imposes no restrictions on the input to or output of the machines.* The former may be constant or variable, and the latter include damping and flux variations. The power differentials ΔP_1 and ΔP_2 may be represented by functions with any arbitrary number of discontinuities. Hence the criterion inherently provides for the inclusion of switching, fault clearing, etc. As a matter of fact, its validity definitely depends upon all operations causing discontinuities taking place before the maximum angle is reached.[1]

Switching is always associated with time. It is known how long after the occurrence of a fault the faulty line section or feeder will be disconnected by the action of the automatic circuit breakers. In order that the discontinuity due to switching, however, can be introduced at the proper point in equation (33), the *switching angle* must be determined. This can only be obtained by determining a portion of the angle-time curve sufficient for the purpose of finding the angle corresponding to the time of switching. This part of the angle-time curve must usually be calculated by point-by-point solutions of equations (28) and (29), or of equation (30). The switching angle may also, under certain conditions, be obtained by use of precalculated swing curves as discussed later in this chapter.

If stability is to be present, it is evident that the relative acceleration (or retardation, as the case may be) of the machines with respect to each other must change sign during the first swing apart. In other words, the term $(\Delta P_1/M_1) - (\Delta P_2/M_2)$ must change sign. If it does not, it is evident that the integral in equation (33) can never be zero, the machines will continuously swing apart, and stability is definitely lost.

If one assumes that the acceleration changes sign, and also that the condition given by equation (33) is satisfied so that the system is stable, there are two possibilities in regard to the value of the acceleration $(\Delta P_1/M_1) - (\Delta P_2/M_2)$ at maximum angle, *viz.:*

[1] If this is not the case, however, it is possible to extend the interpretation of the criterion. Not only must the machines come to rest with respect to each other during their first swing apart, but also during subsequent swings together or apart until all discontinuities are included. Not until then may actual conclusions be drawn in regard to the stability situation.

1. The relative acceleration may have a finite value.
2. The relative acceleration may be zero, as indicated by

$$\left(\frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2}\right) = 0 \tag{34}$$

In case 1, above, the system is stable with a margin, *i.e.*, the loading is below the critical. Case 2 represents the limiting case. The loading is critical and equal to the transient power limit of the system for the disturbance in question.

There are two methods of applying the stability criterion, *viz.*:

1. Use equation (33) and examine whether it yields a solution for the maximum angle δ_m. If it does, the system is stable; if no solution exists, the system is unstable.

2. Find the critical angle δ'_m from equation (34). Evaluate and examine the sign of equation (33) for this maximum angle $\delta_m = \delta'_m$. If negative, the system is stable; if zero, critical load conditions exist; and if positive, the system is unstable.

Concentration of Inertia at One End of the System.—In a two-machine system the inertia may be concentrated at one end of the system[1] without changing the actual circuit and the actual machine inputs provided

1. The network connecting the machines contains reactance only

2. One machine operates as a generator and the other as a motor

The general expressions for the acceleration of the machines in a two-machine system are as given by equations (28) and (29)

Equation (30) applicable to the general two-machine system may be written

$$\frac{d^2\delta}{dt^2} = \frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2} = \frac{M_2(P_{1i} - P_1) - M_1(P_{2i} - P_2)}{M_1M_2} \tag{35}$$

where as before P_{1i} and P_{2i} designate the input to, and P_1 and P_2

[1] Extension of this theorem will be found in Chap XIV. The *general two-machine system* may also be represented by an *equivalent system* with the inertia concentrated in an *equivalent* machine at one end of an *equivalent circuit* and with an infinite bus at the other The input to the equivalent machine will differ from that of either machine and is a function of the actual inputs and the inertia constants Similarly, the equivalent circuit (including the equivalent machine) and the corresponding equivalent power-angle curve depend on the actual circuit and the inertia constants of the machines

the output of, the two machines, respectively, both being considered as generators. If the receiving-end machine is a motor and the circuits only contain reactance

$$\left.\begin{aligned} P_{1i} &= -P_{2i} = P_i \\ P_2 &= -P_1 = P \end{aligned}\right\} \tag{36}$$

Under these conditions equation (20) reduces to

$$\frac{d^2\delta}{dt^2} = \frac{M_1 + M_2}{M_1 M_2}(P_i - P) = \frac{1}{M_0}(P_i - P) \tag{37}$$

which is the equation for acceleration that would have been obtained for the system under consideration *were the inertia concentrated at one end with an infinite bus at the other.* The inertia constants of the two machines combine as if they were "in parallel," and the resultant inertia constant is given by

$$M_0 = \frac{M_1 M_2}{M_1 + M_2} \tag{38}$$

If the expression for the output, which obviously in this case is a simple undisplaced sinusoid, is introduced, the swing equation on which the solution may be based is

$$M_0 \frac{d^2\delta}{dt^2} = P_i - P_m \sin \delta \tag{39}$$

It is here assumed that each machine is represented by a single reactance and that the voltage behind this reactance is constant (for instance, the direct-axis transient reactance and voltage behind this reactance). Furthermore, it neglects damping.

The Equal-area Method.—The simplified stability criterion for two-machine systems expressed by equation (33) may be translated into an "equal-area" conception. Evidently

$$\int_{\delta_0}^{\delta_m} \Delta P d\delta = \int_{\delta_0}^{\delta_m} (P_i - P) d\delta \tag{40}$$

represents an area, *viz.*, the area between input and output curves, plotted versus angle and bounded by the initial and maximum angle. Hence equation (33) may be modified to

$$\frac{1}{M_1}\left(\begin{array}{l}\text{net area (1)} \\ \text{between } \delta_0 \text{ and } \delta_m\end{array}\right) = \frac{1}{M_2}\left(\begin{array}{l}\text{net area (2)} \\ \text{between } \delta_0 \text{ and } \delta_m\end{array}\right) \tag{41}$$

Hence the system is stable when the two areas thus defined and divided by the respective inertia constants are equal.

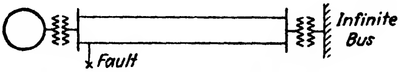


Fig. 178.—Generator supplying power to a large system (infinite bus) over a two-circuit transmission line with sending- and receiving-end transformers.

Explicit use of the equal-area method, however, as a criterion of stability in a two-machine system is convenient and practical only when

1. One end of the system is an infinite bus.
2. The inertia can be concentrated at one end so that an infinite bus is obtained at the other.

To gather a more comprehensive understanding of the significance of the equal-angular method its application to the case of a line fault will be discussed. Again consider the simple system in Fig. 178 consisting of a generating station feeding into an infinite bus over a double-circuit transmission line. The power-angle characteristic for this system based on reactance only is given in Fig. 179. The power represented is the output of the generating station for constant flux linkages in the machines, and the angles are the displacements between the voltage behind the transient reactance of the generators and the voltage of the infinite bus. Curve 1 is the power-angle curve with the two circuits of the line complete; curve 2, with one circuit disconnected; and curve 3 corresponds to a condition of fault as indicated in Fig. 178.

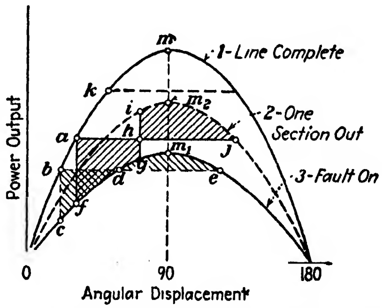


Fig. 179.—Power-angle curve for the system in Fig. 178.

Assume that point *a* is the initial operating point before the fault occurs. If the fault is not cleared, it is obvious that instability will result, since the power input (assumed constant) to the generators for any angular position would be larger than the output. If *b* represented the initial operating point, the system might be stable even if the fault were sustained. The criterion

of stability is that the area dm_1e must be larger than the area *bcd*. The latter in the absence of damping represents energy stored in the rotating parts of the generators and the former, energy given up by the rotating parts.

Assume once more that point *a* represents initial conditions. If a short circuit suddenly occurs, the output will drop to that given by point *f* and a power differential represented by *af* will be available for acceleration. Assume that when the machine has swung ahead to point *g* the short circuit is cleared. The output is then immediately increased to that corresponding to point *i*. This output is larger than the input and retarding action will take place. The criterion of stability for this short circuit and its subsequent clearing is that the area him_2j should be larger than the area *afgh*.

If the original operating point with both lines intact had been at *k*, a short circuit would of necessity result in instability, even if the faulty section were tripped out in a very short time. This must obviously be true since the initial power exceeds the maximum output of the generator with one circuit of the transmission line disconnected. It is obvious from the foregoing considerations and the diagram (Fig. 179) that the closer the power-angle curves for normal and fault conditions approach each other, the less severe will be the shock impressed on the system when short circuit occurs.

Application of the Equal-area Method to Some Simple Cases.—To illustrate the use of the equal-area principle in the solution of transient stability problems a few simple cases[1] will be handled by this method, *viz.*:

1. Tripping of a section of a multicircuit line as a normal switching operation.
2. Sustained fault or a fault with long-delayed switching.
3. Feeder fault with its subsequent clearing by disconnection of the feeder.
4. Line fault with its subsequent clearing by dropping of the faulty line section.

The system will in each case consist of a generator connected to an infinite bus through a circuit whose reactance only is considered. The input to and the voltage behind transient reactance in the generator will be assumed strictly constant.

[1] Park and Bancker, *loc. cit.*

It should be noted that the apparently explicit formal solutions for the transient power limits [equations (45), (48), and (51)] are *actually implicit* since the angles involved depend upon these same values of power. A quantitative answer, however, is readily obtained in a given case by a cut-and-try calculation.

1. *Tripping of a Section of a Multicircuit Line as a Normal Switching Operation.*—The system is shown in Fig. 180(*a*) and

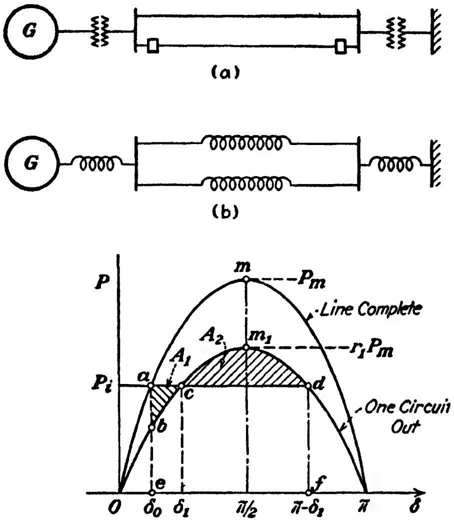


FIG. 180.—(*a*) Generator supplying a large system (infinite bus) over a two-circuit transmission line with sending- and receiving-end transformers. (*b*) The equivalent reactance circuit of (*a*). (*c*) Power-angle curves illustrating the effect of suddenly tripping out one of the two transmission circuits.

its reactance-circuit diagram in (*b*). The power-angle curves with the transmission line intact and with one section disconnected are indicated in Fig. 180(*c*). The notation employed is apparent from this figure.

The system is stable if the area A_1 is smaller than or (at the limit) equal to the area A_2. Instead of comparing these areas, however, it is more convenient to use the rectangle *eadf* and the irregular area $ebcm_1df$. The input to the generator is given by

$$P_i = P_m \sin \delta_0 = r_1 P_m \sin \delta_1 \qquad (42)$$

and the above-mentioned areas become

$$\text{Rectangle} = P_i(\pi - \delta_1 - \delta_0) \tag{43}$$

$$\text{Irregular area} = r_1 P_m \int_{\delta_0}^{\pi - \delta_1} \sin \delta \, d\delta = r_1 P_m(\cos \delta_0 + \cos \delta_1) \tag{44}$$

Equating these areas the transient power limit of the system for the given disturbance is

$$\begin{aligned} P(\max) = P_i &= r_1 P_m \frac{\cos \delta_0 + \cos \delta_1}{\pi - \delta_0 - \delta_1} \\ &= r_1 P_m \frac{\cos \delta_0 + \cos\left[\sin^{-1}\left(\dfrac{\sin \delta_0}{r_1}\right)\right]}{\pi - \delta_0 - \sin^{-1}\left(\dfrac{\sin \delta_0}{r_1}\right)} \end{aligned} \tag{45}$$

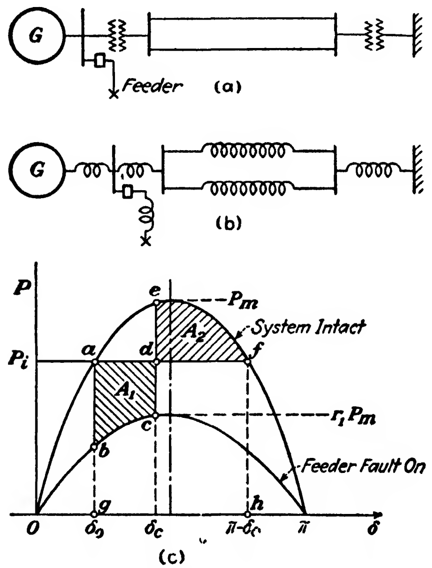


FIG. 181.—(a) Generator supplying a large system (infinite bus) over a two-circuit transmission line with sending- and receiving-end transformers. Fault on local feeder. (b) The equivalent circuit of (a) considering reactance only. (c) Power-angle curves illustrating the effect of the feeder fault and its subsequent clearing.

2. *Sustained Fault or Fault with Long-delayed Switching.*—This problem is evidently analogous to the preceding one when the second power-angle curve [Fig. 180(c), curve 2] is considered to represent the system with the fault applied rather than the

system with one line section disconnected. The transient power limit, therefore, is given by equation (45).

3. *Feeder Fault with Its Subsequent Clearing by Disconnection of the Feeder.*—The system with the faulted feeder is shown in Fig. 181(*a*), its reactance-circuit diagram in (*b*), and the power-angle curves for the normal and the faulted system in (*c*).

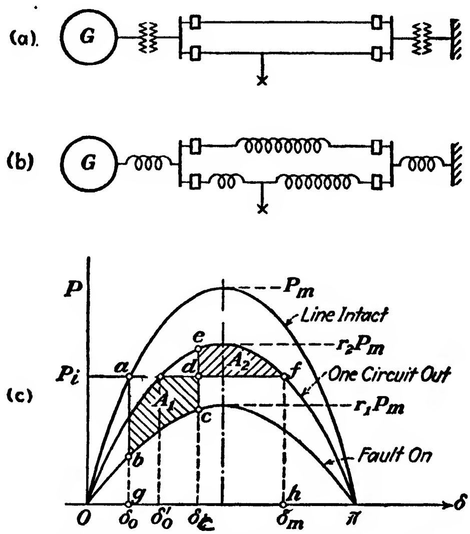


FIG. 182.—(*a*) Generator supplying a large system (infinite bus) over a two-circuit transmission line with sending- and receiving-end transformers. Fault on one of the transmission circuits. (*b*) The equivalent circuit of (*a*) considering reactance only. (*c*) Power-angle curves illustrating the effect of the fault and its subsequent clearing by disconnecting the faulty transmission circuit.

If the fault is cleared at an angle δ_c, it is obvious that for stability the area A_1 must be smaller than or—at the limit—equal to the area A_2. Instead of A_1 and A_2 the rectangle *gafh* and the irregular area *gbcdefh* will be compared. It will be assumed that the clearing angle has been calculated from the specified clearing time and hence is known.

The input to the generator (original output) is given by equation (42). The areas of interest are

$$\text{Rectangle} = P_i(\pi - 2\delta_0) \tag{46}$$

$$\begin{aligned}\text{Irregular area} &= r_1P_m\int_{\delta_0}^{\delta_c} \sin\delta\,d\delta + P_m\int_{\delta_c}^{\pi-\delta_0} \sin\delta\,d\delta \\ &= P_m[(1+r_1)\cos\delta_0 + (1-r_1)\cos\delta_c]\end{aligned} \tag{47}$$

Equating the above areas the transient power limit of the system for the disturbance caused by the feeder fault becomes

$$P(\max) = P_i = P_m \frac{(1 + r_1) \cos \delta_0 + (1 - r_1) \cos \delta_c}{\pi - 2\delta_0} \tag{48}$$

4. *Line Fault with Its Subsequent Clearing by Dropping of the Faulty Line Section.*—The system with its faulted line is shown in Fig. 182(*a*), its reactance-circuit diagram in (*b*), and the power-angle curves for the system with the transmission line intact, with the line faulted, and with the faulty section cleared in (*c*).

As in the previous case it will also here be assumed that the clearing angle δ_c has been calculated from the clearing time and hence is known. For stability the area A_1 must be smaller or—at the limit—equal to the area A_2. Instead of the areas A_1 and A_2 the rectangle *gafh* and the irregular area *gbcdefh* will be compared.

The input to the generator (original output) may be expressed by

$$P_i = P_m \sin \delta_0 = r_2 P_m \sin \delta_0' \tag{49}$$

from which the angle δ_0' may be determined so that the maximum angle becomes

$$\delta_m = \pi - \delta_0' = \pi - \sin^{-1}\left(\frac{\sin \delta_0}{r_2}\right) \tag{50}$$

The areas to be compared are

$$\text{Rectangle} = P_i(\delta_m - \delta_0) \tag{51}$$

$$\begin{aligned} \text{Irregular area} &= r_1 P_m \int_{\delta_0}^{\delta_c} \sin \delta \, d\delta + r_2 P_m \int_{\delta_c}^{\delta_m} \sin \delta \, d\delta \\ &= P_m[r_1(\cos \delta_0 - \cos \delta_c) + r_2(\cos \delta_c - \cos \delta_m)] \end{aligned} \tag{52}$$

Equating the above areas the transient power limit of the system for the disturbance caused by the line fault is given by

$$\begin{aligned} P(\max) = P_i &= P_m \frac{r_1(\cos \delta_0 - \cos \delta_c) + r_2(\cos \delta_c - \cos \delta_m)}{\delta_m - \delta_0} \\ &= P_m \frac{r_1 \cos \delta_0 + (r_2 - r_1) \cos \delta_c + r_2 \cos\left[\sin^{-1}\left(\frac{\sin \delta_0}{r_2}\right)\right]}{\pi - \delta_0 - \sin^{-1}\left(\frac{\sin \delta_0}{r_2}\right)} \end{aligned} \tag{53}$$

The Use of Precalculated Swing Curves.—Consider a system consisting of a synchronous machine connected to an infinite bus through a circuit containing reactance only as shown in Fig. 183(*a*). The machine is represented by its direct-axis transient reactance behind which constant voltage is assumed.

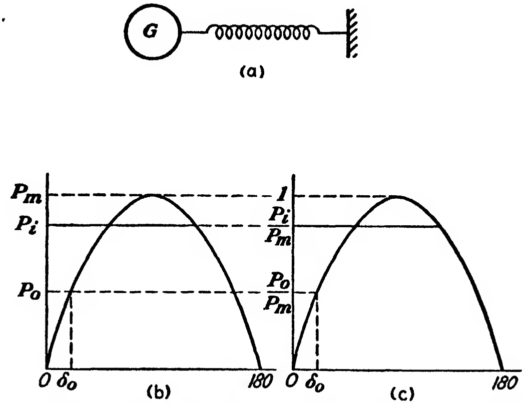


FIG. 183.—(*a*) Generator supplying a large system (infinite bus) over a transmission line or feeder. Reactive circuit elements only are considered. (*b*) and (*c*) Power-angle curves for the system in (*a*).

Neglecting damping the equation for the acceleration of the machine is

$$M\frac{d^2\delta}{dt^2} = P_i - P_m \sin \delta \tag{54}$$

Introducing a quantity τ related to the time t by

$$\tau = \sqrt{\frac{P_m}{M}}t = \sqrt{\frac{180f}{H}\frac{P_m}{P_n}}t \tag{55}$$

(P_n being the rating of the machine) and inserting this in the differential equation, equation (54) may be written

$$\frac{d^2\delta}{d\tau^2} = \frac{P_i}{P_m} - \sin \delta = P - \sin \delta \tag{56}$$

An equation has thus been obtained which is independent of the inertia of the synchronous machine and of the absolute constants of the circuit. The solution of it (for constant input and constant flux linkages) is governed only by the ratio of power input to the amplitude of the power-angle curve and, of course, by the initial angle. The latter again is defined by the

ratio of the initial output (at time $t = 0+$, *i.e.*, immediately after the discontinuity) to the above-mentioned amplitude. [See the power-angle curves, Fig. 183(*b*) and (*c*).] Hence when the ratios P_i/P_m and P_0/P_m are specified, equation (56) yields a definite solution for the angle as a function of the modified time τ. If equation (56) is solved for a suitable range of values of these ratios, it is evident that a family of curves will be obtained which represent the universal solution of the synchronous machine—infinite bus system considered.[1] By entering these curves with the proper ratios and modifying the time scale in accordance with equation (55) the correct angle-time curve may readily be determined.

In determining τ from the first part of equation (55), the units of power used for P_m and M are immaterial but must, of course, be consistent for the two quantities. Thus if P_m is expressed in per unit on a selected base, M must be expressed in per unit per electrical degree per second squared on the same base. In the second part of the equation the units of P_m and P_n obviously are arbitrary, if the same. The stored energy H, however, should always be expressed in kilowatt-seconds per kilovolt-ampere.

The family of curves reproduced in Figs. 184 to 193, inclusive, is the result of a large number of integraph solutions of equation (56).[2] Each curve sheet represents a constant P_0/P_m (or

[1] See papers by Park and Bancker, *loc. cit.;* I. H. Summers and J. B. McClure, "Progress in the Study of System Stability," *Trans. A.I.E.E.*, p. 132, 1930. See also H. L. Byrd, and S. R. Pritchard, Jr., "Solution of the Two-machine Stability Problem," *Gen. Elec. Rev.*, p. 81, February, 1933. In this paper the precalculated curves are arranged in a different manner particularly convenient for determining maximum permissible switching time corresponding to a given condition of initial power.

[2] These curves were obtained on the M.I.T. integraph in 1929 by Summers and McClure and are included in their paper, *loc. cit.* They are reproduced in this treatise by courtesy of Mr. McClure and the General Electric Company.

The abscissa (modified time) used is $\tau' = \sqrt{\pi/180}\,\tau$ where τ is as defined by equation (55). As will be seen, this gives a very convenient scale, although actually the difference results from their use of radians rather than electrical degrees in the acceleration term of the swing equation.

For the benefit of those who may refer to Park and Bancker's paper, it may be mentioned that their curves are plotted versus a modified time $\tau'' = \sqrt{2}\,\tau' = \sqrt{\pi/90}\,\tau$.

T_0/T_m), and hence the curves on each sheet start at the same angle. The range of P_0/P_m covered is from 0 to 0.90 in 10 per cent steps. The individual curves are for constant $P = P_i/P_m$ (or T_i/T_m). On each sheet this ratio varies from a minimum

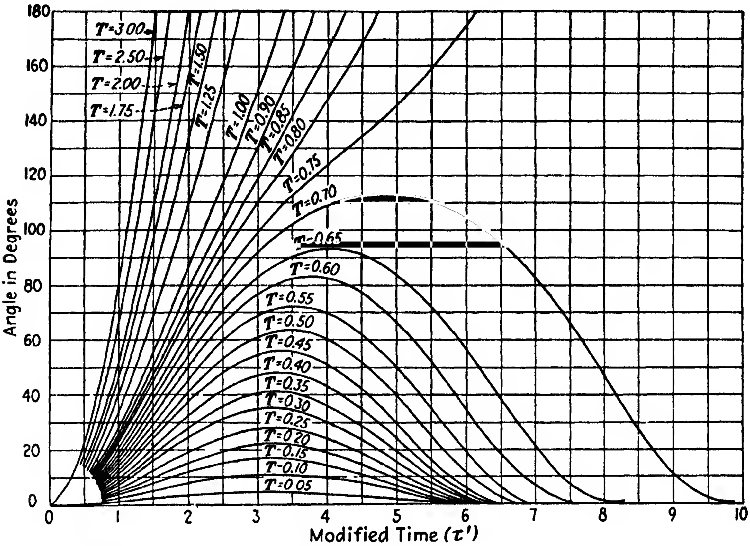


FIG. 184.—Angle-time curves for straightaway reactance system for $T_0/T_m = 0$.

T (or P) $= T_i/T_m$ (or P_i/P_m)

$$\tau' = \sqrt{\frac{\pi f}{H} \cdot \frac{P_m}{P_n}}\, t$$

T_i (or P_i) = shaft input

T_0 (or P_0) = initial output (value of torque or power at initial operating angle as given by torque- or power-angle curve applying while machine is swinging)

T_m (or P_m) = maximum output (as given by torque- or power-angle curve applying while machine is swinging)

T_n (or P_n) = rating of machine

H = stored energy in kw.-sec. per kva.

f = frequency in cycles per second.

The unit used for the torque and power quantities T_i, P_i, T_0, P_0, T_m, P_m, T_n, and P_n is immaterial if consistently applied (*e.g.*, kilowatts, per cent, per unit).

slightly larger than the P_0/P_m ratio for that sheet up to a maximum of 3.00.

The most important use of these precalculated swing curves is in connection with transient-stability analyses by the equal-area method. It will be remembered that in problems involving the clearing of a fault at the lapse of a specified time, it was necessary to determine the corresponding clearing angle before

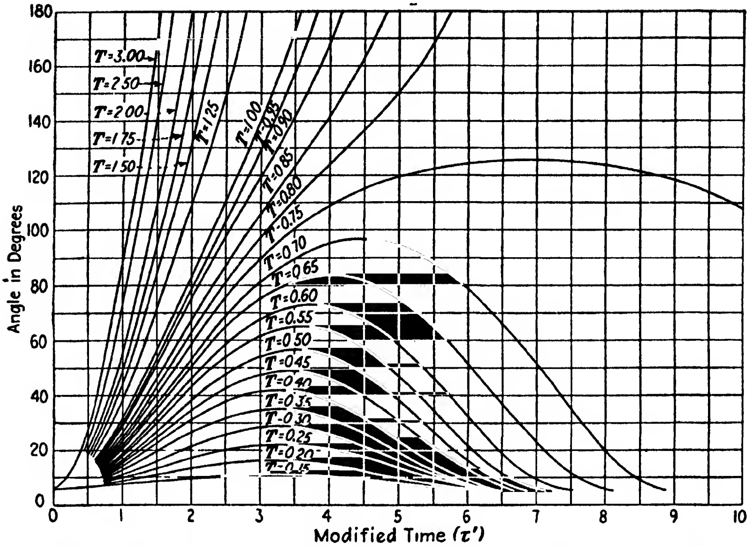


FIG. 185.—Angle-time curves for straightaway reactance system for T_0/T_m = 0.10. For legend see Fig. 184.

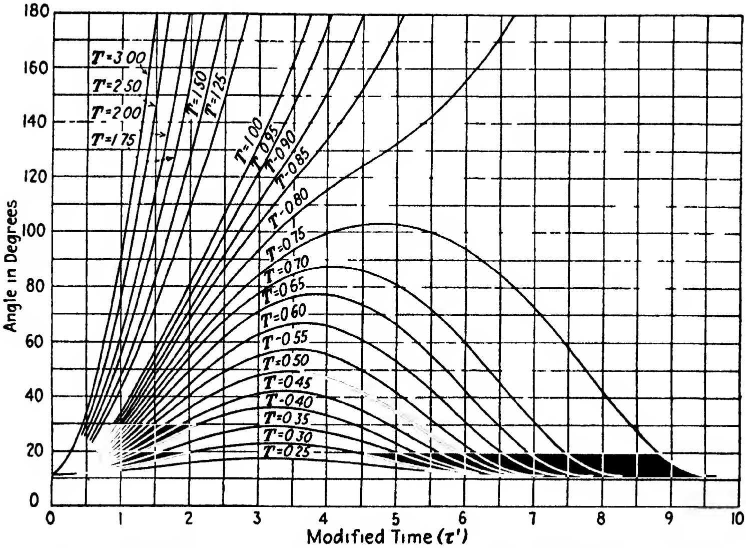


FIG. 186.—Angle-time curves for straightaway reactance system for T_0/T_m = 0.20. For legend see Fig. 184.

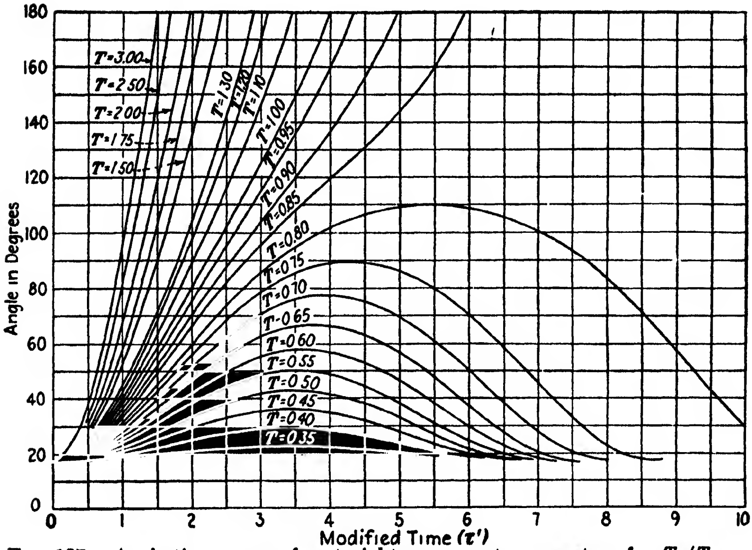


FIG. 187.—Angle-time curves for straightaway reactance system for T_0/T_m = 0.30. For legend see Fig. 184.

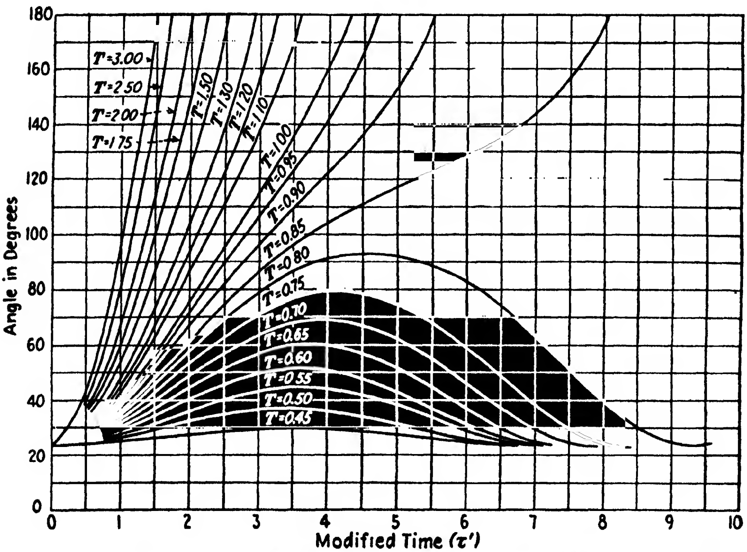


FIG. 188.—Angle-time curves for straightaway reactance system for T_0/T_m = 0.40. For legend see Fig. 184.

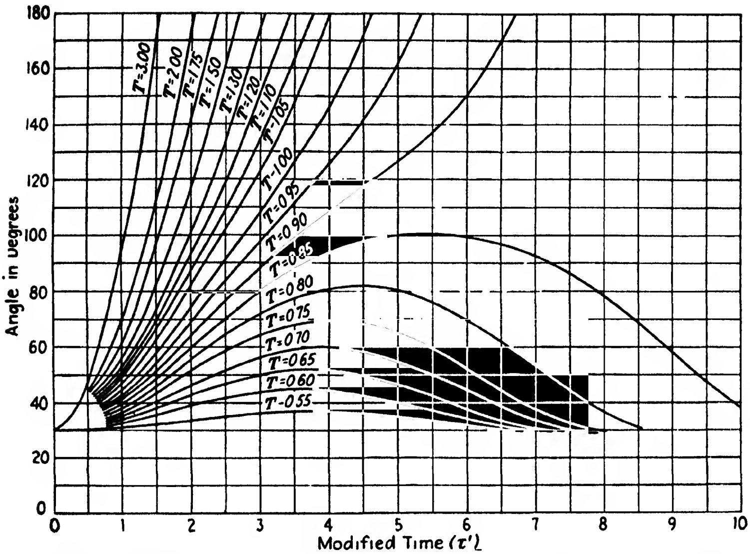


FIG. 189.—Angle-time curves for straightaway reactance system for T_0/T_m = 0.50. For legend see Fig. 184.

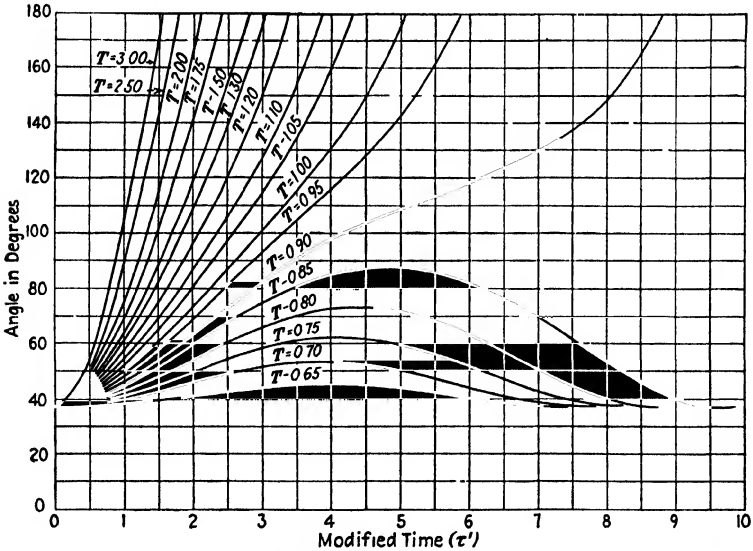


FIG. 190.—Angle-time curves for straightaway reactance system for T_0/T_m = 0.60. For legend see Fig. 184.

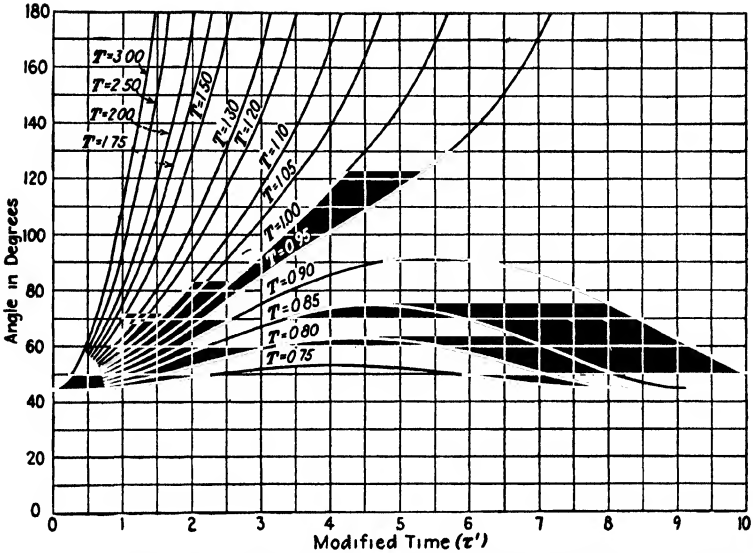


FIG. 191.—Angle-time curves for straightaway reactance system for T_0/T_m = 0 70. For legend see Fig 184.

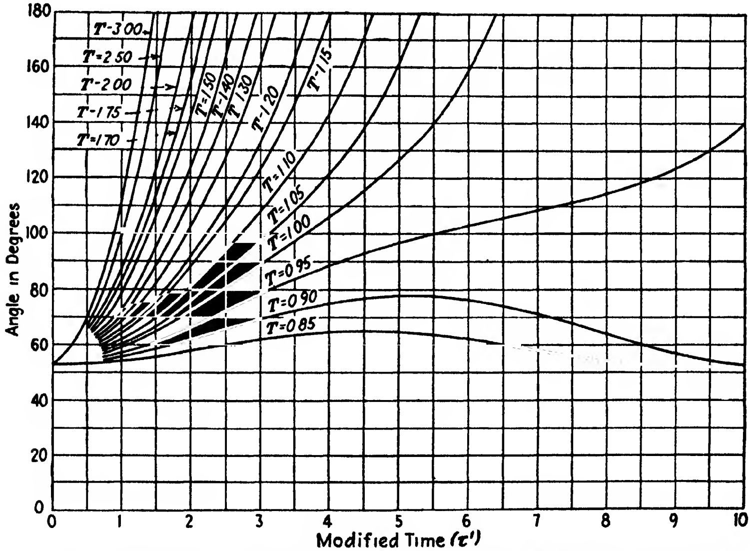


FIG. 192.—Angle-time curves for straightaway reactance system for T_0/T_m = 0.80. For legend see Fig. 184.

applying the equal-area criterion. The clearing angle corresponding to any clearing time can obviously be rapidly obtained from the precalculated curves

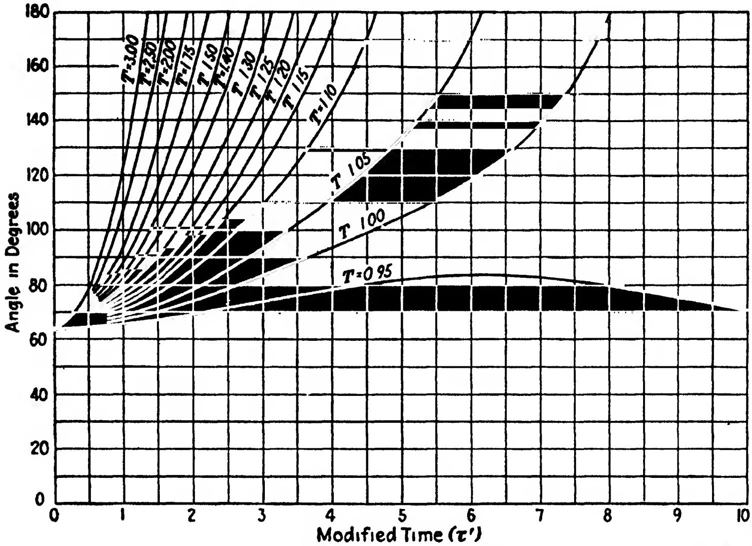


FIG 193 —Angle-time curves for straightaway reactance system for T_0/T_m = 0 90 For legend see Fig 184

EXAMPLE 1

Statement of Problem

A simple system consists of a generating station connected to an infinite bus over a reactance tie [Fig 194(a)] The generator is rated 5,000 kva and has 30 per cent transient reactance The stored energy of the rotating parts (including prime mover) is 3 0 kw-sec per kva The tie has 30 per cent reactance (on generator base)

With the system operating in the steady state, the generator delivering 5,000 kw. with a voltage of 120 per cent behind transient reactance and 100 per cent at the infinite bus, a three-phase fault occurs on a local originally unloaded feeder just outside the protective reactance The latter is 10 per cent (on generator base) The fault is cleared in 0 2 sec

Determine by the equal-area method whether the system possesses stability under the above-mentioned condition of fault, making use of precalculated swing curves to correlate switching time and switching angle

Solution

If per-unit quantities are used, the power output equation prior to the occurrence of the fault and also after the fault has been cleared is

$$P = \frac{E_1E_2}{x_t}\sin\delta = \frac{1.2 \times 1.0}{0.6}\sin\delta = 2.0\sin\delta$$

and when the system is faulted

$$P = \frac{1.2 \times 1.0}{\dfrac{2 \times 0.3 \times 0.1 + 0.3 \times 0.3}{0.1}}\sin\delta$$

$$= \frac{1.2 \times 1.0}{1.5}\sin\delta = 0.8\sin\delta$$

From the above power expressions, the initial operating angle and the output immediately following the occurrence of the fault may be evaluated. Thus

$$P_i = 1.0 = 2\sin\delta_0$$

giving $\sin\delta_0 = 0.5$ and $\delta_0 = 30$ deg.

$$P_0 = 0.8\sin\delta_0 = 0.8 \times 0.5 = 0.4$$

In order to make use of a precalculated swing curve, the following quantities are needed:

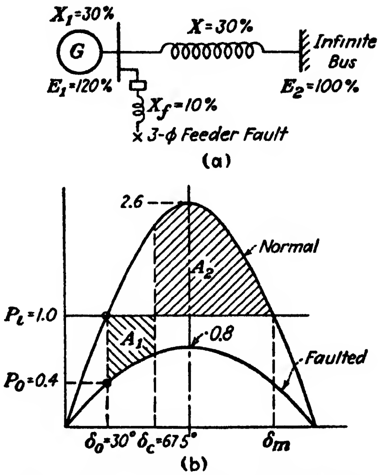


FIG. 194.—(*a*) System analyzed in Example 1. (*b*) Power-angle curves for (*a*) illustrating the effect of a local feeder fault at the generating station and its subsequent clearing.

$$P = \frac{P_i}{P_m} = \frac{1.0}{0.8} = 1.25$$

$$\frac{P_0}{P_m} = \frac{0.4}{0.8} = 0.5$$

Referring to equation (55) and footnote 2 on page 414, the relationship between actual and modified time becomes:

$$\tau' = \sqrt{\frac{\pi}{180}}\tau = \sqrt{\frac{\pi}{180}}\sqrt{\frac{180f}{H}\frac{P_m}{P_n}}t = \sqrt{\frac{\pi f}{H}\frac{P_m}{P_n}}t = \sqrt{\frac{\pi \times 60 \times 0.8}{3.0}}t = 7.10t$$

which for a clearing time of 0.2 sec. gives $\tau' = 1.42$.

Entering the curve sheet for $P_0/P_m = 0.5$ (Fig. 189) at $\tau' = 1.42$ and $P = 1.25$, the clearing angle is obtained as $\delta_c = 67.5$ deg.

Locating this angle in the power-angle diagram [Fig. 194(*b*)], which is plotted to scale, stability requires that the area A_2 be larger than the area A_1. It is evident by inspection that this condition is satisfied. Consequently, the system is stable and will survive the shock caused by the occurrence of the three-phase feeder fault and its subsequent clearing in 0.2 sec.

EXAMPLE 2

Statement of Problem

Figure 195 represents an equivalent diagram of a substation supplying a synchronous motor over a feeder circuit. From the substation bus, there is also another feeder A which in this problem is assumed to be unloaded. Each feeder, as well as the equivalent circuit connecting the substation back to a point which may be considered an infinite bus, has a reactance of 10 per cent based on the synchronous-motor rating. The infinite-bus voltage is 100 per cent.

Synchronous-motor Data

Rating.	10,000 kva.
Speed.	600 r.p.m.
Frequency.	60 cycles per sec.
Synchronous reactance.	60 per cent (approximate saturated value)
Transient reactance. . . .	30 per cent
WR^2.	500,000 lb.-ft.^{2} (including shaft load)

The excitation of the motor is adjusted so that unity power factor is obtained at the motor terminals when the motor is operating at full load.

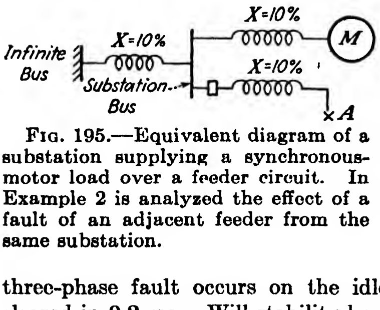


FIG. 195.—Equivalent diagram of a substation supplying a synchronous-motor load over a feeder circuit. In Example 2 is analyzed the effect of a fault of an adjacent feeder from the same substation.

a. What is the steady-state power limit?

b. Can a shaft load of 10,000 kw. be suddenly applied to the motor when it is initially operating at no load? If not, how much load can be applied?

c. While the motor is carrying a load of 8,000 kw., a symmetrical three-phase fault occurs on the idle feeder A, as shown. The fault is cleared in 0.2 sec. Will stability be maintained?

1. Use the equal-area criterion. Obtain clearing angle by point-by-point computation of the necessary part of the swing curve.

2. Use the equal-area criterion in connection with the use of precalculated swing curves.

Solution

a.—The excitation voltage is determined by reference to the vector diagram (Fig. 196) as follows:

$$P = VI = 1.00 \qquad \text{or} \qquad V = \frac{1}{I}$$

$$E^2 = V^2 + (0.20I)^2 = \frac{1}{I^2} + (0.20I)^2 \tag{a}$$

$$E_m^2 = V^2 + (0.60I)^2 \tag{b}$$

By subtraction

$$E_m^2 - E^2 = (0.36 - 0.04)I^2 = 0.32I^2 \qquad (c)$$

From (a)

$$1.00 = \frac{1}{I^2} + 0.04I^2 \qquad \text{giving} \qquad I^2 = 1.044$$

Substituting this value in equation (c) gives

$$E_m^2 = 1.00 + 0.32 \times 1.044 = 1.334$$

Hence

$$E_m = 1.155 \text{ or } 115.5 \text{ per cent}$$

The steady-state power limit is consequently:

$$P_m = \frac{EE_m}{x} = \frac{1.00 \times 1.155}{0.8} = 1.444 \text{ or } 14{,}440 \text{ kw.}$$

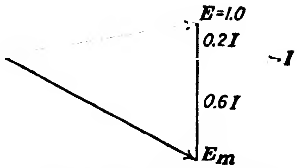


FIG. 196.—Vector diagram for the system in Fig. 195 under normal operating conditions (Example 2).

b.—Figure 197 refers to no-load conditions. If the vector diagram is solved, the voltage behind transient reactance (E') is obtained.

$$E_m - E = 0.8I = 1.155 - 1.0 = 0.155 \qquad \text{giving} \qquad I = 0.194$$
$$E' = E + 0.50I = 1.0 + 0.5 \times 0.194 = 1.097.$$

The electrical power input to the motor is consequently given by

$$P = \frac{EE'}{x_t} \sin \delta = \frac{1.0 \times 1.097}{0.5} \sin \delta = 2.194 \sin \delta$$

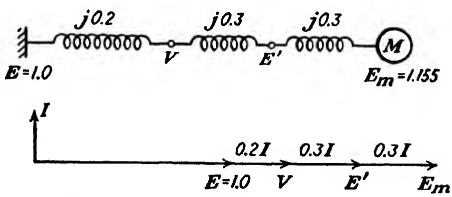


FIG. 197.—Vector diagram for the system in Fig. 195. Synchronous motor floating at no load (Example 2).

from which the power-angle curve (Fig. 198) is plotted. Since full-load shaft power (10,000 kw.) is suddenly to be applied,

$$\sin \delta_1 = \frac{P_i}{P_m} = \frac{1.00}{2.194} = 0.455$$
$$\delta_1 = 27.1 \text{ deg.} = 0.473 \text{ rad.}$$
$$\delta_m = 152.9 \text{ deg.} = 2.669 \text{ rad.}$$

For stability, area A_1 must be smaller than area A_2. The values of these areas are:

$$A_1 = P_i\delta_1 - \int_0^{\delta_1} P_m \sin\delta\, d\delta$$
$$= 1.00 \times 0.473 - 2.194(1.00 - \cos 27.1°)$$
$$= 0.473 - 0.242 = 0.231$$
$$A_2 = \int_{\delta_1}^{\delta_m} P_m \sin\delta\, d\delta - P_i(\delta_m - \delta_1)$$
$$= 2.194\ (\cos 27.1 - \cos 152.9) - 1.00(2.669 - 0.473)$$
$$= 3.91 - 2.20 = 1.71.$$

It may thus be concluded that from the standpoint of stability it would be possible suddenly to apply full load to the motor shaft.

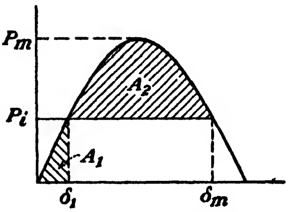


FIG. 198.—Power-angle curve for the system in Fig. 195 for normal operating conditions. No fault on the adjacent feeder (Example 2).

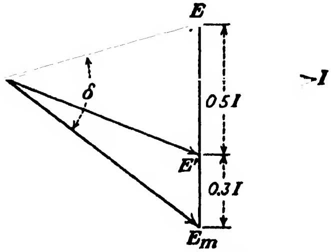


FIG. 199.—Vector diagram for the system in Fig. 195 under conditions of normal operation with 8,000 kilowatts on the synchronous motor (Example 2).

c.—Consider first initial operating conditions (see vector diagram, Fig. 199) in order to find the voltage behind transient reactance. For steady-state operation with 8,000-kw. load, the power-angle relation is

$$P = 0.8 = \frac{EE_m}{x}\sin\delta = \frac{1.0 \times 1.155}{0.8}\sin\delta = 1.444\sin\delta$$

from which

$$\sin\delta = \frac{0.8}{1.444} = 0.554 \qquad \delta = 33.6 \text{ deg.}$$

Taking the motor excitation voltage E_m as reference axis, the following equations may be written:

$$E_m + j0.3I = E' \qquad (d)$$
$$E_m + j0.8I = E \qquad (e)$$

From the second is obtained

$$I = \frac{E - E_m}{j0.8}$$

which, substituted in (*d*), gives

$$E' = \tfrac{5}{8}E_m + \tfrac{3}{8}E$$

Hence

$$E' = \tfrac{1}{8}[5(1.155 + j0) + 3 \times 1.00\ (\cos 33.6 + j \sin 33.6)] = 1.054$$

Assuming constant voltage behind transient reactance, the power-angle curve prior to the occurrence of the fault is

$$P = \frac{EE'}{x_t} \sin \delta = \frac{1.00 \times 1.054}{0.50} \sin \delta = 2.108 \sin \delta$$

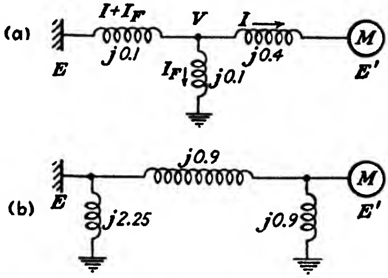


FIG. 200.—Equivalent circuits for system in Fig. 195 with fault on substation feeder (Example 2).

from which the initial operating angle (between the voltage behind transient reactance and the voltage at the infinite bus) is found to be

$$\sin \delta_0 = \frac{0.8}{2.108} = 0.379 \qquad \delta_0 = 22.8 \text{ deg.}$$

When the system is faulted, the circuit diagrams in Fig. 200(*a*) and (*b*) apply. By reference to the latter, the power-angle relation may be written

$$P = \frac{EE'}{x} \sin \delta = \frac{1.00 \times 1.054}{0.9} \sin \delta = 1.171 \sin \delta$$

from which

$$P_0 = 1.171 \sin \delta_0 = 1.171 \sin 22.8 = 0.444$$

also

$$\sin \delta_1 = \frac{0.8}{1.171} = 0.683 \qquad \delta_1 = 43.1 \text{ deg.}$$

(1) *Point-by-point Determination of Angle at Time of Clearing.*—Referring to the swing equation, equation (1), the inertia constant M needed is[1]

[1] Computed by equation (3), Chap. XVIII.

$$M = \frac{1.28 \times 10^{-9} \times WR^2 \times n^2}{f} = \frac{1.28 \times 10^{-9} \times 5 \times 10^6 \times 600^2}{60}$$

$$= 3.84 \text{ kw./elec. deg./sec.}^2 \text{ or } 3.84 \times 10^{-4} \text{ p.u./elec. deg./sec.}^2$$

Point-by-point Method I will be used. Consequently the following equations [equations (7) and (8)] apply:

$$\omega_n = \omega_{n-1} + k'\Delta P_{n-1} \qquad (f)$$

$$\delta_n = \delta_{n-1} + \Delta t \omega_{n-1} + k''\Delta P_{n-1} \qquad (g)$$

In these, using a time interval of 0.05 sec.

$$k' = \frac{\Delta t}{M} = \frac{0.05 \times 10{,}000}{3.84} = 130.3$$

$$k'' = \frac{\Delta t^2}{2M} = \frac{0.05^2 \times 10{,}000}{2 \times 3.84} = 3.256$$

The necessary computations carried out in accordance with the equations above are included in the following table:

Interval number	t sec.	P_i p.u.	P p.u.	ΔP p.u.	$\omega\Delta t$	$k'\Delta P$	$k''\Delta P$	ω elec. deg./sec.	δ elec. deg.
0	0	0.800	0.444	0.356	0	46.4	1.160	0	22.27
1	0.05	0.800	0.466	0.334	2.32	43.5	1.088	46.4	23.43
2	0.10	0.800	0.529	0.271	4.50	35.3	0.882	89.9	26.84
3	0.15	0.800	0.625	0.175	6.26	22.8	0.570	125.2	32.22
4	0.20	0.800	.....	.....	....	....	.....	148.0	39.05

Hence the angle at the time of clearing, δ_c, is equal to 39.1 deg.

Figure 201 shows the power-angle curves with the significant numerical values indicated. The values of the appropriate areas are:

$$A_1 = P_i(\delta_c - \delta_0) - \int_{\delta_0}^{\delta_c} 1.171 \sin \delta \, d\delta$$

$$= 0.8(39.1 - 22.3)\frac{\pi}{180} - 1.171(\cos 22.3° - \cos 39.1)$$

$$= 0.234 - 1.171 \times 0.149 = 0.06$$

$$A_2 = \int_{\delta_c}^{\delta_m} 2.108 \sin \delta \, d\delta - P_i(\delta_m - \delta_c)$$

$$= 2.108(\cos 39.1 - \cos 157.7) - 0.8(157.7 - 39.1)\frac{\pi}{180}$$

$$= 2.108 \times 1.702 - 1.657 = 1.93$$

Since the area A_2 is much larger than the area A_1, the system is stable.

In view of the ample margin, this conclusion obviously might safely have been arrived at merely by inspection of Fig. 201 when the curves are drawn approximately to scale.

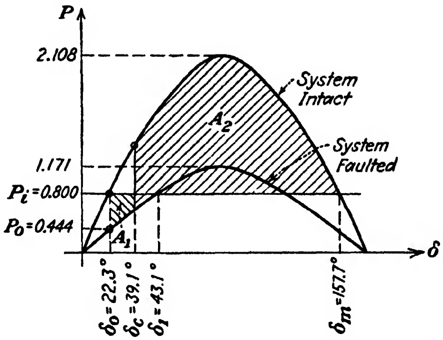


FIG. 201.—Power-angle curves for the system in Fig. 195 illustrating the effect of a substation feeder fault and its subsequent clearing (Example 2).

(2) *Using Precalculated Curves.*—Referring to equation (55) and footnote 2 on page 414, the relationship between modified time and actual time is given by

$$\tau' = \sqrt{\frac{\pi}{180}}\tau = \sqrt{\frac{\pi P_m}{180M}}t = \sqrt{\frac{\pi \times 11{,}710}{180 \times 3.84}}t = 7.29t$$

Hence at time of clearing $t_c = 0.02$ second, $\tau' = 1.46$. Also

$$\frac{P_0}{P_m} = \frac{0.444}{1.171} = 0.379 \qquad \text{and} \qquad \frac{P_i}{P_m} = \frac{0.80}{1.171} = 0.682$$

Interpolation between the curve sheets, Figs. 187 and 188 in this case, becomes necessary. It is found that

$$\text{If} \quad \frac{P_0}{P_m} = 0.30 \qquad \delta_c = 38 \text{ deg.}$$

$$\text{If} \quad \frac{P_0}{P_m} = 0.40 \qquad \delta_c = 38 \text{ deg.}$$

The clearing angle obtained by this method therefore is 38 deg. This checks well with that previously found (39.1 deg.). The conclusion as to stability of course remains as previously stated.

EXAMPLE 3

Statement of Problem

A generator G of moderate size is supplying 8,000 kw. over a double-line circuit to a large system (to be treated as an infinite bus of 100 per cent

voltage), as shown in Fig. 202. The power is supplied at unity power factor at the generator terminals. A symmetrical three-phase fault occurs at the mid-point of one line. The fault hangs on for 0.2 sec. and is, at the end of this time, cleared by simultaneous operation of the automatic circuit breakers at the ends of the faulted line.

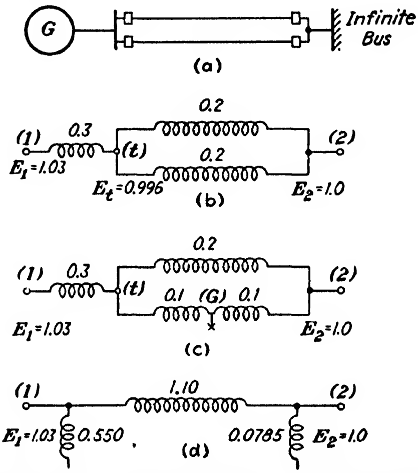


FIG. 202.—(a) Generator supplying power to a large system (infinite bus) over a double-circuit line (Example 3). (b) Circuit diagram of (a) for normal operating conditions. (c) Circuit diagram of (a) with one of the lines faulted. (d) Equivalent circuit for the fault condition illustrated in (c).

The line reactances (on a 10,000-kva. base) are as shown in the figure. The following data apply to the generator:

Rating	10,000 kva.
Speed	200 r.p.m.
Frequency	60 cycles per sec.

Direct-axis reactances:

Synchronous	100 per cent (unsaturated value)
Synchronous	60 per cent (approximate saturated value)
Transient	30 per cent

Quadrature-axis reactances:

Synchronous	60 per cent
Transient	60 per cent

$$WR^2 = 3 \times 10^6 \text{ lb.-ft.}^2 \text{ (including turbine)}$$

a. By means of a point-by-point analysis plot curves of accelerating power differential ΔP, rotational speed above or below synchronous speed ω, and angle with respect to the infinite bus δ versus time. Using time intervals of 0.05 sec. (unless otherwise is specifically stated below), carry out the analysis for a full second.

b. Apply the equal-area criterion for stability to the above problem.

For the purpose of comparing the effect of different assumptions and different methods of point-by-point analysis, as well as the effect of length of time interval, the following solutions are to be presented:

(1) Point-by-point Method I, Assumption *A*.
(2) Point-by-point Method I, Assumption *A*, time interval 0.02 sec.
(3) Point-by-point Method I, Assumption *A*, time interval 0.01 sec.
(4) Point-by-point Method I, Assumption *B*.
(5) Point-by-point Method I, Assumption *A*, using actual rotor angle.
(6) Point-by-point Method II, Assumption *A*.
(7) Point-by-point Method II, Assumption *B*.
(8) Point-by-point Method III, Assumption *A*.
(9) Point-by-point Method III, Assumption *B*.
(10) Point-by-point Method IV, Assumption *A*.
(11) Point-by-point Method IV, Assumption *B*.
(12) Point-by-point Method IV, Assumption *A*, using actual rotor angle.

The point-by-point methods I to IV are those outlined in detail in the first part of this chapter.

Assumption A considers the generator represented by its direct-axis transient reactance and the voltage behind this reactance constant, and uses the angle to this voltage rather than the actual rotor angle (unless otherwise is specifically stated).

Assumption B considers the direct-axis flux linkages (*i.e.*, the voltage e_d') constant.

Solution

Only solution (1) based on Point-by-point Method I, Assumption *A*, will be presented in detail.

Pre-fault Conditions

Referring to Fig. 202(*b*)

$$E_tI = 0.8$$

$$E_2 = 1.0 = \sqrt{E_t^2 + \left(\frac{0.8}{E_t}0.1\right)^2}$$

from which

$$E_t = 0.996$$

$$E_1 = \sqrt{E_t^2 + \left(\frac{0.8}{E_t}0.3\right)^2}$$

giving

$$E_1 = 1.026 = 1.03$$

The power equation is

$$P = \frac{E_1E_2}{X_t}\sin\delta = \frac{1.03 \times 1.00}{0.40}\sin\delta = 2.58\sin\delta$$

and the initial operating angle

$$\delta_0 = \sin^{-1} \frac{0.8 \times 0.4}{1.0 \times 1.03} = 18.1 \text{ deg.}$$

Fault Conditions

Referring to Fig. 202(*c*), the Y-circuit 1*t*, 2*t*, *Gt* is transformed to a Δ as follows:

$$X_{12} = \frac{0.3 \times 0.1 + 0.3 \times 0.2 + 0.2 \times 0.1}{0.1} = 1.10$$

$$X_{1G} = \frac{N}{0.2} = 0.550$$

$$X_{2G} = \frac{N}{0.3} = 0.367$$

Paralleling X_{2G} with 0.1 gives

$$X'_{2G} = \frac{0.1 \times 0.367}{0.1 + 0.367} = 0.0785$$

and the final equivalent π-circuit shown in Fig. 202(*d*). Hence

$$X_t = 1.10 \text{ p.u.}$$

and

$$P = \frac{1.03 \times 1.00}{1.10} \sin \delta = 0.936 \sin \delta$$

Fault Cleared Conditions

$$X_t = 0.50 \text{ p.u.}$$

$$P = \frac{1.03 \times 1.00}{0.50} \sin \delta = 2.06 \sin \delta$$

Inertia Constant

Using equation (3), Chap. XVIII, the inertia constant becomes

$$M = \frac{1.28 \times 10^{-9} \times WR^2 \times n^2}{f}$$

$$= \frac{1.28 \times 10^{-9} \times 3 \times 10^6 \times 200^2}{60} = 2.56 \text{ kw. per elec. deg. per sec.}^2$$

or, in per unit on a 10,000-kva. base,

$$M = \frac{2.56}{10{,}000} = 2.56 \times 10^{-4} \text{ p.u. per elec. deg. per sec.}^2$$

$$\left(H = \frac{180 \times 60 \times 2.56}{10{,}000} = 2.76 \text{ kw.-sec. per kva.}\right)$$

Swing-curve Calculations.—Method I is used. The constants involved are

$$k' = \frac{\Delta t}{M} = \frac{0.05}{2.56 \times 10^{-4}} = 195.3$$

$$k'' = \frac{(\Delta t)^2}{2M} = \frac{0.05 \times 0.05}{2 \times 2.56 \times 10^{-4}} = 4.88$$

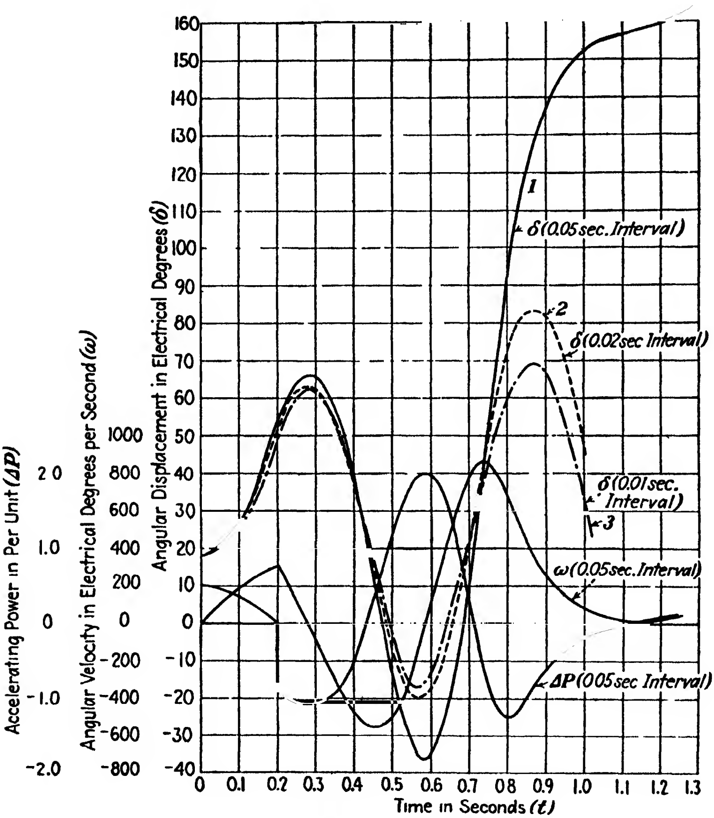


FIG. 203.—Curves of angular displacement, angular velocity, and accelerating power versus time for the system in Fig. 202 illustrating the effect of a line fault and its subsequent clearing in 0.2 second. These curves are determined by point-by-point analyses using Method I and are based on Assumption A (see Example 3). Note the effect of length of time interval upon the calculated oscillations. Appreciable cumulative error is inherent in Method I, unless a very short time interval is used.

The point-by-point calculations are carried out in Table 54. The swing curve and curves of ΔP and ω are plotted in Fig. 203. Swing curves based on

TABLE 54.—POINT-BY-POINT CALCULATION OF SWING CURVES BY METHOD I (Example 3)

(1) Int. no.	(2) t	(3) P_i	(4) P	(5) ΔP	(6) $\Delta t \cdot \omega$	(7) $k'\Delta P$	(8) $k''\Delta P$	(9) ω	(10) δ
0	0	0.800	0.291	0.509	0	99.4	2.48	0	18.1
1	0.05	0.800	0.330	0.470	4.97	91.8	2.29	99.4	20.6
2	0.10	0.800	0.438	0.362	9.57	70.7	1.77	191.2	27.9
3	0.15	0.800	0.592	0.208	13.1	40.6	1.02	261.9	39.2
4	0.20	0.800	1.652	−0.852	15.1	−166.7	−4.16	302.5	53.3
*									
5	0.25	0.800	1.854	−1.054	6.78	−206	−5.14	135.8	64.2
6	0.30	0.800	1.880	−1.080	−3.51	−211	−5.27	−70.2	65.8
7	0.35	0.800	1.729	−0.929	−14.0	−181.7	−4.53	−281.2	57.0
8	0.40	0.800	1.273	−0.473	−23.2	−92.4	−2.31	−462.9	38.5
9	0.45	0.800	0.463	0.337	−27.8	65.8	1.65	−555.3	13.0
10	0.50	0.800	−0.467	1.267	−24.5	247	6.18	−489.5	−13.1
11	0.55	0.800	−1.074	1.874	−12.1	366	9.14	−242.5	−31.4
12	0.60	0.800	−1.163	1.963	6.17	384	9.58	123.5	−34.4
13	0.65	0.800	−0.657	1.457	25.4	284	7.10	507.5	−18.6
14	0.70	0.800	0.494	0.306	39.6	59.8	1.49	791.5	13.9
15	0.75	0.800	1.688	−0.888	42.6	−173.6	−4.33	851.3	55.0
16	0.80	0.800	2.06	−1.26	33.9	−246	−6.14	677.7	93.3
17	0.85	0.800	1.763	−0.963	21.6	−188.1	−4.70	431.7	121.1
18	0.90	0.800	1.379	−0.579	12.2	−113.2	−2.83	243.6	138.0
19	0.95	0.800	1.110	−0.310	6.52	−60.6	−1.51	130.4	147.4
20	1.00	0.800	0.954	−0.154	3.49	−30.1	−0.75	69.8	152.4
21	1.05	0.800	0.867	−0.067	1.99	−13.1	−0.32	39.7	155.1
22	1.10	0.800	0.811	−0.011	1.33	−2.2	−0.1	26.6	156.8
23	1.15	0.800	0.768	0.032	1.22	6.3	0.2	24.4	158.1
24	1.20	0.800	0.720	0.080	1.54	15.6	0.4	30.7	159.5
25	1.25	0.800	....	....	....	....	......	46.3	161.4

ΔP_n:—Col. $(5)_n = (3)_n - (4)_n$ $\qquad k' = 195.3$

ω_n:—Col. $(9)_n = (9)_{n-1} + (7)_{n-1}$ $\qquad k'' = 4.88$

δ_n:—Col. $(10)_n = (10)_{n-1} + (6)_{n-1} + (8)_{n-1}$ $\qquad P_n = \dfrac{1.0 \times 1.03}{X_t} \sin \delta_n$

$= 0.936 \sin \delta_n$ *(during fault)*

$= 0.06 \sin \delta_n$ *(after clearing)*

* Quantities below this line hold *only* after (or *just* after) clearing.

time intervals of 0.02 and 0.01 sec. are also shown. The effect of the cumulative error involved in the 0.05-sec. interval is clearly evident.

Check on Clearing Angle by Use of Precalculated Swing Curves

$$\frac{P_0}{P_m} = \frac{0.291}{0.936} = 0.311$$

$$\frac{P_i}{P_m} = \frac{0.8}{0.936} = 0.855$$

$$\tau' = \left(\sqrt{\frac{60\pi}{2.76}\,\frac{0.936}{1.0}}\right)0.2 = 1.60$$

From precalculated curves (Figs. 187 and 188),

$$\delta_c = 53.5 \text{ deg. (as compared with 53.3 deg. from the swing curve)}$$

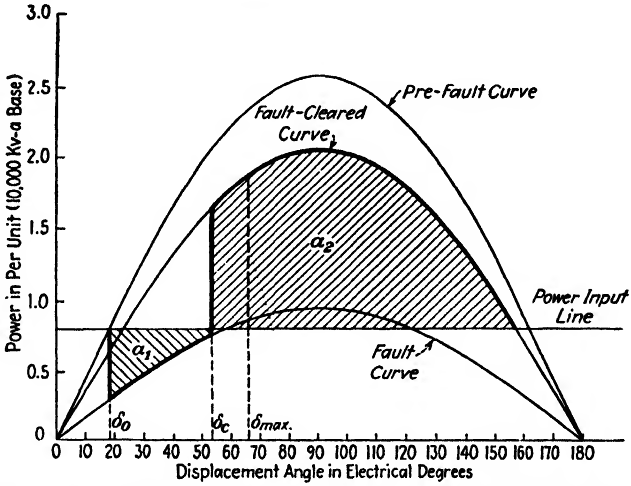


FIG. 204.—Power-angle curves for the system in Fig. 202 illustrating the effect of a line fault and its subsequent clearing in 0.2 second (Example 3). Comparison of the relative size of the two cross-hatched areas indicates stability with an ample margin.

Application of Equal-area Criterion.—The power-angle curves are given in Fig. 204. It is evident by inspection that the equal-area criterion indicates stability with a considerable margin.

EXAMPLE 4

Statement of Problem

Solve Example 3 using a clearing time of 0.25 sec., using (1) point-by-point Method I, Assumption *A*, and (2) point-by-point Method IV (3), Assumption *A*.

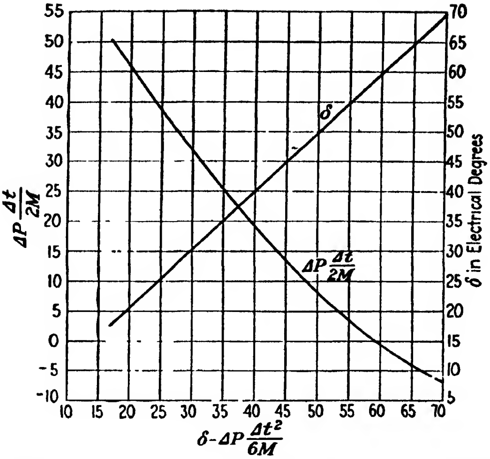


FIG. 205.—Auxiliary curves used in determining the oscillations of the generators in the system in Fig. 202 by point-by-point analysis Method IV. The curves shown apply when the fault is on the system (Example 4).

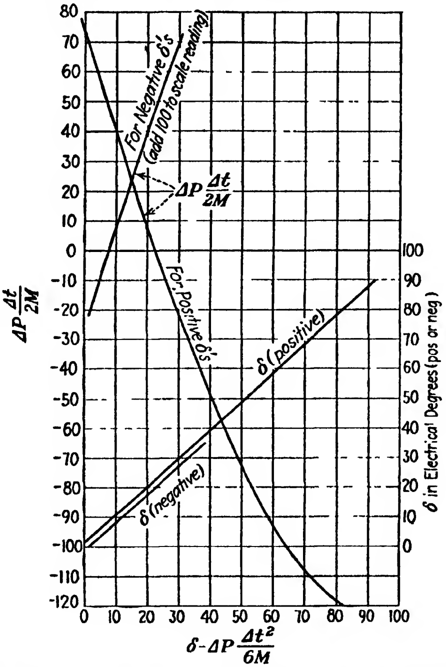


FIG. 206.—Auxiliary curves used in determining the oscillations of the generators in the system in Fig. 202 by point-by-point analysis Method IV. The curves apply after the fault has been cleared (Example 4).

Solution

Part 1.—Details of the solution to this part will not be presented, since they are identical in principle with that presented in Example 3, the only difference being the increase in clearing time. The resulting time curves, however, are plotted in Fig. 207.

Part 2.—Auxiliary curves to be used in accordance with equations (26) and (27) must first be plotted. If reference is made to the solution of Example 3, Part (1), the power differentials with the line faulted and with the fault cleared may be written

$$\Delta P = \begin{cases} (8{,}000 - 9{,}322 \sin \delta) \text{ kw.—line faulted} \\ (8{,}000 - 20{,}508 \sin \delta) \text{ kw.—fault cleared} \end{cases}$$

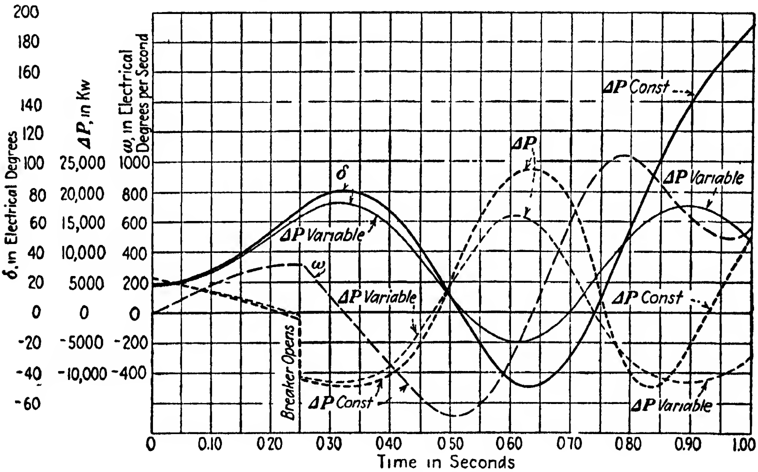


FIG. 207.—Curves of angular displacement, angular velocity, and accelerating power for the system in Fig. 202, illustrating the effect of a line fault and its subsequent clearing in 0.25 second. The curves marked ΔP *Constant* are determined by point-by-point analysis Method I and those marked ΔP *Variable* by Method IV. All curves are based on Assumption A (Example 4).

Since

$$\Delta t = 0.05 \text{ sec.} \qquad \text{and} \qquad M = 2.564 \text{ kw. per elec. deg. per sec.}^2$$

$$\frac{\Delta t^2}{6M} = 1.624 \times 10^{-3}$$

Consequently

$$\frac{\Delta t^2}{6M}\Delta P = \begin{cases} 1.3 - 1.513 \sin \delta\text{—line faulted} \\ 1.3 - 3.335 \sin \delta\text{—line cleared} \end{cases}$$

From the above, using assumed values of δ, the quantities $\Delta t \Delta P/(2M)$ and $\delta - \Delta t^2 \Delta P/(6M)$ may be computed, and δ and $\Delta t \Delta P/(2M)$ plotted

against $\delta - \Delta t^2 \Delta P/(6M)$. The necessary values are collected in the tables below.

LINE FAULTED

δ elec. deg.	$\sin\delta$	$1.513 \sin\delta$	$1.3 - 1.513 \sin\delta$	$\frac{3}{\Delta t}(1.3 - 1.513 \sin\delta)$	$\delta - \Delta P\frac{\Delta t^2}{6M}$ = Col. 1 − Col. 4
18	0.309	0.468	0.832	50.0	17.2
30	0.500	0.757	0.543	32.6	29.5
40	0.643	0.975	0.325	19.5	39.7
50	0.766	1.160	0.140	8.40	49.9
60	0.866	1.310	−0.010	−0.60	60.0
70	0.940	1.411	−0.111	−6.67	70.1

FAULT CLEARED (δ POSITIVE)

δ elec. deg.	$\sin\delta$	$3.335 \sin\delta$	$1.3 - 3.335 \sin\delta$	$\frac{3}{\Delta t}(1.3 - 3.335 \sin\delta)$	$\delta - \Delta P\frac{\Delta t^2}{6M}$ = Col. 1 − Col. 4
0	0	0	1.3	78.0	−1.3
10	0.1736	0.58	0.72	43.2	9.28
20	0.342	1.14	0.16	9.6	19.84
30	0.500	1.67	−0.37	−22.2	30.37
40	0.643	2.14	−0.84	−50.4	40.84
50	0.766	2.56	−1.26	−75.6	51.26
60	0.866	2.89	−1.59	−95.4	61.59
70	0.940	3.14	−1.84	−110.4	71.84
80	0.985	3.29	−1.99	−119.4	81.99
90	1.000	3.34	−2.04	−122.4	92.04

When the fault is cleared, the angle δ also assumes negative values. The value of the expression

$$\delta - \frac{\Delta t^2}{6M}\Delta P = \delta - 1.3 + 3.335 \sin\delta$$

for δ negative is $-(2.6 +$ value for δ positive). Hence the values computed for positive δ's can be used by simply adding 2.6 and reversing the sign.

The expression

$$\frac{\Delta t}{2M}\Delta P = \frac{3}{\Delta t}(1.3 - 3.335 \sin\delta)$$

for negative values of δ becomes $-\left(\text{value for } \delta \text{ positive} - 2.6\frac{3}{\Delta t}\right)\cdot$ Hence

TABLE 55.—POINT-BY-POINT CALCULATION OF SWING CURVES BY METHOD IV(3)
(Example 4)

t sec.	δ_{n-1} deg.	$\frac{1}{20}\omega_{n-1}$	$\frac{1}{30}\Delta P_{n-1}\frac{\Delta t}{2M}$	$\delta_n - \Delta P_n\frac{\Delta t^2}{6M}$	$\Delta P_{n-1}\frac{\Delta t}{2M}$	ω_{n-1} deg./sec	ΔP_{n-1} kw.
0	18.2	0	1.65	19.9	49.5	0	5,080
0.05	20.5	4.77	1.53	26.8	45.8	95.4	4,700
0.10	27.3	8.86	1.20	37.4	36.0	177.2	3,690
0.15	37.5	11.77	0.74	50.0	22.2	235.4	2,280
0.20	50.0	13.31	0.3	63.6	8.1	266.3	831
0.25	63.6	13.56	−0.1	77.1	−3.1	271.3	−318
0.25+	63.6	13.56	−3.39	73.8	−101.8	271.3	−10,440
0.30	72.0	2.82	−3.76	71.1	−112.7	56.5	−11,550
0.35	69.3	−8.30	−3.64	57.4	−109.8	−166.0	−11,260
0.40	56.0	−18.19	−2.93	34.9	−88.0	−363.8	−9,030
0.45	34.2	−24.34	−1.17	8.7	−35.0	−486.8	−3,590
0.50	9.7	−23.84	1.50	−12.6	45.0	−476.8	4,610
0.55	−10.5	−15.8	3.86	−22.4	115.8	−316.0	11,880
0.60	−20.0	−2.66	4.90	−17.8	147.0	−53.2	15,080
0.65	−15.3	11.29	4.4	0.4	132.0	225.8	13,540
0.70	1.6	21.49	2.4	29.5	72.0	429.8	7,380
0.75	29.0	24.09	−0.7	52.4	−20.0	481.8	−2,050
0.80	51.0	19.19	−2.6	67.6	−78.0	383.8	−8,000
0.85	65.8	10.04	−3.5	72.3	−105.0	200.8	−10,780
0.90	70.5	−0.76	−3.7	66.0	−111.0	−15.2	−11,400
0.95	64.2	−11.45	−3.43	49.3	−102.8	−229.0	−10,540
1.00	48.0	−20.14	−2.37	25.5	−71.0	−402.8	−7,280
1.05	25.4	−24.11	−0.28	1.0	−8.5	−482.3	−872
1.10	2.1	−21.02	2.34	−16.6	70.3	−420.5	7,220
1.15	−14.3	−11.08	4.28	−21.1	128.5	−221.7	13,180
1.20	−18.8	2.49	4.80	−11.5	143.0	49.8	14,680
1.25	−9.5	15.24	3.73	9.5	112.0	304.8	11,500
1.30	10.0	22.49	1.43	33.9	43.0	459.8	4,410
1.35	33.1	23.52	−1.08	55.5	−32.4	470.4	−3,300
1.40	54.0	17.67	−2.82	68.9	−84.5	353.5	−8,670
1.45	67.0	8.11	−3.56	71.5	−106.8	162.2	−10,960
1.50	69.8	−2.75	−3.68	63.4	−110.5	−55.1	−11,340

the values for positive angles can also here be used by subtracting 156 and reversing the sign. This gives

FAULT CLEARED (δ NEGATIVE)

δ elec. deg.	$\Delta P \dfrac{\Delta t}{2M}$	$\delta - \Delta P \dfrac{\Delta t^2}{6M}$
−10	112.8	−11.82
−20	146.4	−22.44
−30	178.2	−32.97

The auxiliary curves for the system faulted and after the fault has been cleared will be found in Figs. 205 and 206, respectively.

The details of the point-by-point computations, making use of the auxiliary curves in accordance with the method previously outlined (page 398), will be found in Table 55. The curves of power differential and angle are plotted from the tabulated values in Fig. 207.

CHAPTER XIV

TRANSIENT STABILITY. COMPOUND SYSTEMS

The General Two-machine System.—The general two-machine system involves two synchronous machines connected by a dissipative network which may or may not contain static loads (or their equivalent).[1] It represents an important case, and quite frequently practical transient-stability analyses are based on this layout. It may be identical with the system actually at hand but is more often the result of a simplification of the actual system.[2]

FIG. 208.—General two-machine system.

The stability of such a system (for instance, the one in Fig. 208), when its equilibrium is upset by a disturbance, may, in general, be examined by inspecting the swing curves of the machines. Such swing curves are obtained by solving the differential equation

$$M\frac{d^2\delta}{dt^2} = P - P_i = \Delta P \tag{1}$$

for each machine by means of a point-by-point process, as previously has been described.[3] Either the angle of each machine with respect to some standard reference line or else the angle between the machines (the difference between the separate angles) may be plotted versus time. The former indicate the actual movement of each machine, the latter, their relative movement which, of course, is of principal interest from the standpoint of stability. Swing curves for a general two-machine system are illustrated in Fig. 212.

[1] DAHL, O. G. C., "Stability of the General Two-machine System," *Trans. A.I.E.E.*, p. 185, 1935; PRITCHARD, S. R., JR., and EDITH CLARKE, "Calculation of Two-Machine Stability, with Resistance," *Gen. Elec. Rev.*, p. 87, February, 1934.

[2] See Chap. IX.

[3] See Chap. XIII.

The primary thing during a point-by-point analysis is the determination of the output at each point as the calculations progress. If damping is neglected and it is assumed that the entire circuit, including the machines, is linear, *i.e.*, the saturation in the machines is neglected, the output may be determined:

1. By direct calculation.
2. From power-angle curves.
3. From circle diagrams.

1. *Output Determined by Direct Calculation.*—The outputs of the machines[1] are given by

$$P_1 = \frac{E_1^2}{Z_{11}} \cos \theta_{11} - \frac{E_1E_2}{Z_{12}} \cos (\delta + \theta_{12}) \tag{2}$$

and

$$P_2 = \frac{E_2^2}{Z_{22}} \cos \theta_{22} - \frac{E_1E_2}{Z_{12}} \cos (\delta - \theta_{12}) \tag{3}$$

Three sets of these equations must in general be available: (*a*) for operation with the system intact, (*b*) for operation with the system faulted, and (*c*) for operation with the faulty section cleared.[2] Discontinuities in the outputs occur when the fault comes on and when it is cleared. The initial operating angle is obtained from the first set. Using this angle in the second set gives the outputs immediately after the occurrence of the fault. These outputs in connection with the inputs to the machines fix the initial power differentials, assumed constant during the first interval.[3] The outputs for successive intervals are obtained from the same set of equations by substituting the proper system angles. When the time elapsed equals the clearing time, the

[1] The given equations [see Chap. VIII, equations (8) and (9) in particular] apply only when the machines are *represented by a single impedance.* Fundamentally, therefore, they apply to non-salient-pole machines but may also be used for salient-pole machines provided properly corresponding impedances and internal voltages represent the latter. This matter is fully covered in Chap. XII.

[2] If the faulty section is cleared by sequential switching, part *c* may require two sets of equations. In systems with reactance compensation by series capacitors additional sets may also be necessary.

[3] Modification in regard to this assumption is evidently possible and depends, in general, on the accuracy desired. Discussed in detail in Chap. XIII.

third set of equations supersedes the second. The outputs immediately following the clearing of the fault are determined by using the "clearing angle," *i.e.*, the system angle at the instant of clearing, in these equations.

2. *Output Determined from Power-angle Curves.*—The procedure is as before except that the outputs are determined from power-angle curves instead of by calculation. Data for the power-angle curves are obtained from the output equations or from circle diagrams. Power-angle curves for a general two-machine system are shown in Fig. 209. The output of each machine is plotted versus displacement angle between them with machine 1 considered leading.

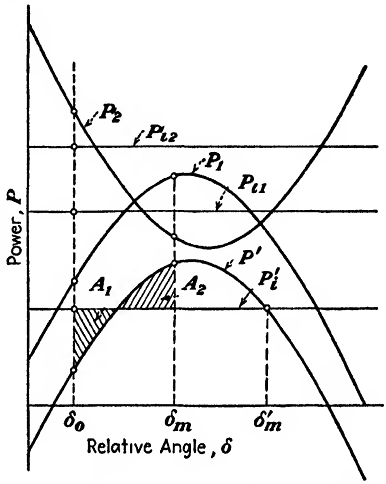


FIG. 209.—Actual and equivalent power-angle curves and input line for a general two-machine system. The maximum angle δ_m is reached when the areas A_1 and A_2 are equal.

3. *Output Determined from Circle Diagrams.*—The charts giving the output of the machines—input to each end of the circuit with machine impedances included—are best constructed from formulas in terms of general circuit constants (Chap. X, Vol. I). Let the general circuit constants (excluding C as not needed in the present application) be

$$\begin{aligned} A &= |A|\underline{/\alpha} = a_1 + ja_2 \\ B &= |B|\underline{/\beta} = b_1 + jb_2 \\ D &= |D|\underline{/\delta^*} = d_1 + jd_2 \end{aligned} \tag{4}$$

assuming that the "sending" end is at generator 1. Since the synchronous machines are considered to operate as generators, both charts should evidently be of the sending-end type. The horizontal and vertical displacements of the center and the radii of the circles are consequently given by:

* This symbol δ adopted and systematically used in Vol. I for the angle of the general circuit constant D, when written in polar form, must not be confused with the angle δ representing synchronous-machine displacement.

For generator 1:

$$A_{G1} = l'V_1^2 = \frac{d_1b_1 + d_1b_2}{b_1^2 + b_2^2}V_1^2 \tag{5}$$

$$B_{G1} = m'V_1^2 = \frac{d_2b_1 - d_1b_2}{b_1^2 + b_2^2}V_1^2 \tag{6}$$

$$C_{G1} = n'V_1^2\frac{E_2}{E_1} = \frac{V_1^2}{\sqrt{b_1^2 + b_2^2}}\frac{E_2}{E_1} \tag{7}$$

For generator 2:

$$A_{G2} = lV_2^2 = -\frac{a_1b_1 + a_2b_2}{b_1^2 + b_2^2}V_2^2 \tag{8}$$

$$B_{G2} = mV_2^2 = \frac{a_2b_1 - a_1b_2}{b_1^2 + b_2^2}V_2^2 \tag{9}$$

$$C_{G2} = nV_2^2\frac{E_1}{E_2} = \frac{V_2^2}{\sqrt{b_1^2 + b_2^2}}\frac{E_1}{E_2} \tag{10}$$

The charts are, as seen, of the "modified" type universally applicable independent of voltage. The active- and reactive-

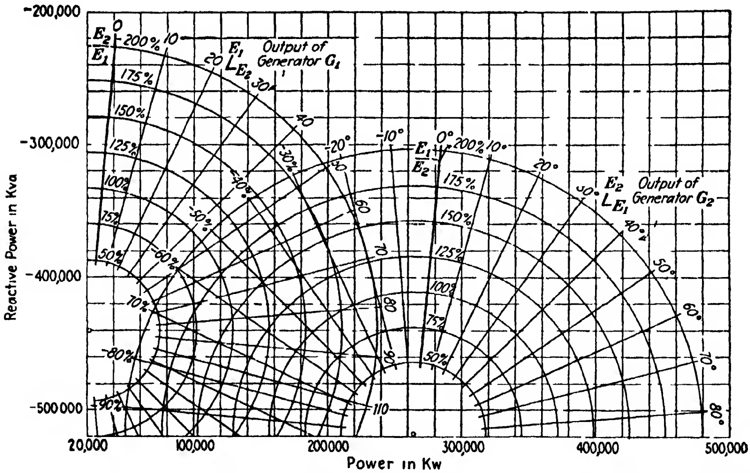


Fig. 210.—Elements of circle diagrams giving the output of the generators of a general two-machine system under fault condition. Note the abnormal amount of reactive power supplied.

power scales are correct and direct-reading at the nominal value of voltage at the end in question. The circles represent ratios of the voltages at the two ends. The power unit used is immaterial.

Charts for the three operating conditions previously discussed may be constructed. Since the first set in general is used only for determining the initial operating conditions, it may be omitted if desired and if not required for any other purpose. This may possibly save some time. The two other sets, *i.e.*, for operation with the system faulted and for operation with the faulty section cleared, are needed during the point-by-point analysis proper. The outputs for the respective intervals are readily ascertained by entering the charts with the voltage ratio and the angle.

Elements of charts for the two machines are shown in Fig. 210. They are for operation with the fault on the system (note the abnormal amounts of reactive power supplied). The zero-angle lines are located in the usual manner by means of the angles of the general circuit constants A and D. Arrows indicate the direction in which the voltage vector swings in the two charts, respectively, when the machines move apart. The operating point immediately after the occurrence of the fault is located by means of the initial angle δ_0.

Representation of Performance of the General Two-machine System by a Single Equivalent Power-angle Curve.—It has been previously shown (Chap. XIII) that in the special case of a two-machine system where one machine operates as a generator and the other as a motor and the circuit contains reactance only, it was permissible to concentrate an equivalent mass at one end of the system and consider the other as an infinite bus.[1] The circuit itself was to be retained as in the actual layout, and the input to the equivalent machine the same as the input to the actual machine.

A somewhat similar procedure is possible also with the general two-machine system. It is possible also here to locate a machine with an equivalent mass at one end and to consider the other end as an infinite bus and hence enable the solution to be based on a *single power-angle curve*. In the general case, however—and this distinction should be carefully noted—it is necessary (1) *to modify the circuit so as to obtain an equivalent power-angle curve*, and (2) *to use an equivalent input to the machine.*

[1] See papers by R. H. Park and E. H. Bancker, "System Stability as a Design Problem," *Trans. A.I.E.E.*, p. 170, 1929; I. H. Summers and J. B. McClure, "Progress in the Study of System Stability," *Trans. A.I.E.E.*, p. 132, 1930; and others.

The relative acceleration of the two machines is given by

$$\frac{d^2\delta}{dt^2} = \frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2} = \frac{P_{i1} - P_1}{M_1} - \frac{P_{i2} - P_2}{M_2} \tag{11}$$

which for the purpose of getting an equation of standard form [corresponding to equation (1)] may be written

$$\frac{M_1M_2}{M_1 + M_2}\frac{d^2\delta}{dt^2} = \frac{M_2\Delta P_1 - M_1\Delta P_2}{M_1 + M_2} = \frac{M_2P_{i1} - M_1P_{i2}}{M_1 + M_2} - \frac{M_2P_1 - M_1P_2}{M_1 + M_2} = P_i' - P' \tag{12}$$

or, more simply, as

$$M_0\frac{d^2\delta}{dt^2} = P_i' - P' = \Delta P' \tag{13}$$

Here the equivalent input P_i' is given by

$$P_i' = \frac{M_2P_{i1} - M_1P_{i2}}{M_1 + M_2} \tag{14}$$

and the equivalent output P' by

$$P' = \frac{M_2P_1 - M_1P_2}{M_1 + M_2} \tag{15}$$

while the equivalent inertia constant M_0, as in the simpler case, is obtained by "paralleling" the actual inertia constants of the two machines and is hence given by

$$M_0 = \frac{M_1M_2}{M_1 + M_2} \tag{16}$$

Equation (13) is, as seen, of exactly the same form as that for a machine connected to an infinite bus through a circuit of arbitrary constants. The stability of the general two-machine system may be correctly examined by the solution of this equation by point-by-point or other methods. Hence the general two-machine case at hand has been reduced to that of an equivalent generator supplying power over an equivalent circuit to an infinite bus. The input to the equivalent machine is a function of the inputs to the actual machines and the inertia constants, while the output of the equivalent machine similarly depends upon the outputs of the actual machines and the inertia constants.

From the standpoint of relative motion, this representation, as well as equation (13) based on it, is exact in every respect.[1] The inputs may be constant or variable, and the outputs may or may not include the effect of damping. Its use in stability solutions based on simplified criteria, however, is practical only when the inputs are constant and the damping neglected. Under these conditions, the equivalent input is fixed, and the output is a function of the internal voltages of the machines and the displacement angle between them (*i.e.*, the displacement angle between the equivalent machine and the infinite bus).

Introducing the expressions for the outputs from equations (2) and (3) and collecting constant and variable terms, equation (13) becomes

$$M_0\frac{d^2\delta}{dt^2} = P_i' - \frac{M_2\dfrac{E_1^2}{Z_{11}}\cos\theta_{11} - M_1\dfrac{E_2^2}{Z_{22}}\cos\theta_{22}}{M_1 + M_2} + \frac{E_1E_2}{(M_1 + M_2)Z_{12}}[M_2\cos(\delta + \theta_{12}) - M_1\cos(\delta - \theta_{12})] \quad (17)$$

By combining the trigonometric functions, this equation again reduces to

$$M_0\frac{d^2\delta}{dt^2} = P_i' - P' = P_i' - [P_c' - P_m'\cos(\delta + \psi)] \quad (18)$$

Here the displacement term, the amplitude, and the phase angle of the equivalent sinusoidal power-angle curve are given by

$$P_c' = \frac{M_2\dfrac{E_1^2}{Z_{11}}\cos\theta_{11} - M_1\dfrac{E_2^2}{Z_{22}}\cos\theta_{22}}{M_1 + M_2} \quad (19)$$

$$P_m' = \frac{E_1E_2}{(M_1 + M_2)Z_{12}}\sqrt{M_1^2 + M_2^2 - 2M_1M_2\cos 2\theta_{12}} \quad (20)$$

and

$$\psi = \tan^{-1}\left(\frac{M_1 + M_2}{M_2 - M_1}\tan\theta_{12}\right) \quad (21)$$

In Figs. 209 and 211 are shown actual and equivalent power-angle

[1] Subject, of course, to the usual limitation resulting from the assumption of constant actual speed of rotation (*i.e.*, constant relationship between torque and power).

curves and input lines for a representative general two-machine system. Figure 209 involves a single (initial) discontinuity only and may, for instance, represent the case of a fault with either

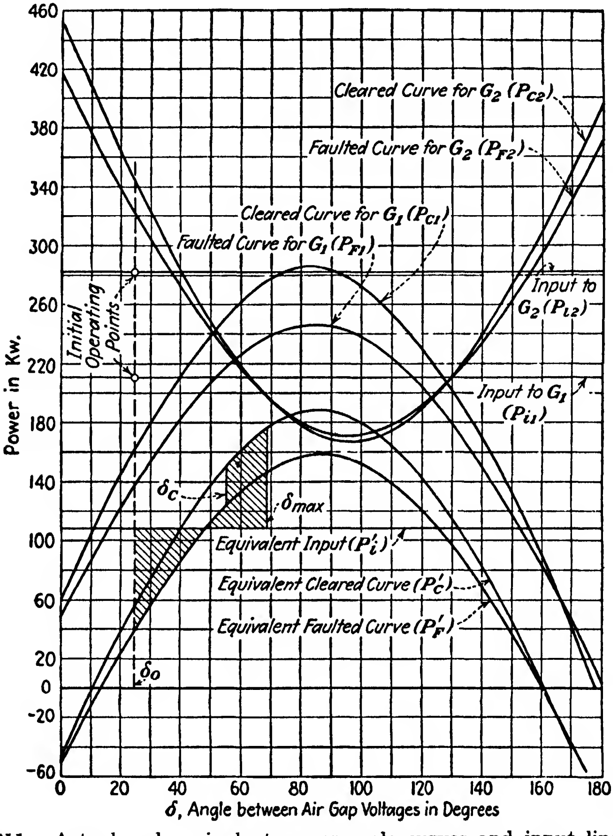


FIG. 211.—Actual and equivalent power-angle curves and input lines for a general two-machine system. The curves illustrate the effect of a line fault and its subsequent clearing (at angle δ_c). The maximum angle $\delta_{max.}$ is reached when the cross-hatched areas on either side of the equivalent input line (the accelerating and decelerating areas) are equal. (The curves here shown are results from an actual examination of the stability of a large power-transmission system.)

long-delayed or instantaneous clearing. Figure 211 involves two discontinuities (one initial) and may be taken to illustrate the case of a fault and its subsequent clearing after the lapse of a certain time.

Simplified Stability Criterion Applied to the General Two-machine System.—This criterion, which has already been outlined in the preceding chapter, assumes that if the machines come to rest with respect to each other during their first swing apart, the system is stable. The mathematical formulation of this criterion, as given by equation (33), Chap. XIII, is

$$\int_{\delta_0}^{\delta_m}\left(\frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2}\right)d\delta = \int_{\delta_0}^{\delta_m}\frac{\Delta P'}{M_0}d\delta = 0 \tag{22}$$

introducing in the second term the equivalent power differential and inertia constant in accordance with equation (13). In the previous detailed discussion of this criterion and its application, it was stated that in a stable system there were two possibilities in regard to the value of the relative acceleration

$$\left(\frac{\Delta P_1}{M_1}\right) - \left(\frac{\Delta P_2}{M_2}\right) = \frac{\Delta P'}{M_0}$$

at maximum angle, *viz.:*

1. The relative acceleration may have a finite value.
2. The relative acceleration may be zero.

In the equivalent formulation of the general two-machine system, it is evident that the equivalent power differential is proportional to the relative acceleration. Hence zero acceleration occurs when $\Delta P' = 0$. Graphically this corresponds to the point of intersection between the equivalent input line and power-angle curve.

Equivalent power-angle curves as well as time curves of angle, rate at which the machines swing apart (relative slip), and acceleration are plotted for the two cases in Figs. 212 and 213, respectively. These are based on a single (initial) discontinuity only and are merely intended to illustrate principles. Either case indicates stability. In the former the machines come to rest with respect to each other while the acceleration still is negative. This would cause the machines to swing together again and oscillate with respect to each other before finally merging into new steady-state conditions. In the second case, where the load is critical, the machines come to rest with respect to each other, and the acceleration becomes zero simultaneously. The system is aperiodic, and the new steady-state conditions, reached

at the end of the first swing, are represented by the maximum angle.[1]

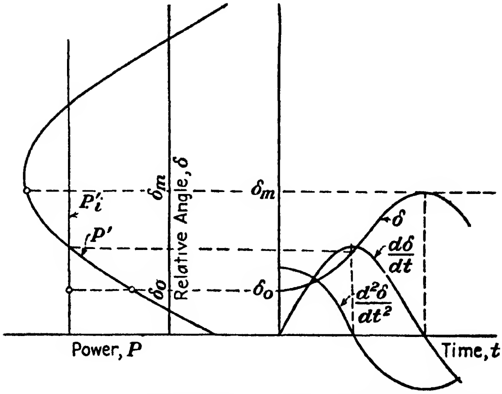


FIG. 212.—Equivalent power-angle curve and time curves of relative angle, slip, and acceleration for a general two-machine system with loading below the critical.

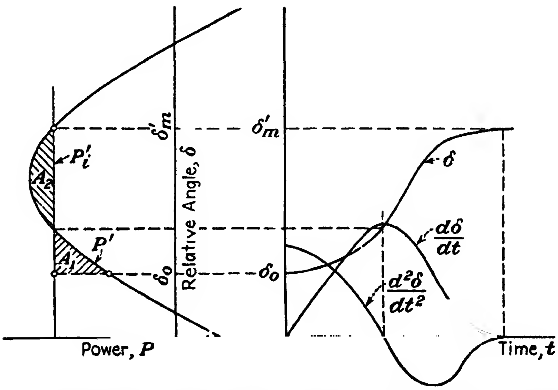


FIG. 213.—Equivalent power-angle curve and time curves of relative angle, slip, and acceleration for a general two-machine system for critical load conditions. Stability requires that the area A_1 be smaller than or, at the limit, equal to the area A_2.

For the purpose of further illustration and comparison, curves for two unstable cases are plotted in Figs. 214 and 215. In the

[1] It is evidently necessary that some margin actually be present. If exactly the critical load were carried and, in accordance with the above, the system attempted to settle at the maximum angle, the operation here would be statically unstable and could not be sustained except under conditions of dynamic equilibrium.

former the equivalent input line lies above the maximum of the equivalent power-angle curve. Hence the power differential and the acceleration, although decreasing for a while and passing

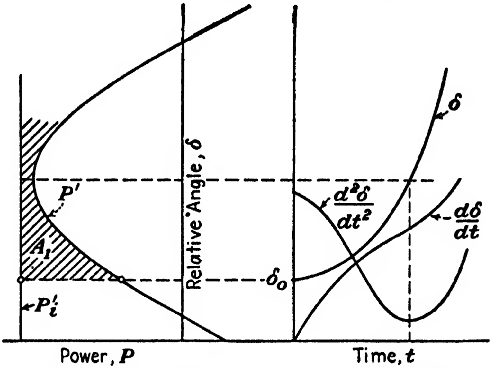


FIG. 214.—Equivalent power-angle curve and time curves of relative angle, slip, and acceleration for a general two-machine system with loading above the critical. Stability is definitely lost. Since the equivalent input line lies entirely above the power-angle curve the sign of the acceleration remains unchanged and there is no tendency to check the swinging apart of the machines.

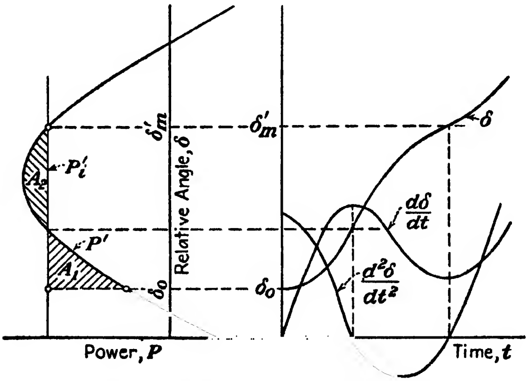


FIG. 215.—Equivalent power-angle curve and time curves of relative angle, slip, and acceleration for a general two-machine system with loading above the critical. Stability is definitely lost. Since the equivalent input line intersects the power-angle curve the acceleration changes sign and there is hence a tendency, although insufficient, to check the swinging apart of the machines. Area A_1 is larger than area A_2, indicating that synchronism cannot be maintained.

through a minimum, do not change sign. The angle and relative slip continuously increase. In Fig. 215 the power differential and the acceleration change sign but the kinetic energy stored

during the (positive) acceleration period is not fully given up prior to reaching the optimum angle. The relative speed after reaching a maximum passes through a minimum at the critical angle. The angle continuously increases as in the preceding case. (It is of interest to note that the angle curve in Fig. 215 has two points of inflection, while in Fig. 214 there are none.)

It has already been shown that the simplified stability criterion for two-machine systems may be converted into an "equal-area" method [equations (40) and (41), Chap. XIII]. To apply this equal-area criterion of stability directly on the power-angle curves, however, is not practical when separate power-angle curves are used because the optimum angle to which the system may be permitted to swing is not inherently defined geometrically by the power-angle curves and the input lines (see Figs. 209 and 211). With separate power-angle curves, the graphical determination of optimum angle requires an auxiliary plot of relative acceleration $(\Delta P_1/M_1) - (\Delta P_2/M_2)$ versus angle. Explicit use of the equal-area method, therefore, as a criterion of stability in the two-machine system is especially convenient and practical when the inertia is concentrated at one end so that an infinite bus is obtained at the other, *i.e.*, when the equivalent system previously described is utilized.

With respect to the equivalent power-angle curve, the equal-area method is directly applicable as such. The optimum angle is here fixed by the intersection of the equivalent power-angle curve and the equivalent input line. Referring to Fig. 213, stability requires that the area A_1 be smaller than or—at the limit—equal to the area A_2.

General Multimachine System.—The complexity of the transient-stability analysis increases with the number of synchronous machines or generating stations which have to be considered as separate entities. A multimachine system therefore should always be reduced as much as practicable and consistent with accuracy prior to the analysis. Machines or stations which are closely connected electrically and which therefore may be considered to oscillate together as a group may be combined into a single equivalent machine. Sometimes inspection of the system will reveal to what extent this can be done and what machines or stations may be joined together into groups. Obviously, no general rule can be given. The

grouping will depend on the layout of the particular system, on the position of the faults to be considered, and on the relative size and characteristics of the machines. With respect to the last, it is principally the transient characteristics and their inertia which are of importance. The proper combination into groups would in each case depend upon the judgment of the engineer handling the problem and is greatly facilitated when he has experience in handling multimachine stability problems and can draw upon this experience and utilize facts from previous analyses as a guide.

Maximum simplicity results if and when the multimachine system can be reduced to a two-machine setup. In the case of an isolated hydroelectric station supplying power over a long-distance transmission line to a metropolitan load area, such a reduction is usually practicable and gives sufficient accuracy when the faults to be studied are on the transmission line. From the steady-state point of view, reduction of this system has been discussed in detail in Chap IX utilizing the conceptions of equivalent generators and equivalent loads developed in Chap. VI. Much of the treatment there given has a bearing also on the methods of reduction which may be used in transient-stability problems.

If reducible to a two-machine system, the methods for checking transient stability are as outlined above. In other words, in this case a point-by-point analysis may be dispensed with and simplified criteria utilized. In the multimachine problem on the other hand (*i.e.*, in any case where the system to be analyzed contains more than two machines, it being immaterial in this connection whether this system represents the actual system or a simplification thereof), complete point-by-point analyses must in general be made. In other words, it is necessary to determine angle-time curves of all machines or at least of those machines (or groups of machines) which are critical under the examined fault conditions. These curves must be extended over a time interval long enough for conclusions to be reached as to the stability of the system. Furthermore, in the multimachine case, actual machine angles with respect to a common reference axis (which, of course, may coincide with the field structure of one of the machines) must be used rather than the relative angles between the various machines. Of course, the latter

angles may easily be obtained and plotted after the former have been determined.

Equivalent Representation of a Machine Group with Associated Circuits and Loads.—When several machines associated with a certain part of the network and certain loads are to be grouped together, the fundamental criterion must be that the equivalent used to represent them has essentially correct transient characteristics (electrical as well as inertia characteristics) at the point of connection to the system. This requirement can seldom be accurately met. If the proper machines are combined, however, and the combination judiciously handled, an acceptable approximation may be obtained. Since the machines are assumed to swing together, the relative angles among them remain at their initial values during the transients. Furthermore, in the reduction it is assumed that some internal voltage in the machines (usually the voltage behind transient reactance) also remains fixed and equal to the initial value. In regard to the loads, these should strictly be represented by their transient characteristics which may or may not be the same as during steady-state operation. Frequently, however, they are replaced by constant impedances which eliminates any possibility of nonlinear relations and makes the reductions simpler from the standpoint of computation.

With regard to inertia, it is usual practice to assign to the equivalent machine an inertia constant equal to the sum of the inertia constants of the actual machines involved in the equivalent representation.

In order to obtain the transient operating characteristics of a combination of machines, circuits, and loads at the point of connection with the system, a procedure similar to that outlined in Chap. VI for steady-state conditions may be followed. Comments or details at this point therefore do not seem necessary. It should be borne in mind, however, that transient reactances must be used for the machines and transient characteristics for the loads.

When machine saturation is taken into account and the load characteristics are nonlinear, the characteristics at the tie-in bus are a family of curves of power versus reactive power at various values of voltage. Curves of constant angle (displacement angle between junction-point voltage and "internal" equivalent-

machine voltage) must also be located in this diagram. As an alternative to this, the performance may be represented by a family of power-angle and reactive-power-angle curves at various values of voltage. The use of these curves in the subsequent stability analysis usually necessitates graphical processes.

When machine saturation is not considered and the load characteristics are linear (*i.e.*, replaced by constant impedances), the above-mentioned curves become circles. This means that the group of machines with their associated circuits and loads actually may be reduced to a three-terminal network (T- or π-circuit). One terminal of this equivalent circuit represents the point of connection to the system (the system bus), another the equivalent internal machine bus, and the third the neutral (or ground). The voltage applied to the internal machine bus is an equivalent internal voltage, depending upon the actual internal voltages of the machines and the (fixed) phase displacement among them. The performance of this equivalent circuit is representable by a standard circle diagram or computable by simple equations.[1]

As in the steady state, the characteristics at the system bus or the equivalent representation is valid only for a given condition of initial loading. If this is changed, new transient characteristics or a new equivalent circuit must be determined.

In obtaining the equivalent representation of a group of machines and their associated circuits and loads, it was assumed above that the angular displacement among machines was fixed and equivalent to their actual values under initial steady-state operating conditions. In order further to simplify the situation, the internal voltages of the machines may be considered in phase and of the same magnitudes. When in addition linear characteristics are assumed, the problem of obtaining the equivalent circuits is much simplified, this being true whether the reduction is performed by computation or by representation and measurements on a Network Analyzer. In the above considerations, it was also tacitly assumed that the part of the system to be reduced ties in with the remainder at a single point only. This resulted in single families of performance characteristics or equivalent three-terminal networks. If there are two or more points of connection, *serious complications* are introduced when nonlinear-

[1] Equations (23) and (24) reduced to a two-machine case apply.

ity is involved. With linear characteristics, on the other hand, the situation is readily handled. With two points of connection to the system, the equivalent representation becomes a four-terminal network, with three points of connection, a five-terminal network, and so on. It is believed that most practical cases seldom require the retention of more than two tie-in points.

The method for determining the equivalent of a machine group with its associated circuits and loads actually most frequently used in transient-stability studies assumes that all machine, circuit, and load characteristics are linear and that the voltages behind transient reactance of the various machines are equal and in phase. It should be noted, however, that also with this method it becomes necessary to reevaluate the equivalent circuit when the load conditions are changed.

The analytical reduction can be performed by successive applications of Δ-Y and Y-Δ transformations. Sometimes also the more general star-mesh transformations may be necessary. This has been fully covered in Chap. VI in connection with reductions in the steady state. The process may be somewhat lengthy and complicated, but the principles involved are obviously straightforward.

In a relatively complicated setup, much time and effort may be saved by making use of a Network Analyzer. The group of machines with its associated circuits and loads are set up on the analyzer, and all the machine reactances (on their internal side) are connected together to a common bus. The identity of this as well as the identity of the connection points with the system is to be preserved. By proper voltage and current measurements from which impedances may be computed at these buses, the elements of the equivalent circuit may readily be determined.

Representation by an Equivalent Generator and an Equivalent Load.—In discussing the representation of a receiving-end system in connection with the problem of steady-state stability (Chap. VI), the advantage was pointed out of using a separate generator and equivalent load both connected directly to the system bus. In general, this is practical only when connection to the system occurs at a single point and eliminates the necessity of completely redetermining the equivalent circuit when load conditions change. Obviously the same idea may be extended to the transient case. The very same methods may be applied

here, the difference being, of course, that the machines must be represented by their transient reactances instead of their synchronous reactances. Also, if the loads are not to be represented by constant impedances transient load characteristics must be introduced (assuming that there is a difference between these and the steady-state characteristics).[1]

Equivalent Generator.—The size of the equivalent generator is determined from the consideration that the short-circuit kilovolt-amperes which it would supply to a symmetrical three-phase short circuit at its terminals are the same that the machine group to be represented would supply to a similar short circuit at the point of interconnection to the system.

These short-circuit kilovolt-amperes may be determined on three bases (similar to those used for steady-state conditions) *viz.*, (1) with the voltages behind transient reactance of all generators phase-displaced by the proper amount and having the proper values to supply the loads on the system and with the loads attached; (2) with the voltages behind transient reactance of all generators in phase but equal to their actual values with load on the system without, however, considering the load in obtaining the short-circuit kilovolt-amperes; and (3) with the voltages behind transient reactance of all generators in phase and equal to unity and omitting the loads.

It is not considered necessary to discuss the procedure in further detail, reference being made to Chap. VI. Whether

[1] Network-analyzer studies of receiving-end-system representation for transient-stability analyses have been made at the Massachusetts Institute of Technology in the following Master's theses:

Keefe, W. E., and W. E. Peterson, "Equivalent Circuits Representing Receiving-End Systems for Transient Stability Studies," 1935.

Lebenbaum, Paul, Jr., "Equivalent Circuits for the Representation of Large Metropolitan-area Loads and Generating Stations in Transient Stability Studies," 1936.

Hori, Taichiro, "Equivalent Circuits for the Representation of Large Metropolitan-area Loads and Generating Stations in Transient Stability Studies," 1936.

Falls, O. B., Jr., and C. Hobson, "Equivalent Circuits for the Representation of Large Metropolitan-area Loads and Generating Stations in Transient Stability Studies," 1937.

Perouse, M. J. and R. M. Stinglhamber, "Equivalent Circuits for the Representation of Large Metropolitan Areas in Transient Stability Studies," 1937.

These studies are being continued.

carried through by computation or on a Network Analyzer, the ultimate result will be an equivalent generator of a certain size with a certain equivalent transient reactance attached directly at the point of interconnection to the system.

Equivalent Load.—The equivalent load is also obtained by methods similar to those applying to steady-state conditions as discussed in Chap. VI (except that transient characteristics may have to be used). The ultimate result is a single equivalent load at the point of interconnection to the system. The equivalent load may be given approximate load characteristics if desired, or considered as a constant-impedance load. In either case adjustments as a consequence of varying load conditions may be directly reflected in the equivalent load.

An Alternative Approximate Method for Determining the Equivalent Generator and the Equivalent Load.—This method involves the use of the size of equivalent generator and equivalent load determined on the basis of steady-state considerations. To this generator is then arbitrarily assigned a transient reactance of reasonable value. It may even be taken equal to what may be considered normal for the types of machine involved, or a value may be selected based upon the weighted average of the actual transient reactances of the machines in the group.

With respect to the size of the equivalent load, this will be the same as for steady-state conditions. If actual load characteristics are to be applied to it, these would be transient characteristics rather than the steady-state characteristics if there is any difference between the two.

Determination of Output.—As previously stated one of the essentials in a point-by-point stability analysis is the determination of the outputs of the machines for each step in the analysis. In the multimachine system this may be done by (1) calculation, and (2) measurements on a Network Analyzer. In a complicated system, the latter is usually the only practical procedure from the point of view of time and labor.

Output by Calculation.—The output of any machine depends upon the internal voltages of all machines in the system, the phase position of all machines with respect to a common reference axis (or the relative displacement angles among the machines, which in effect amounts to the same thing), and the electrical constants of the connecting network as well as of the loads.

Power-flow equations can be readily established only when each machine is represented by a single impedance and the loads by constant impedances. The general type of output equation is then the same as would hold for steady-state conditions[1] and may for the nth machine be written

$$
\begin{aligned}
P_n + jQ_n &= \frac{E_n^2}{Z_{nn}} \cos \theta_{nn} - \frac{E_nE_1}{Z_{n1}} \cos (\delta_n - \delta_1 + \theta_{n1}) - \frac{E_nE_2}{Z_{n2}} \cos (\delta_n - \delta_2 + \theta_{n2}) - \cdots \\
&\quad + j\left[-\frac{E_n^2}{Z_{nn}} \sin \theta_{nn} + \frac{E_nE_1}{Z_{n1}} \sin (\delta_n - \delta_1 + \theta_{n1}) + \frac{E_nE_2}{Z_{n2}} \sin (\delta_n - \delta_2 + \theta_{n2}) + \cdots \right] \\
&= \frac{E_n^2}{Z_{nn}} \sin \alpha_{nn} + \frac{E_nE_1}{Z_{n1}} \sin (\delta_n - \delta_1 - \alpha_{n1}) + \frac{E_nE_2}{Z_{n2}} \sin (\delta_n - \delta_2 - \alpha_{n2}) + \cdots \\
&\quad + j\left[-\frac{E_n^2}{Z_{nn}} \cos \alpha_{nn} + \frac{E_nE_1}{Z_{n1}} \cos (\delta_n - \delta_1 - \alpha_{n1}) + \frac{E_nE_2}{Z_{n2}} \cos (\delta_n - \delta_2 - \alpha_{n2}) + \cdots \right] \qquad (23)
\end{aligned}
$$

or, since $\delta_n - \delta_1 = \delta_{n1}$, etc.,

$$
\begin{aligned}
P_n + jQ_n &= \frac{E_n^2}{Z_{nn}} \cos \theta_{nn} - \frac{E_nE_1}{Z_{n1}} \cos (\delta_{n1} + \theta_{n1}) - \frac{E_nE_2}{Z_{n2}} \cos (\delta_{n2} + \theta_{n2}) - \cdots \\
&\quad + j\left[-\frac{E_n^2}{Z_{nn}} \sin \theta_{nn} + \frac{E_nE_1}{Z_{n1}} \sin (\delta_{n1} + \theta_{n1}) + \frac{E_nE_2}{Z_{n2}} \sin (\delta_{n2} + \theta_{n2}) + \cdots \right] \\
&= \frac{E_n^2}{Z_{nn}} \sin \alpha_{nn} + \frac{E_nE_1}{Z_{n1}} \sin (\delta_{n1} - \alpha_{n1}) + \frac{E_nE_2}{Z_{n2}} \sin (\delta_{n2} - \alpha_{n2}) + \cdots \\
&\quad + j\left[-\frac{E_n^2}{Z_{nn}} \cos \alpha_{nn} + \frac{E_nE_1}{Z_{n1}} \cos (\delta_{n1} - \alpha_{n1}) + \frac{E_nE_2}{Z_{n2}} \cos (\delta_{n2} - \alpha_{n2}) + \cdots \right] \qquad (24)
\end{aligned}
$$

The first of these involves, as will be noted, angles with respect to a common axis while the second makes use of relative angles.

[1] The first and second part of equation (23), above, are identical with equations (9) and (12) respectively, of Chap. VIII.

It is evident that the latter are positive when the nth machine *leads* the other machine in question. The voltage symbols represent magnitudes of the internal voltages, and the impedance symbols represent magnitudes of the driving-point and transfer impedances, respectively. The θ's are the angle of the corresponding impedances, while the α's represent the complements of the impedance angles. In most stability analyses, it is usually only the active-power output which is of interest and needs evaluation.

If each machine is represented by its direct-axis transient reactance, a method which is frequently adopted for salient- as well as non-salient-pole machines, all internal voltages should be those behind transient reactance, in other words (using the nomenclature introduced in Chap. XII) $E_n = e_{in}$, $E_1 = e_{i1}$, etc.

All driving-point and transfer impedances entering into the equations should be evaluated from a setup utilizing direct-axis transient reactances for the machines. The angles involved will then, according to the equation used, either represent the phase displacements of these voltage vectors with respect to a common reference or the relative phase displacement among them. With this representation, the true angular position of the machine rotors is not obtained.

If each machine is represented by its quadrature-axis reactance, a procedure sometimes followed with salient-pole machines in order that the correct angles of the field structures may be properly reflected, voltages behind quadrature-axis reactance must be used. Thus: $E_n = e_{Dn}$, $E_1 = e_{D1}$, etc. These voltage vectors, as previously demonstrated, rigidly follow the field structure of a salient-pole machine. If the machine is a non-salient-pole machine and a representation is desired which will give the correct rotor angles, direct-axis synchronous reactances and voltages behind these (*i.e.*, the excitation voltages) should be used. It is to be recalled that these voltages (voltage behind quadrature-axis reactance and excitation voltage) may never be considered as constant during the oscillations of the machines, neither when the usual simplifying assumptions are made (see Chap. XII) nor when exciter action is included in the analysis.

It should not be inferred from the above that in a multi-machine system all machines *must* be represented in the same

manner. This is, of course, not necessary. On the contrary, it may at times be desirable to apply a different type of representation to some of the machines than is used for others. This may be due to the fact that the system contains both salient- and non-salient-pole machines, assuming that it is desired to distinguish between their characteristics beyond what is reflected by differences in the value of a single constant (for instance, the direct-axis transient reactance). Different representation is more likely to be introduced, however, in order to increase the accuracy of treatment of those machines which are of particular interest in the problem at hand (for instance, the machines in a contemplated station, machines which are critical under the examined fault conditions, etc.) especially when refinements, like exciter response, etc., are to be taken into account.

It is obvious that the use of the output equations rapidly becomes complicated in a multimachine system. Furthermore, and this usually is still more serious, it is a very complicated and laborious task to obtain the necessary driving-point and transfer impedances entering into the equations by purely computational processes.

Output by Network Analyzer.—The system in its final form for the point-by-point analysis, *i.e.*, after the required amount of preliminary reduction, grouping of machines, etc., is set up on the analyzer. If within analyzer limits, it may sometimes be convenient to represent the system more completely even though in the actual analysis certain machines will be considered to swing together as a group. The group feature can obviously be taken care of by merely connecting the reactances of the machines which go with the group to the same generating unit (phase shifter). Such complete representation may be of advantage if the effect of a number of faults in different parts of the system is to be studied, since for the various faults it may not always be feasible to consider the same group arrangements.

On the network analyzer, each machine must be represented by a single reactance (previously discussed in Chap. XII). This will, for salient-pole machines, be either the direct-axis transient reactance or the quadrature-axis reactance and, for non-salient-pole machines, the direct-axis transient reactance or direct-axis synchronous reactance.

Further Comments on the Use of a Network Analyzer in Connection with Transient-stability Analyses of Multimachine Systems.—The positive-sequence system is represented on the analyzer and the necessary adjustments made to secure the proper steady-state operating conditions. The fault is then applied at the proper point. Its subsequent clearing at the correct time is accomplished by merely opening the appropriate connections. If the fault is a solid three-phase symmetrical short circuit, the proper representation on the analyzer is a metallic connection between the fault point and the neutral or ground bus. If the arc resistance of the fault is to be considered, the connection just referred to should include this resistance.

If the fault to be represented is dissymmetrical (as for instance a line-to-ground, a line-to-line, or a double-line-to-ground, short circuit), the proper representation is an impedance between the fault point and the ground bus. This impedance usually referred to as the *fault shunt* depends upon the type of fault and on the total negative- and zero-sequence impedances of the system as viewed from the point of fault.[1] For a line-to-ground fault, the impedance equals the negative- and zero-sequence system impedances in series; for a double-line-to-ground fault, the negative- and zero-sequence impedances in parallel; and for a line-to-line fault, the negative-sequence impedance alone.

The necessary negative- and zero-sequence system impedances may be determined by direct measurements on an analyzer setup of the system's negative- and zero-sequence networks, respectively. This may be done (and this is undoubtedly the preferable procedure when the system is a complicated one) prior to the stability analysis. If the system is less elaborate so as to permit the simultaneous setup of the positive-, negative-, and zero-sequence networks without exceeding the limitations of the board, the fault effect may be obtained by interconnecting the sequence networks in the proper manner.[1] This method, when applicable for physical reasons, possesses an advantage in such problems where it is desired not only to measure positive-sequence quantities (and only these quantities can be obtained from the positive-sequence network alone) but also certain negative- and zero-sequence quantities. This may be desirable in such problems, for instance, as involve determination of short-

[1] See Chap. XI.

circuit currents or deal with circuit-breaker duties and relaying.

If it is not possible simultaneously to set up all three sequence networks, the required negative- and zero-sequence quantities may still be determined relatively easily provided values of the proper transfer constants are determined by previous measurements on the negative- and zero-sequence networks separately. These transfer constants (complex numerics) relate either the negative- or zero-sequence fault current with the corresponding sequence quantity in some other part of the system. During the measurements on the positive-sequence setup, the current flowing in the fault shunt is recorded. From this the negative- or zero-sequence fault current can readily be obtained and thus the above-mentioned transfer constant applied to give the desired quantity at the other point.

The application of the fault shunt (or the corresponding interconnection of the sequence networks) as outlined above changes the voltages as well as the active- and reactive-power distribution in the system. The analyzer readings of machine outputs together with the proper inputs determine the power differentials tending either to accelerate or decelerate the machines, as the case may be. These differentials are then used in the ordinary manner for computing the amount of angular swing of each machine during a short interval of time. Hence, the angular position of all machines is obtained at the end of the interval permitting a corresponding displacement of all phase shifters to be made on the analyzer. Direct measurement now once more gives the output of the machines and hence new values of accelerating or decelerating power differentials so that the further angular travel of each machine during the second interval of time may be computed. By continuing the point-by-point analysis in this manner, obtaining each time the network solution from the analyzer, the displacements of the various machines may be mapped out over a sufficient length of time so as to permit a reliable conclusion as to whether or not the system will be stable under the given conditions of fault.

Schedules for Point-by-point Analysis of Transient Stability.—As previously pointed out, the point-by-point analysis is best carried out according to a definite schedule. There are several ways in which such schedules may be arranged. The number of items to be included depends upon the results desired and upon the degree of refinement to be used.

If the aim of the analysis is merely to determine whether or not the system is stable, output power and angular displacement are the principal quantities which require recording. The absolute angular velocities with which the machines oscillate are necessary if the inputs are influenced by governor action, whereas

TRANSIENT STABILITY ANALYSIS
Point-by-point Analysis Using Network Analyzer
SCHEDULE A: CONSTANT VOLTAGE "BEHIND TRANSIENT REACTANCE"; DAMPING NEGLECTED

Generating plant:
Units operating:
Transformer banks operating:
Loads:
Remarks:

Row number	Item	Where read or how obtained	Initial conditions		
			Before fault	After fault	
1	n	(No of time instant; interval)			
2	t	$= n\Delta t$ (seconds)			
3	e'_d	Read: Bus . to N			
4	e_t	Read: Bus to N			
5	i	Read: Phase shifter G			
6	i_F	Read: Fault shunt			
7	P_s	Read: Phase shifter G			
8	ΔP	$= P_i - P = P_i - P_s$			
9	$k\Delta P_{n-1}$	$k = 2k'' = \dfrac{(\Delta t)^2}{M}$			
10	$\Delta\delta$	$\Delta\delta_n = \Delta\delta_{n-1} + k\Delta P_{n-1}$			
11	δ	$\delta_n = \delta_{n-1} + \Delta\delta_n$			
12	ω	$\omega_{n-1/2} = \dfrac{\Delta\delta_n}{\Delta t}$			

NOTE Swing computations proper (rows 9–12, inclusive) are based on Method II.

inclusion of (low-frequency) damping usually requires recording of relative velocities (or slips). If regulator and exciter action is to be included, the terminal voltages (or, in general, the voltages actuating the automatic regulators) must also be kept track of. In addition, it may be necessary to calculate (or measure, if use is made of a network analyzer) and record currents in such problems which also bear on breaker duties and relaying.

Certain basic schedules were suggested in the preceding chapter. These were of the "horizontal" type with items of entry as column headings. A "vertical" schedule, however, may be

TRANSIENT STABILITY ANALYSIS
Point-by-point Analysis Using Network Analyzer
SCHEDULE B: CONSTANT FLUX LINKAGES IN DIRECT AXIS; DAMPING NEGLECTED

Generating plant:. . . . .
Units operating:. . . . .
Transformer banks operating:. . . . .
Loads:. . . . .
Remarks:. . . . .

Row number	Item	Where read or how obtained	Initial conditions		
			Before fault	After fault	
1	n	(No. of time instant; interval)			
2	t	$= n\Delta t$ (seconds)			
3	e_D	Read: Bus. . . .to $N = e'_d + i_d(x_q - x'_d)$			
4	e_t	Read: Bus. . . .to N			
5	i_d	Read: Phase shifter G			
6	$i_d(x_q - x'_d)$				
7	i_F	Read: Fault shunt			
8	P_s	Read: Phase shifter G			
9	ΔP	$= P_i - P = P_i - P_s$			
10	$k\Delta P_{n-1}$	$k = 2k'' = \dfrac{(\Delta t)^2}{M}$			
11	$\Delta\delta$	$\Delta\delta_n = \Delta\delta_{n-1} + k\Delta P_{n-1}$			
12	δ	$\delta_n = \delta_{n-1} + \Delta\delta_n$			
13	ω	$\omega_{n-1/2} = \dfrac{\Delta\delta_n}{\Delta t}$			

NOTE: Swing computations proper (rows 10–13, inclusive) are based on Method II.

more practical in many instances, especially where the point-by-point analysis has to be carried out over an appreciable period of time.[1] In this type of schedule, the items are arranged vertically as rows.

[1] The height of this schedule can usually be kept equal to standard letter size (11 in.). Thus, independent of the length of the analysis and its horizontal dimension, this schedule will require folding in one dimension only and will bind conveniently in letter-size files.

Examples of schedules which have been found practical are presented. Each of these applies to some one synchronous machine in a multimachine system and involves different basic

TRANSIENT STABILITY ANALYSIS

Point-by-point Analysis Using Network Analyzer

SCHEDULE C: EXCITER ACTION INCLUDED (NEGLECTING EFFECT OF SATURATION); DAMPING NEGLECTED

Generating plant:....................................

Units operating:....................................

Transformer banks operating:........................

Loads:..

Remarks:..

Row number	Item	Where read or how obtained	Initial conditions		
			Before fault	After fault	
1	n	(No. of time instant; interval)			
2	t	$= n\Delta t$ (seconds)			
3	e_D	Read: Bus....to N			
4	e_t	Read: Bus....to N			
5	i_d	Read: Phase shifter G			
6	$i_d(x_q - x'_d)$				
7	i_F	Read: Fault shunt			
8	e_f	$e_{f(0)} = i_{f(0)}$. Other values from exciter-response curve.			
9	i_f	$= e'_d + i_d(x_d - x'_d)$			
10	$\Delta e'_d$	$= \frac{\Delta t}{T_d}(e_f - i_f)_{n-1}$			
11	e'_d	$= e'_{d(n-1)} + \Delta e'_d$			
12	P_s	Read: Phase shifter G			
13	ΔP	$= P_i - P = P_i - P_s$			
14	$k\Delta P_{n-1}$	$k = 2k'' = \frac{(\Delta t)^2}{M}$			
15	$\Delta\delta$	$\Delta\delta_n = \Delta\delta_{n-1} + k\Delta P_{n-1}$			
16	δ	$\delta_n = \delta_{n-1} + \Delta\delta_n$			
17	ω	$\omega_{n-\frac{1}{2}} = \frac{\Delta\delta_n}{\Delta t}$			

NOTE: Swing computations proper (rows 14–17, inclusive) are based on Method II.

assumptions and varying degrees of refinements. It should be borne in mind, of course, that they are all subject to such modification and changes (in number and order of items, etc.) as may

be dictated by specific requirements in a particular analysis. The following schedules are included:

Schedule A.—Constant voltage behind transient reactance. Damping neglected.

Schedule B.—Constant flux linkages in the direct axis. Damping neglected.

Schedule C.—Exciter action included (neglecting effect of saturation in the alternator). Damping neglected.

Schedule D.—Exciter action included (neglecting effect of saturation in the alternator). High-resistance damping included.

Schedule E.—Exciter action included (neglecting effect of saturation in the alternator). High- and low-resistance damping included.

These schedules as set up are particularly adapted to Network-analyzer solutions. They are, however, essentially equally applicable to solutions which are exclusively based on analytical methods. In the latter case the entries which are now indicated as "read" in some manner on the analyzer would be calculated by the appropriate formulas.

It is considered logical to include the schedules at the end of the present chapter in spite of the fact that certain details entering into *Schedules C, D,* and *E* have not as yet been covered. Thus the problem of exciter action will be comprehensively discussed in the three chapters which are to follow. In order fully to appreciate the manner in which the effect of regulator action and exciter response is brought into the point-by-point analysis, a study of these chapters is desirable. Chapter XVII is particularly pertinent in this respect.

Damper action, which is included in *Schedules D* and *E,* is treated in some detail in Chap. XIX. Consideration of the effect of high-resistance damper windings in a certain machine requires merely the keeping track of the negative-sequence currents supplied by this machine as the point-by-point analysis progresses. This can be done by calculation using appropriately developed formulas for the case at hand, or by measurement on the Network Analyzer, as previously pointed out.

The swing computations themselves, *i.e.*, the determination of the angular displacement of a certain machine during a short interval of time when the accelerating or decelerating power differential is known, depends upon the method adopted. It

will be recalled that four methods were described and their relative merits discussed in Chap. XIII. The swing computa-

TRANSIENT STABILITY ANALYSIS
Point-by-point Analysis Using Network Analyzer
SCHEDULE D EXCITER ACTION INCLUDED (NEGLECTING EFFECT OF SATURATION); HIGH-RESISTANCE DAMPING INCLUDED

Generating plant
Units operating
Transformer banks operating
Loads·
Remarks

Row number	Item	Where read or how obtained	Initial conditions: Before fault	Initial conditions: After fault	
1	n	(No of time instant, interval)			
2	t	$= n\Delta t$ (seconds)			
3	e_D	Read Bus to N			
4	e_t	Read Bus to N			
5	i_d	Read Phase shifter G			
6	$i_d(x_q - x'_d)$				
7	i_F	Read Fault shunt			
8	e_f	$e_{f(0)} = i_{f(0)}$ Other values from exciter-response curve			
9	i_f	$= e'_d + i_d(x_d - x'_d)$			
10	$\Delta e'_d$	$= \frac{\Delta t}{T_d}(e_f - i_f)_{n-1}$			
11	e'_d	$= e'_{d(n-1)} + \Delta e'_d$			
12	P_s	Read Phase shifter G			
13	i^-	$i^- = k^- i_F$			
14	P_d^-	$= P_0^-(i^-)^2 \cong (R^- - R^1)(i^-)^2$			
15	ΔP	$= P_i - P = P_i - P_d^- - P_s$			
16	$k\Delta P_{(n-1)}$	$k = 2k'' = \frac{(\Delta t)^2}{M}$			
17	$\Delta\delta$	$\Delta\delta_n = \Delta\delta_{n-1} + k\Delta P_{n-1}$			
18	δ	$\delta_n = \delta_{n-1} + \Delta\delta_n$			
19	ω	$\omega_{n-\frac{1}{2}} = \frac{\Delta\delta_n}{\Delta t}$			

NOTE Swing calculations proper (rows 16–19, inclusive) are based on Method II

tions in the schedules presented herein are predicated upon Method II which is both simple and essentially devoid of cumula-

Transient Stability Analysis

Point-by-point Analysis Using Network Analyzer

SCHEDULE E: exciter action included (neglecting effect of saturation); high- and low-resistance damping included

Generating plant:......................................

Units operating:......................................

Transformer banks operating:..........................

Loads:..

Remarks:..

Row number	Item	Where read or how obtained	Initial conditions		
			Before fault	After fault	
1	n	(No. of time instant; interval)			
2	t	$= n\Delta t$ (seconds)			
3	e_D	Read: Bus....to N			
4	e_t	Read: Bus....to N			
5	i_d	Read: Phase shifter G			
6	$i_d(x_q - x'_d)$				
7	i_F	Read: Fault shunt			
8	e_f	$e_{f(0)} = i_{f(0)}$. Other values from exciter-response curve			
9	i_f	$= e'_d + i_d(x_d - x'_d)$			
10	$\Delta e'_d$	$= \frac{\Delta t}{T_d}(e_f - i_f)_{n-1}$			
11	e'_d	$= e'_{d(n-1)} + \Delta e'_d$			
12	P_e	Read: Phase shifter G			
13	i^-	$i^- = k^- i_F$			
14	P_d^-	$= P_0^-(i^-)^2 \cong (R^- - R^+)(i^-)^2$			
15	s_{12}	$(s_{12})_n = \frac{1}{360f}\left(\frac{d\delta_{12}}{dt}\right)_n \cong \frac{1}{360f}$ $\frac{(\Delta\delta_1 - \Delta\delta_2)_n}{\Delta t} = \frac{1}{360f}(\omega_1 - \omega_2)_{n-\frac{1}{2}}$			
16	s_{13}	$(s_{13})_n = \frac{1}{360f}\left(\frac{d\delta_{13}}{dt}\right)_n \cong \frac{1}{360f}$ $\frac{(\Delta\delta_1 - \Delta\delta_3)_n}{\Delta t} = \frac{1}{360f}(\omega_1 - \omega_3)_{n-\frac{1}{2}}$			
17	P_{d12}	$= a_{12}\sin^2(\delta_1 - \delta_2) + b_{12}\cos^2(\delta_1 - \delta_2)$			
18	P_{d13}	$= a_{13}\sin^2(\delta_1 - \delta_3) + b_{13}\cos^2(\delta_1 - \delta_3)$			
19	P_d^+	$= P_{d12}^+ + P_{d13}^+ = P_{d12}s_{12} + P_{d13}s_{13}$			
20	ΔP	$= P_i - P = P_i - P_e - P_d^+ - P_d^-$			
21	$k\Delta P_{(n-1)}$	$k = 2k'' = \frac{(\Delta t)^2}{M}$			
22	$\Delta\delta$	$\Delta\delta_n = \Delta\delta_{n-1} + k\Delta P_{n-1}$			
23	δ	$\delta_n = \delta_{n-1} + \Delta\delta_n$			
24	ω	$\omega_{n-\frac{1}{2}} = \frac{\Delta\delta_n}{\Delta t}$			

Note: The particular arrangement in this schedule for calculation of low-resistance damping (rows 15–19, inclusive) applies to Generator 1 in a three-machine system. Suitable modifications are readily made when a different number of machines is involved (see discussion of schedules in Chap. XIX). Swing calculations proper (rows 21–24, inclusive) are based on Method II.

tive error. If it is desired to use other methods, corresponding modifications in the last four rows of the schedules will be necessary.

EXAMPLE 1

Statement of Problem

Consider the system described in Example 1 of Chap. VI and for which the steady-state stability limit was determined in Example 1, Chap. IX.

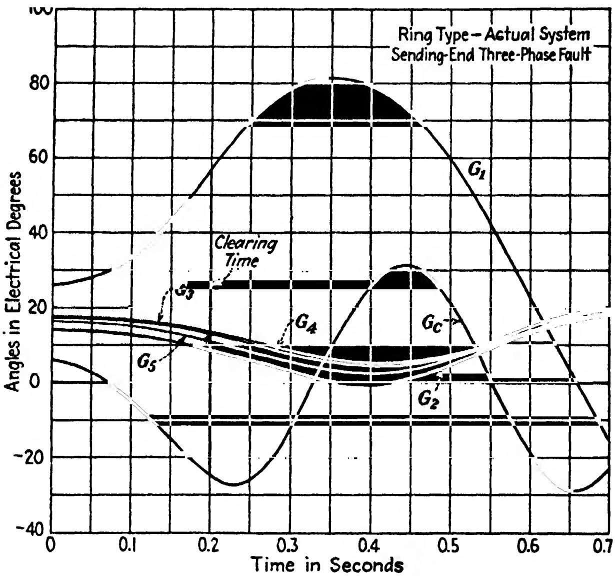


FIG. 216.—Swing curves for the synchronous machines of the system analyzed in Example 1, illustrating the effect of a transmission line fault and its subsequent clearing in 0.2 second. The curves were determined by a point-by-point analysis based on Method II associated with measurements on the M.I.T. Network Analyzer.

Determine by means of a point-by-point analysis, in conjunction with measurements on the Network Analyzer, swing curves for the synchronous machines in this system and estimate the transient-power limit of the hydroelectric station when a solid three-phase fault occurs near the sending-end high-tension bus of the transmission line and complete clearing takes place at 0.2 sec. by simultaneous action of the automatic circuit breakers at the two ends of the faulted section:

a. Handling the system as a multimachine system.

b. Representing the metropolitan system to which the transmission line feeds by an equivalent generating station and an equivalent load directly at the receiving-end bus.

The synchronous machines are to be represented by their direct-axis transient reactances and the voltages behind these assumed constant. The

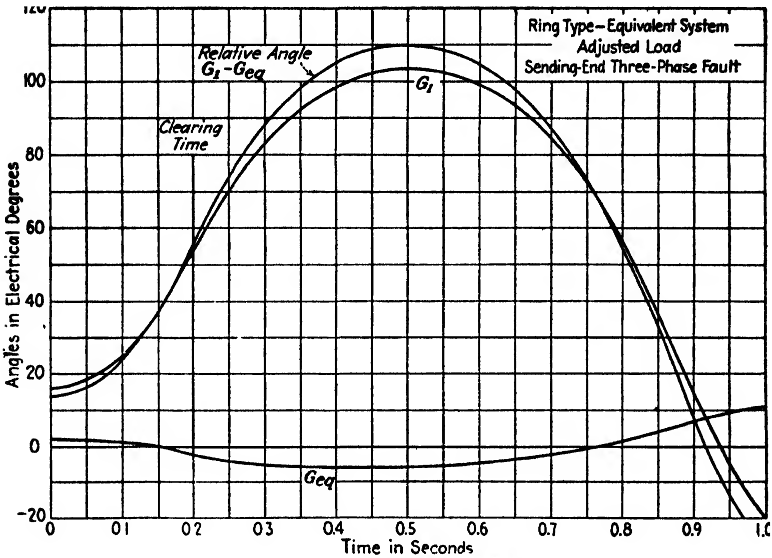


FIG. 217.—Swing curves for the system analyzed in Example 1 when the receiving-end machines are represented by an equivalent generator directly at the receiving-end bus. (Additional explanation in caption to Fig. 216.)

load characteristics as given in Example 1, Chap. VI, are to be used in (*a*) and (*b*). Part (*b*) is to be repeated using a constant-impedance load. Additional data necessary for the transient analysis are as follows:

Generating station	Rating, kva.	Direct-axis synchronous reactance, per unit	Direct-axis transmission reactance, per unit	Negative-sequence reactance, per unit	Inertia constant, kw./elec. deg./sec.[2]
G_1	262,500	0.90	0.30	0.36	63.9
G_2	100,000	1.00	0.25	0.13	101.9
G_3	75,000	1.00	0.25	0.13	76.4
G_4	60,000	1.00	0.25	0.13	61.1
G_5	60,000	1.00	0.25	0.13	61.1
Synchronous condenser....	125,000	1.80	0.40	0.25	17.3

Solution[1]

a. Pertinent steady-state conditions in the system at a certain system load (corresponding to less than half the normal transmission-line loading) are:

	G_1	G_2	G_3	G_4	G_5	G_6	Transmission line
P kw	110,000	48,000	55,000	43,000	43,000	0	101,000
Q kva	51,000	−28,200	−39,000	−27,500	−34,000	−33,000	−52,000
e_i per cent	95	112	120.5	118.5	121.5	115	
δ_i elec. deg.	26	14.2	17.6	17.4	16.4	6	

Here δ_i represents the angular displacement between the voltage behind direct-axis transient reactance and a standard synchronous reference.

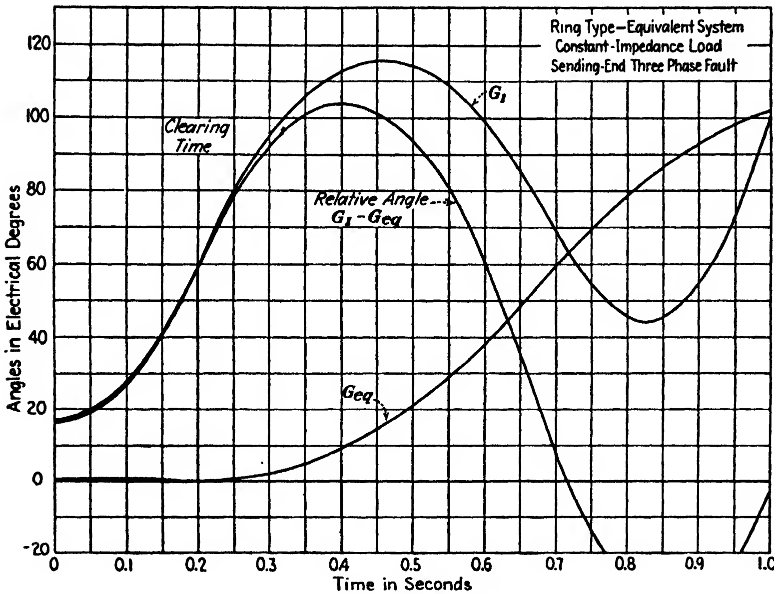


FIG. 218.—Swing curves for the system analyzed in Example 1 when the receiving-end machines are represented by an equivalent generator directly at the receiving-end bus. (Additional explanation in caption to Fig. 216.)

Method II (see preceding chapter) was used in carrying out the point-by-point analysis. The swing curves so obtained are plotted in Fig. 216.

[1] Results quoted herein taken from thesis by Pérouse and Stinglhamber, *loc. cit.*

b. The steady-state conditions for the equivalent system at two values of system load (the reason for using these particular loads will appear later) are:

	G_1		G_{eq}		Transmission line	
	Load 1	Load 2	Load 1	Load 2	Load 1	Load 2
P kw	129,000	145,000	185,000	186,000	119,000	132,000
Q kva	50,000	48,000	−176,000	−196,000	−15,000	−12,000
e_i per cent	96	97	121	123		
δ_i elec. deg	16	17	2	0.5		

The swing curves for load 1 using the actual load characteristics are given in Fig. 217. The swing curves for load 2 using a constant-impedance load are given in Fig. 218.

In each of the above cases the load was so chosen that an addition of 5 per cent to the system load will make the remote station G_1 unstable. Actually point-by-point solutions were made for these additions in load, and the load limit was determined by interpolation. The table below gives the results.

	Output of G_1, kw.
Case *a*. Actual load characteristics:	
Stable	110,000
Limit	125,000
Unstable	129,000
Case *b*. Actual load characteristics:	
Stable	129,000
Limit	134,000
Unstable	145,000
Case *b*. Constant-impedance load:	
Stable	145,000
Limit	156,000
Unstable	166,000

CHAPTER XV

ANALYSIS OF UNLOADED EXCITERS

When automatic voltage regulators are installed, the exciters and their action play an important part in the problem of transient stability. Their operating characteristics, especially with reference to their terminal voltage variations when the regulator inserts or short-circuits resistance in their field circuit, influence the stability of the system in no small degree. In order to appreciate the magnitude and nature of this effect, it is necessary to understand the transient processes in the exciter itself and to know the factors which govern them.

Were the iron in the exciters and the generators unsaturated, *i.e.*, their magnetization curves straight lines, the transient behavior could be analyzed by such methods and equations as apply to linear circuits. Only when operation actually takes place below the point where saturation effects are noticeable, will such methods give rigorous results, while, in general, they may be applied as approximations only. In the discussion of exciter behavior, therefore, which is to follow, cognizance will be taken of the presence of saturation.

While linear circuits give rise to differential equations with *constant coefficients*, variable coefficients characterize those representing the nonlinear circuits. There are in general four methods available for the solution (integration) of these differential equations, *viz.:*

1. Formal analytical solutions.
2. Graphical solutions.
3. Point-by-point solutions.
4. Integraph solutions.

In order to obtain an analytical solution by formal methods, it is necessary (1) that the variable coefficients be expressed mathematically in terms of one of the variables, and (2) that the differential equation as it then stands actually be integrable. In connection with the exciter problem, this means that the

magnetization curve, or a function thereof, must be represented mathematically and this expression properly inserted in the differential equation. As a rule it is the Fröhlich equation which serves this purpose, although there are alternatives that also may be employed.

In transient electrical problems, the time variation of the electrical quantities are of interest, and in the solutions it is, as a rule, more convenient to have these quantities explicitly expressed in terms of the time than vice versa. When formal methods are used to determine exciter transients, explicit solutions for the electrical quantities, *i.e.*, voltage and current, cannot always be attained. More often the result is an explicit solution for the time in terms of one of the electrical quantities.

The graphical solutions involve the carrying out of the integration of the differential equation by graphical means. Upon having adapted the form of the differential equation to this method of solution, the integrand is plotted and the integral evaluated by measuring areas. This can be done by a planimeter, if such an instrument is available, or else with sufficient ease and accuracy by counting little squares on the cross-section paper. The methods of graphical integration are very effective in connection with analysis of exciter problems.

The point-by-point solutions are applicable to any transient problem, and it makes no difference whether the circuit is linear or nonlinear. The point-by-point methods are often convenient in exciter analysis and may be applied alone or in conjunction with graphical methods, as may be practical in the case of the loaded exciter.

By the integraph method the solutions are obtained by machine integration and recorded graphically. Such solutions may be rapid as well as accurate, but the application of this method obviously depends on whether or not an integraph is accessible, which frequently will not be the case.

Independent of the method used, there is one factor omitted in the treatment which follows: *viz.*, the possible screening effect of eddy currents set up in the iron when the flux varies.[1] This

[1] Weber, Ernst, "Field Transients in Magnetic Systems," *Trans. A.I.E.E.*, Vol. 50, p. 1234, December, 1931; Wagner, C. F., "Transients in Magnetic Systems," *Trans. A.I.E.E.*, Vol. 53, p. 418, March, 1934.

Rüdenberg, R., "Elektrische Schaltvorgänge," 3d ed., p. 327, Julius Springer, Berlin, 1933.

effect is uncertain at best, and depends upon the nature of the magnetic circuit and upon the rapidity of flux variation. The effect is noticeable particularly with thick solid cores of low resistivity, but should be negligibly small with laminated cores of low-loss irons, especially when the flux does not change too rapidly. Since the eddy currents counteract the change in flux, their effect is always to lengthen the transient process and virtually to add to the time constant of the circuit.

Unfortunately, exact calculation of eddy currents is a complicated proposition, and rigorous inclusion in the analysis of their effect, therefore, is not easy. The most practical way of

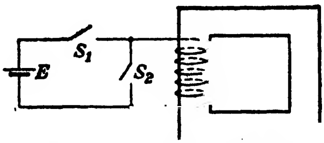


FIG. 219.—Schematic diagram of iron-cored coil.

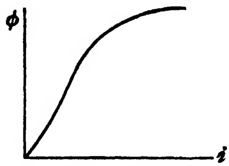


FIG. 220.—Magnetization curve of iron-cored coil.

approximately reflecting their effect is probably to shade the time constants upward.[1] If handled in this manner the various methods of solution developed below remain directly applicable.

Omitting the consideration of eddy currents entirely may give rise to a certain error in some instances. It is believed, however, that this error will not be serious in most exciter and other iron-core-circuit problems where the variations in flux are relatively slow.

The Iron-cored Coil.—As a preliminary to the study of the transient behavior of exciters, it may be of value to consider the simple iron-cored coil. The principles utilized in connection with this problem are, with the necessary modifications, applicable also in the analysis of the exciters.

Referring to Fig. 219, a direct-current voltage can be suddenly applied to the coil by the closing of the switch S_1. Also, the coil can be short-circuited and made to discharge by means of the switch S_2.[2] The magnetization curve for the coil is available, as

[1] RÜDENBERG, *loc. cit.*, p. 327.

[2] The carrying out of this operation in the laboratory would obviously necessitate the protection of the source by a current-limiting resistance or by an automatic breaker.

shown in Fig. 220. Assume that this curve gives the relationship between the *iron flux proper* and the current. In other words, any leakage flux is not included, the data for the curve, for instance, being obtained by use of a search coil of small dimensions wound tightly on the core.

The differential equations for the charge of the coil is then

$$N\frac{d\phi}{dt} + L'\frac{di}{dt} + Ri = E \tag{1}$$

and for the discharge

$$N\frac{d\phi}{dt} + L'\frac{di}{dt} + Ri = 0 \tag{2}$$

The symbols employed have the following meaning:

E = direct-current voltage applied
i = instantaneous current
ϕ = instantaneous value of iron flux
ϕ' = instantaneous value of leakage flux
L' = leakage inductance
N = number of turns $\times 10^{-8}$
R = total resistance
t = time

Analytical Solution.—The magnetization curve will be represented by the standard form of the Fröhlich equation, *viz.*,

$$\phi = \frac{ai}{b + i} \tag{3}$$

This inserted in the differential equation,[1] equation (1), gives

$$\left[\frac{Nab}{(b + i)^2} + L'\right]\frac{di}{dt} + Ri = E \tag{4}$$

[1] In handling the differential equations for a linear circuit, it is customary to solve separately for the transient and steady-state current. The former is obtained by putting the differential equation equal to zero, the latter by letting it equal the applied voltage. Mathematically this is equivalent to determining the *complementary function* and the *particular integral.* If, for instance, the differential equation is

$$\frac{d^2i}{dt^2} + a\frac{di}{dt} + bi = E(t) \tag{a}$$

where a and b are constants, and the total current is considered equal to the sum of the transient and the steady-state current, *i.e.*, $i = i_t + i_s$, it may be written:

which, separating the variables, may be written

$$\frac{Nab}{(E - iR)(b + i)^2}di + \frac{L'}{E - iR}di = dt \tag{5}$$

To accomplish integration of the first term of this equation, it must be split into partial fractions. Thus:

$$\frac{1}{(E - iR)(b + i)^2} = \frac{A}{E - iR} + \frac{B}{(b + i)^2} + \frac{C}{b + i} \tag{6}$$

giving the following three equations for the determination of the coefficients:

$$A - CR = 0 \tag{7}$$
$$A2b - BR + C(E - bR) = 0 \tag{8}$$
$$Ab^2 + BE + CEb = 1 \tag{9}$$

By simultaneous solution of these, the coefficients are found to be

$$A = \frac{R^2}{(E + Rb)^2} \tag{10}$$

$$B = \frac{1}{E + Rb} \tag{11}$$

$$C = \frac{R}{(E + Rb)^2} \tag{12}$$

$$\underbrace{\left[\frac{d^2i_t}{dt^2} + a\frac{di_t}{dt} + bi_t\right]}_{\substack{\text{Transient part } = 0;\\ \text{no voltage to sustain } i_t}} + \underbrace{\left[\frac{d^2i_s}{dt^2} + a\frac{di_s}{dt} + bi_s\right]}_{\substack{\text{Steady-state part; } i_s \text{ sus-}\\ \text{tained by the applied voltage}}} = E(t) \tag{b}$$

When the circuit is nonlinear, the above procedure is *not applicable.* This is due to the fact that the variable parameter (or parameters) depends on the *total value* of some electrical quantity (flux, voltage, current, as the case may be). Assume that in equation (a) the coefficient b, for instance, is a function of the current, *i.e.*, $b = f(i) = f(i_t + i_s)$. If a segregation into parts involving transient and steady-state currents only is attempted the result is

$$\frac{d^2i_t}{dt^2} + a\frac{di_t}{dt} + f(i_t + i_s)i_t + \frac{d^2i_s}{dt^2} + a\frac{di_s}{dt} + f(i_t + i_s)i_s = E(t) \tag{c}$$

which shows that complete splitting into a transient and a steady-state part is impossible. Differential equations for nonlinear circuits, therefore, must be solved without distinguishing at the outset between transient and steady-state quantities.

Substituting in equation (5), the equation to be integrated now becomes

$$\frac{NabR^2}{(E + Rb)^2(E - iR)^2}di + \frac{Nab}{(E + Rb)(b + i)^2}di + \frac{NabR}{(E + Rb)^2(b + i)}di + \frac{L'}{E - iR}di = dt \quad (13)$$

Carrying out the integration gives

$$-\frac{NabR}{(E + Rb)^2}\log(E - iR) - \frac{Nab}{(E + Rb)(b + i)} + \frac{NabR}{(E + Rb)^2}\log(b + i) - \frac{L'}{R}\log(E - iR) = t + K \quad (14)$$

where K is a constant of integration. Combining the first and fourth term equation (14) may be written

$$-\frac{Nab}{(E + Rb)(b + i)} + \frac{NabR}{(E + Rb)^2}\log(b + i) - \left[\frac{NabR}{(E + Rb)^2} + \frac{L'}{R}\right]\log(E - iR) = t + K \quad (15)$$

To simplify, introduce

$$K_1 = \frac{Nab}{E + Rb} \quad (16)$$

$$K_2 = \frac{NabR}{(E + Rb)^2} \quad (17)$$

$$K_3 = \frac{NabR}{(E + Rb)^2} + \frac{L'}{R} \quad (18)$$

Inserting these constants equation (15) reduces to

$$-\frac{K_1}{b + i} + K_2 \log(b + i) - K_3 \log(E - iR) = t + K \quad (19)$$

The integration constant K depends on the initial conditions and may be determined from these. In the present case the current is zero when the switch S_1 is closed, *i.e.*, at zero time. Hence

$$i = 0 \qquad \text{when} \qquad t = 0$$

and consequently

$$-\frac{K_1}{b} + K_2 \log b - K_3 \log E = K \quad (20)$$

This inserted in equation (19) gives

$$t = \frac{K_1}{b} - \frac{K_1}{b + i} + K_2 \log\frac{b + i}{b} + K_3 \log\frac{E}{E - iR} \quad (21)$$

The above expression is the desired solution and gives the relationship between current and time during the build-up process. As seen, the time is obtained explicitly in terms of the current. If a curve is plotted, however, by assuming values of current and calculating the corresponding values of time, the current corresponding to any arbitrary time may at once be selected. The relationship between time and flux may be obtained formally by substituting in equation (21) for the current in terms of flux from the Fröhlich equation. When a graph of current versus time has been plotted, data for a flux-time curve can at once be secured by reference to the magnetization curve.

For the condition of discharge of the coil (switch S_2 suddenly closed) the differential equation (13) still holds with the exception that the external voltage E now is zero. The resistance may or may not be another. Introducing the following constants:

$$K_1' = \frac{Na}{R} \tag{22}$$

$$K_2' = \frac{Na}{Rb} \tag{23}$$

$$K_3' = \frac{Na}{Rb} + \frac{L'}{R} \tag{24}$$

the solution becomes

$$-\frac{K_1'}{b+i} + K_2' \log (b + i) - K_3' \log i = t + K' \tag{25}$$

The initial conditions for the determination of the constant of integration K' are given by the fact that the current at zero time has a certain known value. Hence

$$i = i_0 \qquad \text{when} \qquad t = 0$$

and consequently

$$-\frac{K_1'}{b+i_0} + K_2' \log (b + i_0) - K_3' \log i_0 = K' \tag{26}$$

If this is inserted in equation (25), the final form of solution giving the relationship between current and time during the discharging process becomes

$$t = \frac{K_1'}{b+i_0} - \frac{K_1'}{b+i} - K_2' \log \frac{b+i_0}{b+i} + K_3' \log \frac{i_0}{i} \tag{27}$$

It should be noted that a better representation of the magnetization curve may be secured by the use of a modification of

the Fröhlich equation, *viz.:*

$$\phi = \frac{ai}{b + i} + ci \tag{3a}$$

As seen, this differs from the standard form [equation (3)] merely by the addition of a corrective linear term, which especially tends to increase the degree of coincidence in the region of high values of flux.

With the modified flux-current relationship, the differential equation [equation (4)] becomes

$$\left[\frac{Nab}{(b + i)^2} + (Nc + L')\right]\frac{di}{dt} + Ri = E \tag{4a}$$

Hence the effect is an apparent increase in the leakage inductance by an amount Nc. All solutions based on the standard Fröhlich equation, therefore, are directly applicable also when the modified equation is used. An "apparent" leakage inductance $Nc + L'$ is simply substituted for the actual leakage inductance L' wherever the latter appears in the formulas.

EXAMPLE 1

Statement of Problem

The following data apply to the magnetization curve of an iron-cored coil:

Flux Density, Gausses	Magnetizing Current, Amperes
0	0
5,000	0.10
8,000	0.35
10,000	0.83
12,000	2.15
13,000	3.60

Establish the Fröhlich equation and the modified Fröhlich equation for this magnetization curve. Plot the analytical curves and the experimental one on the same curve sheet so that the degree of coincidence may be observed.

Solution

a. The Fröhlich Equation.—Referring to equation (3), the constants will be determined at flux densities of 8,000 and 12,000 gausses. Hence

$$8,000 = \frac{a \times 0.35}{b + 0.35} \tag{a}$$

$$12,000 = \frac{a \times 2.15}{b + 2.15} \tag{b}$$

Simultaneous solution gives

$$a = 13{,}300$$
$$b = 0.231$$

The desired equation is therefore

$$B = \frac{13{,}300i}{0.231 + i} \text{ gausses} \tag{c}$$

Data for the curve calculated from equation (*c*) are collected in Table 56.

b. The Modified Fröhlich Equation.—Referring to equation (3*a*), the constants will be determined at flux densities of 8,000, 10,000, and 12,000 gausses. Hence

$$8{,}000 = \frac{a \times 0.35}{b + 0.35} + c \times 0.35 \tag{d}$$

$$10{,}000 = \frac{a \times 0.83}{b + 0.83} + c \times 0.83 \tag{e}$$

$$12{,}000 = \frac{a \times 2.15}{b + 2.15} + c \times 2.15 \tag{f}$$

Simultaneous solution gives

$$a = 11{,}120$$
$$b = 0.152$$
$$c = 714$$

TABLE 56.—DATA FOR CALCULATED MAGNETIZATION CURVES
(Example 1)

i amp.	Fröhlich equation			Modified Fröhlich equation				
	$13{,}300i$	$0.231 + i$	B gausses	$11{,}120i$	$0.152 + i$	$714i$	$\frac{11{,}120i}{0.152 + i}$	B gausses
0	0	0.231	0	0	0.152	0	0	0
0.1	1,330	0.331	4,020	11,120	0.252	71.4	4,420	4,491
0.2	2,660	0.431	6,180	22,250	0.352	142.8	6,040	6,182
0.3	3,990	0.531	7,510	3,340	0.452	224.2	7,400	7,624
0.4	5,320	0.631	8,450	4,450	0.552	285.6	8,070	8,355
0.5	6,650	0.731	9,100	5,570	0.652	357.0	8,540	8,897
0.6	7,980	0.831	9,600	6,670	0.752	428.0	8,880	9,308
0.8	10,640	1.031	10,310	8,900	0.952	572.0	9,350	9,902
1.0	13,300	1.231	10,800	11,120	1.152	714.0	9,640	10,354
1.5	19,950	1.731	11,510	16,690	1.652	1,072.0	10,080	11,152
2.0	26,600	2.231	11,920	22,250	2.152	1,428.0	10,330	11,758
2.5	33,250	2.731	12,200	27,750	2.652	1,785.0	10,470	12,255
3.0	39,900	3.231	12,360	33,350	3.152	2,145.0	10,580	12,725
3.5	46,550	3.731	12,490	3,890	3.652	2,500.0	10,650	13,150
4.0	53,200	4.231	12,600	4,440	4.152	2,855.0	10,680	14,535

The desired equation is therefore

$$B = \frac{11{,}120i}{0.152 + i} + 714 \text{ gausses} \tag{g}$$

Data for the curve calculated from equation (g) are collected in Table 56. The experimental magnetization curves as well as the two calculated curves are shown in Fig 221 It will be noted that the modified Frohlich equation fits the experimental data better than the standard over the entire

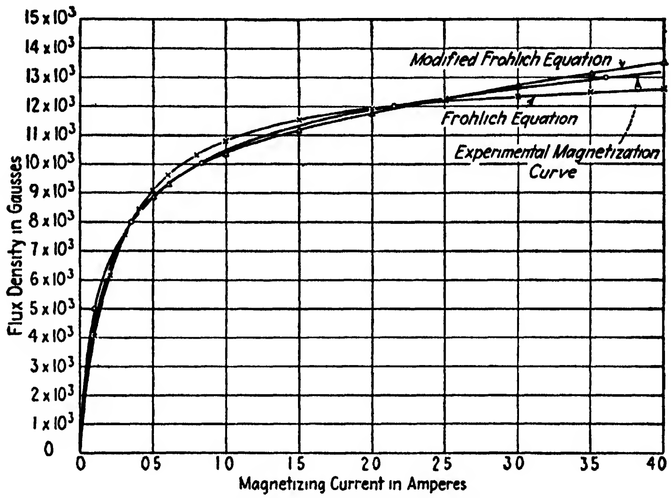


FIG 221 —Experimental magnetization curve of iron-cored coil and its approximation by means of the Frohlich equation and the modified Fröhlich equation (Example 1)

range Exact coincidence of either calculated curve with the experimental one is obtained at the points which served as basis for the constants It is evident that, by judicious selection of these points, almost exact coincidence may be obtained over a limited range of the magnetization curve using either one of the formulas

Solution by Graphical Integration.[1]—When the current has built up to its steady-state value depending upon the impressed voltage and the resistance, and given by

$$I = \frac{E}{R} \tag{28}$$

[1] This particular method of solving the differential equations of saturated direct-current circuits was first suggested by Rüdenberg See RÜDENBERG, R, "Fremd-und Selbsterregung vom magnetisch gesattigten Gleichstromkreisen," *Wiss Veröffentlich Siemens-Konzern*, Vol I, p 179, 1920; "Elektrische Schaltvorgange," J Springer, Berlin, 1923 and 1933

the steady-state value of the flux is simultaneously reached. The final operating point is shown on the magnetization curve in Fig. 222. It is still assumed that the curve gives the relationship between the flux confined to the iron and the current in the coil.

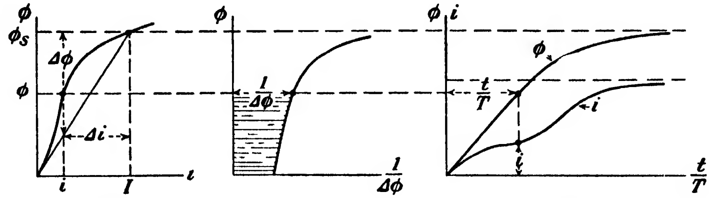


FIG. 222.—Build-up of flux and current in an iron-cored coil. Solution by graphical integration.

If for the present the leakage is entirely neglected, the differential equation (1) may be written

$$N\frac{d\phi}{dt} = E - Ri = R(I - i) = R\Delta i \tag{29}$$

Here, Δi, as seen, represents the difference between the final steady-state current and the instantaneous current actually flowing. Solving equation (29) for the time gives

$$t = \frac{N}{R}\int_{\phi_0}^{\phi}\frac{d\phi}{\Delta i} \tag{30}$$

The lower limit of integration ϕ_0 is the initial flux at zero time. The upper limit is the variable ϕ which evidently may be given any arbitrary value between the initial flux ϕ_0 and the final flux ϕ_s. Equation (30) may be integrated graphically as it stands. It is more convenient, however, first to modify it by introducing a variable $\Delta\phi$, having flux dimensions. which is a linear function of Δi and given by

$$\frac{\Delta\phi}{\phi_s} = \frac{\Delta i}{I} \tag{31}$$

Graphically interpreted $\Delta\phi$ at a certain value of current is the difference between the final flux and the value given by a straight line drawn from the origin through the final operating point on the magnetization curve, as indicated on Fig. 222(a). Substituting for Δi from equation (31) in equation (30) gives

$$t = \frac{N\phi_s}{E}\int_{\phi_0}^{\phi}\frac{d\phi}{\Delta\phi} \tag{32}$$

This equation is in a form suitable for graphical integration. It will be noted that the integral itself represents a pure numeric. Its value depends upon the *limits of integration*, upon the *shape of the magnetic characteristic between these limits*, and upon the *final flux* since this affects $\Delta\phi$. The value of the integral, on the other hand, is not influenced by the circuit constants otherwise, nor by the applied voltage, except insofar as these quantities govern the final operating conditions.

The quotient before the integral depends, as seen, on the number of turns, on the steady-state flux, and on the impressed voltage, and has the *dimensions of time*. It is a *time constant* and may be called the *time constant of the saturated direct-current circuit*. It is given by

$$T = \frac{N\phi_s}{E} = \frac{L_s I}{RI} = \frac{L_s}{R} \tag{33}$$

This time constant is always calculated using the final flux value. It may, as seen, also be expressed in terms of a conventional value of inductance taken as the number of flux linkages per unit current[1] at the final operating point.

The most compact form for equation (32) is therefore

$$t = T\int_{\phi_0}^{\phi} \frac{d\phi}{\Delta\phi} \tag{34}$$

Figure 222 shows how this equation is integrated graphically. It is evident that the integral represents the area between a curve of the reciprocal of $\Delta\phi$ plotted versus flux (ϕ), and the axis of flux, taken, of course, between the proper limits. In Fig. 222(*b*), this area is shown shaded between the initial flux (equal to zero) and an arbitrary flux value. It may be determined by planimeter or by counting squares on the cross-section paper used for the plots and gives, when multiplied by the time constant, the time required for build-up from the initial flux to the particular value of flux considered. By evaluating successive areas, points on a flux-time curve are obtained. In Fig. 222(*c*), the flux is plotted against the ratio of the time to the time constant.

[1] This does not represent the *actual* value of inductance at the final operating point. Further statements regarding inductance of saturated circuits will be found on p. 492 in connection with the discussion of point-by-point solution of circuits containing iron.

Having the flux plot, the corresponding values of current are found by reference to the magnetization curve. Hence the current-time curve may also be drawn.

If the circuit considered had been linear, both flux and current curves would have been exponentials. As it is, the flux curve resembles and, as a matter of fact, does not deviate appreciably from an exponential, although it finally approaches the steady-state value more rapidly. The current, however, is far from being exponential. It generally rises in two distinct steps. This is due to the shape of the magnetization characteristic, the curvature of which may be reversed at low values of flux density.

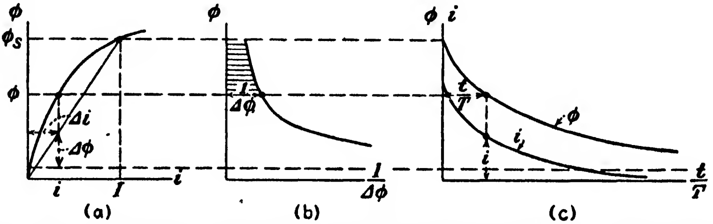


FIG. 223.—Build-down of flux and current in an iron-cored coil. Solution by graphical integration.

During discharge of the coil, when switch S_2 is suddenly closed the differential equation corresponding to equation (29) for build-up, becomes

$$N\frac{d\phi}{dt} = -Ri = R\Delta i \tag{35}$$

Δi is still defined as the difference between the final current—now zero—and the instantaneous current. Introducing again a quantity $\Delta\phi$ as given by equation (31), the differential equation (35) solved for the time may be written.

$$t = T\int_{\phi_s}^{\phi}\frac{d\phi}{\Delta\phi} \tag{36}$$

The graphical interpretation of Δi and $\Delta\phi$ during the build-down process is as indicated in Fig. 223(a). Also here $\Delta\phi$ is the difference between the flux corresponding to the final current (zero current) and the straight line drawn as previously explained. Both Δi and $\Delta\phi$ are now inherently negative. Hence the flux and the current will decrease.

Equation (36) is integrated graphically as explained for the build-up case. The solution is shown in Fig. 222(*a*), (*b*), and (*c*) and requires no further comment as to procedure. In regard to the shape of the time curves, it may be said that again the flux variation approaches the exponential. The current, however, decreases very rapidly at first and then slowly. The current curve differs materially from the exponential.

Leakage Considered.—The problem is susceptible to solution by graphical integration also when the leakage is taken into account. The leakage inductance L' in equations (1) and (2) equals the rate of change of leakage-flux linkages with respect to current. Hence

$$L' = \frac{d(\lambda')}{di} = N\frac{d\phi'}{di} \tag{37}$$

where λ' represents the leakage-flux linkages and ϕ' an *equivalent* leakage flux assumed to link with all turns in the coil. From equation (37) it follows that

$$L'\frac{di}{dt} = N\frac{d\phi'}{dt} \tag{38}$$

and that

$$\phi' = \frac{L'i}{N} \tag{39}$$

Substituting from equation (38) in equation (1) gives

$$N\frac{d(\phi + \phi')}{dt} = E - Ri \tag{40}$$

which in form is identical with equation (29). Consequently, omitting intermediate steps

$$t = T\int\frac{d(\phi + \phi')}{\Delta} \tag{41}$$

Obviously this can be integrated graphically by methods similar to those already described. If the available magnetization curve includes the leakage flux, the procedure is exactly as before. It should be noted, however, that in this case the time constant T will have a slightly different value being then determined from the *total* steady-state flux, *viz.:*

$$T = \frac{N(\phi_s + \phi'_s)}{E} = \frac{L_s + L'}{R} \tag{42}$$

The quantity $\Delta\phi$ is also in this case obtained with reference to the total flux.

If the available magnetization curve does not include the leakage flux, the time constant is based on the core flux only as per equation (33), and $\Delta\phi$ is measured correspondingly. For the purpose of integrating by taking areas, however, the relationship between this $\Delta\phi$ and the *total* instantaneous flux [see equation (41)] must be plotted. The total flux for this purpose may be obtained by adding the leakage flux as calculated by equation (39) to the iron flux given by the magnetization curve for the same values of current. Graphically the total flux is readily obtained by simply displacing the horizontal axis of the magnetization

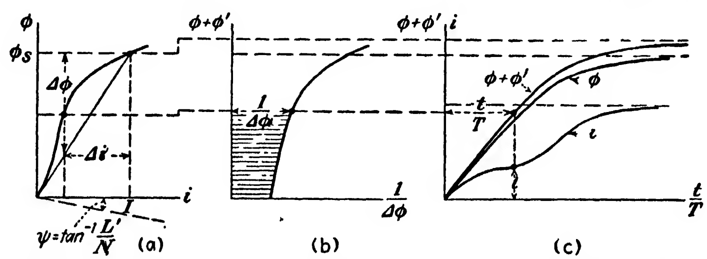


FIG. 224.—Build-up of flux and current in an iron-cored coil when leakage is considered. Solution by graphical integration.

curve by an amount corresponding to the leakage flux, and measuring the ordinates from the so-displaced axis. The angle by which the axis should be displaced is evidently given by

$$\psi = \tan^{-1}\frac{L'}{N} \tag{43}$$

Figure 224 illustrates the solution for the build-up process when leakage is taken into account in this manner.

It follows directly from the preceding how leakage may be taken into account in a solution for the discharge of the coil. It is not considered necessary to discuss this in further detail.

EXAMPLE 2

Statement of Problem

A small experimental transformer[1] has a rating of approximately 2 kw. at 133 volts and 60 cycles. The core is built of silicon-steel laminations of dimensions 8 × 1½ × 0.014 in. At a lamination factor of 95 per cent the net cross-sectional area is 18.4 sq. cm.

[1] Referred to in Examples 2 and 3 of Chap. II of Vol. I.

There are four coils on each leg, numbered 1, 2, 3, and 4, in order of their proximity to the core. In this problem only winding No. 1, consisting of the two No. 1 coils in series, will be considered. The number of turns with this connection is 200. The self-leakage reactance of this winding at 60 cycles is 2.36 ohms and its ohmic resistance is 0.284 ohm.

The direct-current magnetization curve (obtained as the mean of a hysteresis loop) may be constructed from the following data (already given and used in Example 1):

Flux Density, Gausses	Magnetizing Current, Amperes
0	0
5,000	0.1
8,000	0.35
10,000	0.83
12,000	2.15
13,000	3.60

A battery voltage of 2 volts is suddenly applied to the winding, using an external resistance of 0.383 ohm in series.

Determine and plot the curve giving the relationship between the current and the time during the build-up process:

a. By using a method of graphical integration.

b. By analytical solution representing the magnetization curve by the standard and modified Fröhlich equation.

Solution

a. By Graphical Integration.—When leakage is considered, the following equation [see equation (41)] lends itself to the process of graphical integration:

$$t = T\int_0^{\phi+\phi'} \frac{d(\phi + \phi')}{\Delta\phi} \tag{a}$$

where

$$T = \frac{N\phi_s}{E} \tag{b}$$

The given numerical values are:

$$N = 200 \times 10^{-8}, \qquad E = 2 \text{ volts}, \qquad R = 0.667 \text{ ohm}$$

The leakage inductance is:

$$L' = \frac{2.36}{377} = 0.00627 \text{ henry}$$

and the leakage flux

$$\phi' = \frac{L'i}{N} \frac{0.00627i}{200 \times 10^{-8}} = 3{,}130i \text{ maxwells}$$

At a current of 3 amp., the leakage flux becomes 9,400 maxwells. This value was used to locate the leakage line in Fig. 225, which shows the mag-

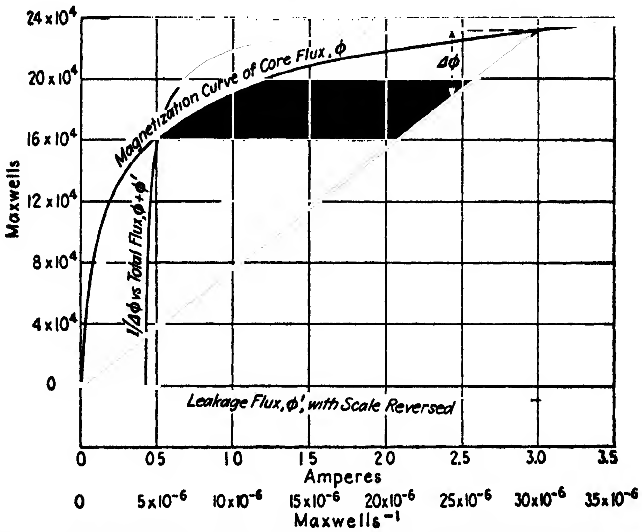


FIG. 225.—Magnetization curve of small experimental transformer considered in Example 2 with leakage-flux and resistance line drawn in. The reciprocal of the flux differential is also plotted versus the total flux.

netization curve and the curve of $1/\Delta\phi$ plotted versus the total flux, $\phi + \phi'$, the latter having been plotted from the following data:

$\phi + \phi'$ Maxwells	$1/\Delta\phi$
0	4.3×10^{-6}
7×10^4	4.4×10^{-6}
14×10^4	4.8×10^{-6}
20×10^4	6.8×10^{-6}
22×10^4	10.4×10^{-6}
23×10^4	18.2×10^{-6}
23.4×10^4	27.4×10^{-6}
23.6×10^4	36.6×10^{-6}

The time constant becomes

$$T = \frac{200 \times 23.2 \times 10^4}{2 \times 10^8} = 0.232 \text{ sec.}$$

The integral [equation (*a*)] is evaluated by measuring areas under the curve of $1/\Delta\phi$ to appropriate limits on the scale of total flux. Such areas, multiplied by T to give time in seconds, are tabulated below. Values of

current corresponding to those of flux are obtained from the magnetization curve.

$\phi + \phi'$ maxwells	t sec.	i amp.
0	0	0
6 $\times 10^4$	0 060	0 05
10 $\times 10^4$	0 101	0 12
14 $\times 10^4$	0 144	0 29
18 $\times 10^4$	0 192	0 73
20 $\times 10^4$	0 220	1 10
22 $\times 10^4$	0 258	1 75
23 $\times 10^4$	0 290	2 30
23 2 $\times 10^4$	0 299	2 40
23 4 $\times 10^4$	0 310	2 54
23 6 $\times 10^4$	0 325	2 65

The build-up curves of flux and current are plotted versus time in Fig 226

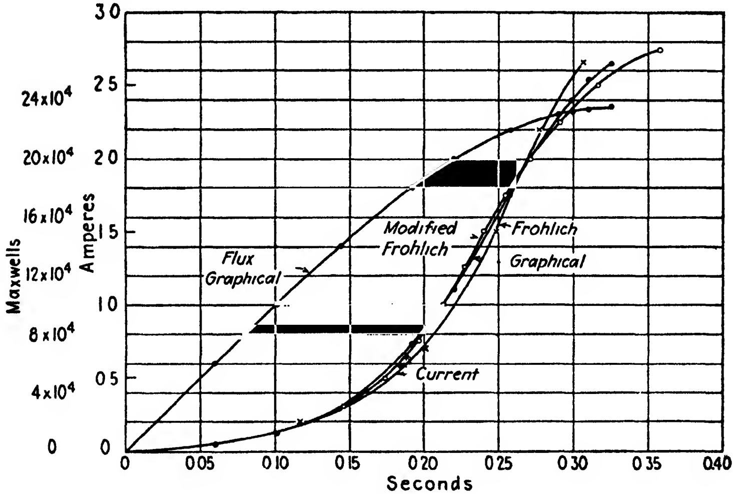


FIG. 226.—Calculated build-up curves of flux and current when a direct-current voltage is suddenly applied to one of the coils of the small experimental transformer considered in Example 2.

b. Analytical Solution Representing the Magnetization Curve by the Standard and Modified Fröhlich Equation.—Referring to equation (21), the solution for either of these cases has the following form:

$$t = \frac{K_1}{b} - \frac{K_1}{b + i} + K_2 \log \frac{b + i}{b} + K_3 \log \frac{E}{E - Ri} \qquad (c)$$

where in accordance with equations (16), (17), and (18),

$$K_1 = \frac{Nab}{E + Rb} \tag{d}$$

$$K_2 = \frac{NabR}{(E + Rb)^2} \tag{e}$$

$$K_3 = K_2 + \frac{L'}{R} \tag{f}$$

Solution Based on Standard Fröhlich Equation.—Using values already determined in Example 1,

$$a = 13{,}300 \times 18.4 = 245{,}000 \text{ maxwell/amp.}$$
$$b = 0.231 \text{ amp.}$$

Hence

$$K_1 = \frac{200 \times 10^{-8} \times 245{,}000 \times 0.231}{2 + 0.667 \times 0.231} = \frac{0.113}{2.154} = 0.0525$$

$$K_2 = \frac{0.113 \times 0.667}{2.154^2} = 0.01625$$

$$K_3 = 0.01625 + \frac{0.00627}{0.667} = 0.0257$$

Using these constants in equation (*c*), the calculations are carried out in Table 57.

Solution Based on Modified Fröhlich Equation.—Using values previously determined in Example 1,

$$a = 11{,}120 \times 18.4 = 205{,}000 \text{ maxwell/amp.}$$
$$b = 0.152 \text{ amp.}$$
$$c = 714 \times 18.4 = 13{,}150 \text{ maxwell/amp.}$$

Hence

$$K_1 = \frac{200 \times 10^{-8} \times 205{,}000 \times 0.152}{2 + 0.667 \times 0.152} = \frac{0.06232}{2.1014} = 0.0297$$

$$K_2 = \frac{0.06232 \times 0.667}{2.1014^2} = 0.00944$$

In formula for K_3, the following must be used for L':

$$Nc + L' = 200 \times 10^{-8} \times 13{,}150 + 0.00627 = 0.0326 \text{ henry}$$

so that

$$K_3 = 0.00944 + \frac{0.0326}{0.667} = 0.0583$$

Employing these constants in equation (*c*), the calculations are carried out in Table 58.

TABLE 57 —CALCULATED FLUX AND CURRENT BUILD-UP CURVES BASED ON STANDARD FROHLICH EQUATION (Example 2)

							(1)	(2)	(3)	(4)	(3) − (4)
i amp	ϕ maxwell	$b + i$	$\frac{b+i}{b}$	$\log \frac{b+i}{b}$	$\frac{E}{E - R_i}$	$\log \frac{E}{E - R_i}$	$K_2 \log \frac{b+i}{b}$	$K_3 \log \frac{E}{E - R_i}$	$\frac{K_1}{b} + (1) + (2)$	$\frac{K_1}{b+i}$	t sec
0 20	12 2 × 10⁴	0 431	1 87	0 626	1 07	0 067	0 010	0 002	0 239	0 122	0 117
0 70	17 7 × 10⁴	0 931	4 03	1 394	1 305	0 266	0 023	0 007	0 257	0 056	0 201
1 50	20 9 × 10⁴	1 731	7 49	2 014	2 00	0 693	0 033	0 018	0 278	0 030	0 248
2 20	22 2 × 10⁴	2 431	10 53	2 350	3 75	1 322	0 038	0 034	0 299	0 022	0 277
2 66	22 7 × 10⁴	2 89	12 51	2 523	8 81	2 176	0 041	0 056	0 324	0 018	0 306

TABLE 58 —CALCULATED FLUX AND CURRENT BUILD-UP CURVES BASED ON MODIFIED FROHLICH EQUATION (Example 2)

							(1)	(2)	(3)	(4)	(3) − (4)
i amp	ϕ maxwell	$b + i$	$\frac{b+i}{b}$	$\log \frac{b+i}{b}$	$\frac{E}{E - R_i}$	$\log \frac{E}{E - R_i}$	$K_2 \log \frac{b+i}{b}$	$K_3 \log \frac{E}{E - R_i}$	$\frac{K_1}{b} + (1) + (2)$	$\frac{K_1}{b+i}$	t sec
0 20	12 2 × 10⁴	0 352	2 320	0 840	1 071	0 0685	0 0079	0 0040	0 2074	0 0844	0 123
0 25	13 2 × 10⁴	0 402	2 645	0 972	1 090	0 0860	0 0092	0 0050	0 2097	0 0739	0 136
0 30	14 0 × 10⁴	0 452	2 915	1 090	1 110	0 1043	0 0103	0 0061	0 2119	0 0656	0 146
0 50	16 2 × 10⁴	0 652	4 29	1 453	1 200	0 1825	0 0137	0 0106	0 2198	0 0455	0 174
0 75	18 0 × 10⁴	0 902	5 93	1 780	1 333	0 288	0 0168	0 0168	0 2291	0 0329	0 196
1 00	19 2 × 10⁴	1 152	7 59	2 025	1 500	0 405	0 0191	0 0236	0 2382	0 0256	0 213
1 25	20 2 × 10⁴	1 402	9 23	2 220	1 718	0 540	0 0210	0 0315	0 2480	0 0212	0 227
1 50	20 9 × 10⁴	1 652	10 88	2 385	2 00	0 693	0 0226	0 0404	0 2584	0 0180	0 240
1 75	21 4 × 10⁴	1 902	12 51	2 525	2 40	0 875	0 0239	0 0510	0 2703	0 0156	0 255
2 00	21 9 × 10⁴	2 152	14 13	2 645	3 00	1 100	0 0250	0 0641	0 2846	0 0138	0 271
2 25	22 2 × 10⁴	2 402	15 80	2 760	4 00	1 390	0 0261	0 0810	0 3025	0 0124	0 290
2 50	22 6 × 10⁴	2 652	17 40	2 850	6 00	1 792	0 0269	0 1045	0 3269	0 0112	0 316
2 75	22 9 × 10⁴	2 902	19 10	2 950	12 00	2 480	0 0278	0 1450	0 3683	0 0102	0 358

The build-up curves of flux and current are plotted in Fig. 226. The curves based on graphical integration may be considered to represent the correct solution. It will be noted that the analytical methods give reasonable accuracy, especially when the modified Fröhlich equation is used to represent the magnetization curve.

Point-by-point Solution of Circuits Containing Iron.—It is assumed that the reader is familiar with the general principles of the point-by-point method for the solution of problems involving transients. In the following its application to a simple circuit containing iron will be briefly reviewed without undue emphasis on details.

Consider an iron-cored coil whose terminals are connected to a source of electromotive force which for the sake of generality may be assumed to be a function of time. This time relationship may be either mathematically or graphically expressed. Assume a discontinuity to be suddenly impressed on this circuit as, for instance, by the sudden application of the above-mentioned voltage or by a sudden change in the circuit constants. The magnetization characteristic of the coil is known.

The differential equation [corresponding to equation (1)] then becomes

$$N\frac{d\phi}{dt} + L'\frac{di}{dt} + Ri = e = E(t) \tag{44}$$

which when a solution in terms of flux is desired may be written

$$\frac{d(\phi + \phi')}{dt} = \frac{E(t) - Ri}{N} \tag{45}$$

and when the current is wanted

$$\frac{di}{dt} = \frac{E(t) - Ri}{L + L'} \tag{46}$$

The relations between leakage inductance and leakage flux have been previously discussed. The inductance due to the flux in the iron equals the rate of change of core-flux linkages with respect to current and is given by

$$L = \frac{d\lambda}{di} = \frac{d(N\phi)}{di} = N\frac{d\phi}{di} \tag{47}$$

Hence this inductance is a variable and depends on the slope of

the magnetization curve. If needed, for particular values of current or flux, tangents must be fitted to the magnetization curve and their slopes determined, as shown in Fig. 227.

The point-by-point method considers the rate of change of current or flux constant during a short interval of time. The length of this interval must be suitably selected but, in order to save labor, should, in general, be taken as long as is consistent with accuracy. Its length, therefore, somewhat depends on the nature of the problem, but usually lies between 0.05 and 0.10 sec. The rate of change is determined for each interval as the analysis progresses, either from conditions at the beginning of or at the midpoint of the interval. The former is the more common, as well as the speedier and will as a rule give sufficient accuracy. The increment in flux or current during any interval evidently equals the rate of change times the length. Knowing the conditions as regards flux or current at the beginning of an interval, the conditions at the end of the same interval are obtained by algebraic addition of the increment. Obviously the conditions at the end of one interval represent the initial conditions for the next one, etc.

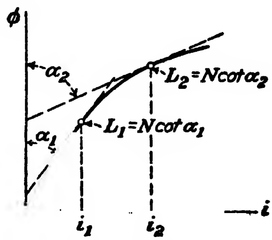


FIG. 227.—Determination of inductance of an iron-cored circuit at different values of saturation.

Considering the nth interval, the manipulations may, in the problem at hand [equations (45) and (46)], be formally expressed as follows:

When the point-by-point analysis is carried out in terms of flux

$$\Delta(\phi + \phi')_n = \left[\frac{d(\phi + \phi')}{dt}\right]_{n-1} \Delta t = \frac{[E(t) - Ri]_{n-1}}{N} \Delta t \quad (48)$$

$$(\phi + \phi')_n = (\phi + \phi')_{n-1} + \Delta(\phi + \phi')_n \quad (49)$$

When the point-by-point analysis is carried out in terms of current

$$\Delta i_n = \left[\frac{di}{dt}\right]_{n-1} \Delta t = \frac{[E(t) - Ri]_{n-1}}{L + L'} \Delta t \quad (50)$$

$$i_n = i_{n-1} + \Delta i_n \quad (51)$$

Since the relationship between flux and current is given by the magnetization curve, the actual point-by-point calculations

need, of course, be carried out for one of these quantities only. The corresponding values of the other are obtained by reference to this curve. It is convenient to arrange the calculations in tabular form as suggested in Tables 59 and 60. In the former the analysis is in terms of flux, in the latter, in terms of current.

TABLE 59.—SCHEDULE FOR POINT-BY-POINT CALCULATION OF FLUX AND CURRENT

Int. no.	t	$E(t)$	Ri	$E(t) - Ri$	$d(\phi + \phi')/dt$	$\Delta(\phi + \phi')$	$\phi + \phi'$	ϕ	i

TABLE 60.—SCHEDULE FOR POINT-BY-POINT CALCULATION OF CURRENT AND FLUX

Int. no.	t	$E(t)$	Ri	$E(t) - Ri$	di/dt	Δi	i	ϕ

EXAMPLE 3

Statement of Problem

The circuit (Fig. 228) represents a transformer T, with N turns on both the primary and secondary, for which the graphical relation $\phi = f(Ni)$ is given. Assume $R_1 = R_2$. A battery having a voltage E is suddenly impressed.

Set up the necessary differential equations and indicate manipulations for finding i_1 and i_2 as functions of time by graphical integration. Illustrate the procedure by means of sketches, and sketch the result. Neglect all leakage fluxes.

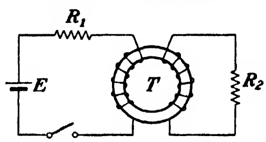


FIG. 228.—Schematic diagram of transformer considered in Example 3.

Solution

In the absence of leakage the following differential equations may be written:

$$N\frac{d\phi}{dt} \times 10^{-8} + R_1 i_1 = E \tag{a}$$

$$N\frac{d\phi}{dt} \times 10^{-8} + R_2 i_2 = 0 \tag{b}$$

from which, by subtraction,

$$E = R_1 i_1 - R_2 i_2 = R(i_1 - i_2) \tag{c}$$

that is,

$$i_1 - i_2 = \frac{E}{R} \tag{d}$$

Adding equations (*a*) and (*b*) gives

$$2N\frac{d\phi}{dt} \times 10^{-8} = E - R(i_1 + i_2) = R(i_{s1} - i_1 - i_2) = R\,\Delta i \tag{e}$$

The core flux as represented by the magnetization curve evidently depends upon the resultant ampere turns, *i.e.*, upon the sum of the primary and

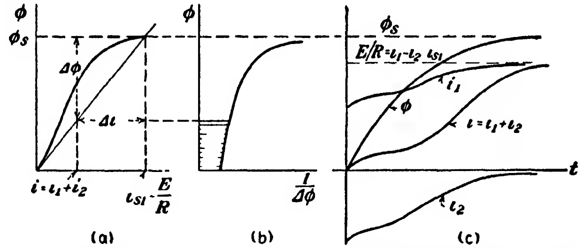


FIG. 229.—Determination of build-up of flux and primary and secondary current in the transformer in Fig. 228 by the method of graphical integration (Example 3).

secondary currents during the build-up process. It is obvious that in the final steady-state conditions the secondary current is zero, so that the final operating point on the magnetization curve, Fig. 229(*a*), corresponds to the final steady-state current in the primary E/R. Defining Δi and $\Delta\phi$ as indicated in Fig. 229, the following relation may be written:

$$\Delta i = \Delta\phi\frac{i_{s1}}{\phi_s} \tag{f}$$

Inserting this in equation (*e*) gives

$$2N\frac{d\phi}{dt} \times 10^{-8} = \frac{Ri_{s1}}{\phi_s}\Delta\phi = \frac{E}{\phi_s}\Delta\phi \tag{g}$$

which, solved for the build-up time, gives

$$t = \frac{2N\phi_s}{E}10^{-8}\int_{\phi_0}^{\phi}\frac{d\phi}{\Delta\phi} = 2T\int_{\phi_0}^{\phi}\frac{d\phi}{\Delta\phi} \tag{h}$$

This is in convenient form for obtaining the build-up of the flux graphically by the standard processes previously described and is carried out as indicated in Fig. 229(*b*) and (*c*). Referring back to the magnetization curve from the flux-time curve gives the time variation of the sum of the primary and secondary currents $i_1 + i_2$. This curve may in mathematical form be represented as

$$i_1 + i_2 = F(t) \tag{i}$$

which, when added to equation (*d*), gives

$$i_1 = \frac{1}{2}\left(F(t) + \frac{E}{R}\right) \tag{j}$$

The primary current is hence obtained graphically by adding the primary steady-state current to the build-up curve of the sum of the primary and secondary currents and dividing by two, as shown in Fig. 229(*c*). Having thus determined the primary current, the secondary current is readily obtained by application of equation (*d*) or (*i*).

General Remarks Concerning Transients in Exciters.—In exciter problems the discontinuity which gives rise to the transients is usually a sudden change in the resistance in the field circuit of the exciter itself. This may be due to hand regulation or to the action of automatic voltage regulators which intermittently short-circuit and reinsert a certain amount of resistance. In an excitation system the transients may in addition be caused by an abrupt resistance variation in the alternator field circuit, *i.e.*, on the output side of the exciter.

In order to analyze the exciter transients taking saturation into account, the magnetization curve must be available. This is an open-circuit characteristic giving the relationship between the terminal voltage on open circuit (equal to the induced armature voltage) and the field current. Data for this curve, if obtained experimentally, should preferably be taken with the machine in question separately excited, even though it may be intended for operation as a self-excited machine, in order to eliminate any effect of armature-resistance drop and armature reaction. However, these factors will as a rule be negligibly small at the low values of current required for excitation purposes only.

The induced voltage is directly proportional to the *armature flux* which, owing to the dispersion, is somewhat smaller than the *field flux*. Hence the question of *leakage* enters. Of course, as

an approximation, the effect of the leakage may be ignored, but since it may be comparatively large in certain designs, it may be better to take it into account. In the preceding analysis of the transients in the iron-cored coil, the leakage flux was considered to be a linear function of the current and hence the leakage inductance constant and independent of the saturation. This is, for instance, generally true in transformers where the path of the leakage flux is predominantly in air and where in most designs any flux of the core-leakage type is comparatively insignificant.[1] In a direct-current machine the situation is somewhat different. On account of the air gap which the entire armature flux has to traverse and in which a very large portion of the total reluctance is concentrated, there is considerable opportunity for leakage fluxes of the core-leakage category to exist. The leakage fluxes extend between the tips of the pole shoes, between the pole cores, and between cores and yoke, and although they have appreciable air-paths, yet the reluctance of their iron-paths is by no means inappreciable. This reluctance depends upon the saturation with the result that the leakage flux, *i.e.*, the flux which does not enter the armature, is not entirely a linear function of the field current.

The conventional theory of direct-current machines assumes the leakage flux to be a *certain constant fraction* of the armature flux. To a machine is assigned a constant *coefficient of dispersion* giving the *ratio of field flux to armature flux.* This field flux is actually an equivalent flux producing complete linkages with all turns in the field coils. The equivalent flux may be looked upon as representing approximately the average value of flux in the field cores. Considering the leakage in this manner is very convenient in calculations, steady-state as well as transient. No doubt, however, the method somewhat overemphasizes the effect of saturation on the leakage flux.

An alternative method—going to the other extreme—is to consider the leakage fluxes wholly independent of saturation and to introduce a strictly constant leakage inductance in the transient analysis. In the following, solutions based on either method will be discussed.

Exciters used in present-day practice may be separately excited as well as self-excited. In the former case excitation is

[1] See Chap. II, Vol. I, for a discussion of transformer leakage.

supplied from a pilot exciter or other constant-potential source, Fig. 230; in the latter from the machine itself, Fig. 231. In the former case the voltage across the field circuit is constant, in the

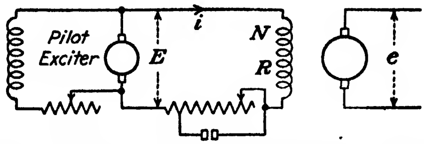


FIG. 230.—Schematic diagram of separately excited exciter.

latter, variable. The transients, and especially the rapidity of the voltage and current variations, therefore, are different for the two types of exciters. The fundamental differential equations which immediately may be written are

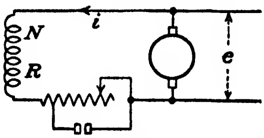


FIG. 231.—Schematic diagram of self-excited exciter.

$$\sigma N\frac{d\phi}{dt} + Ri = E \qquad (\text{or } = e) \quad (52)$$

applicable when use is made of a constant coefficient of dispersion, and

$$N\frac{d\phi}{dt} + L'\frac{di}{dt} + Ri = E \qquad (\text{or } = e) \qquad (53)$$

holding when the leakage inductance is considered constant.

The following notation will be adhered to:

E = voltage of the pilot exciter
e = induced armature voltage
i = field current
ϕ = armature flux
ϕ' = leakage flux
N = number of field turns $\times 10^{-8}$
σ = coefficient of dispersion
L' = leakage inductance of the field
R = net resistance in the field circuit
k = proportionality factor relating armature flux and voltage
t = time
T = time constant based on armature flux
T' = time constant based on leakage flux

Subscripts 1 and 2 attached to the symbols e, i, and ϕ refer to initial and final conditions of operation, respectively.

Since the magnetization curve always is plotted in terms of voltage, it is more convenient to have the differential equations in terms of voltage instead of flux. Let

$$e = k\phi \qquad \text{and} \qquad \frac{de}{dt} = k\frac{d\phi}{dt} \tag{54}$$

the proportionality factor k being given by

$$k = \frac{p}{a} \cdot \frac{nZ}{60} \tag{55}$$

where p = number of poles
a = number of parallel paths in the armature winding
Z = number of armature conductors per pole
n = speed in r.p.m.

Substituting from equation (54) in equations (52) and (53) the fundamental differential equations become

$$\frac{\sigma N}{k}\frac{de}{dt} + Ri = E \qquad (\text{or} = e) \tag{56}$$

and

$$\frac{N}{k}\frac{de}{dt} + L'\frac{di}{dt} + Ri = E \qquad (\text{or} = e) \tag{57}$$

In the solution of the differential equations given above it is in general convenient to introduce the *time constant* of the exciter field winding. As was done for the iron-cored coil time constants corresponding to the *final flux conditions* will be used for the separately excited machine. For the self-excited machine, it makes no difference at what condition of flux the time constant is calculated. With the *formulation used*, the time constant of this machine is independent of saturation.

The time constant, due to the part of the flux which crosses the air-gap and enters the armature, is for the separately excited machine

$$T = \frac{N\phi_2}{Ri_2} = \frac{N\phi_2}{E} = \frac{Ne_2}{kE} \tag{58}$$

and for the self-excited machine

$$T = \frac{N\phi_2}{Ri_2} = \frac{N\phi_2}{e_2} = \frac{Ne_2}{ke_2} = \frac{N}{K} \tag{59}$$

It will be seen that the latter is constant, as mentioned above.

When leakage is considered by introducing a constant leakage inductance, the time constant due to the leakage flux is

$$T' = \frac{L'}{R} \tag{60}$$

The actual time constant of the field winding, or the time constant due to the total number of flux linkages, is the sum of the time constants due to armature and leakage flux. Hence, this time constant is given by

$$T_t = T + T' \tag{61}$$

When leakage is taken into account by assuming a constant coefficient of dispersion, the time constant of the field winding due to the total flux evidently becomes

$$T_t = \sigma T \tag{62}$$

Attention is called to the statement on page 473 with regard to screening effect of eddy currents tending, if present, to prolong the build-up and build-down processes. Particularly appropriate to note at this point is the possibility of reflecting their effect approximately in the analysis by shading the time constants upward.

Separately Excited Machines. *Analytical Solutions.* 1. *Constant Leakage Inductance.*—A Fröhlich equation of the same form as equation (3)[1] may now be used to express the relationship between induced voltage and field current, *i.e.*, to represent the magnetization curve.
Hence

$$e = \frac{ai}{b + i} \tag{63}$$

Inserting from this in the differential equation (57) gives

$$\left[\frac{Nab}{k(b + i)^2} + L'\right]\frac{di}{dt} + Ri = E \tag{64}$$

which when integrated yields a solution in terms of current. Comparison shows that this equation is identical in form with equation (4) for the iron-cored coil, the only actual difference

[1] The modified Fröhlich equation [equation (3*a*)] may, of course, also be used. This will not change the form of the solutions (see statements on p. 479).

being the factor k in the denominator of the first term. The general solution, therefore, is given by equation (19), and the constants can without difficulty be written down by inspection with reference to equations (16), (17), and (18), inclusive. However, in exciter problems, it may be convenient to obtain the solution for time in terms of the time constant. Factoring out the time constant corresponding to the armature flux (which in this case is the iron flux proper) as given by equation (58), the three constants evidently become

$$K_1 = \frac{Eab}{e_2(E + Rb)} \tag{65}$$

$$K_2 = \frac{EabR}{e_2(E + Rb)^2} = \frac{RK_1}{E + Rb} \tag{66}$$

$$K_3 = \frac{EabR}{e_2(E + Rb)^2} + \frac{kEL'}{Ne_2R} = K_2 + \frac{T'}{T} \tag{67}$$

Assuming that the field current flowing when the discontinuity is impressed is i_1, *i.e.*, that

$$i = i_1 \qquad \text{when} \qquad t = 0$$

the integration constant K [see equation (19)] is given by

$$-\frac{K_1}{b + i_1} + K_2 \log (b + i_1) - K_3 \log (E - i_1R) = \frac{K}{T} \tag{68}$$

Substituting this in equation (19) gives the final solution as

$$t = T\left(\frac{K_1}{b + i_1} - \frac{K_1}{b + i} + K_2 \log \frac{b + i}{b + i_1} + K_3 \log \frac{E - Ri_1}{E - Ri}\right) \tag{69}$$

When the time-current relationship has been found, the variation in voltage with time is obtained by reference to the magnetization curve. It is entirely possible, however, to establish a formal solution for the voltage by the same methods as were used above for the current. The Fröhlich equation may be solved for current, *viz.*:

$$i = \frac{be}{a - e} \tag{70}$$

Making use of this in the differential equation (57) the latter becomes

$$\left[\frac{N}{k} + \frac{L'ab}{(a - e)^2}\right]\frac{de}{dt} + \frac{Rbe}{a - e} = E \tag{71}$$

Separating the variables gives

$$\left[\frac{N}{k\left(E - \dfrac{Rbe}{a - e}\right)} + \frac{L'ab}{(a - e)^2\left(E - \dfrac{Rbe}{a - e}\right)}\right]de = dt \qquad (72)$$

which again may be written

$$\frac{Na}{k[Ea - (E + Rb)e]}de - \frac{Ne}{k[Ea - (E + Rb)e]}de + \frac{L'ab}{(a - e)[Ea - (E + Rb)e]}de = dt \qquad (73)$$

In order to integrate the last term on the left-hand side partial fractions will be resorted to. Thus

$$\frac{1}{(a - e)[Ea - (E + Rb)e]} = \frac{A}{a - e} + \frac{B}{Ea - (E + Rb)e} \qquad (74)$$

giving the following simultaneous equations for the determination of the coefficients

$$AEa + Ba = 1 \qquad (75)$$

$$A(E + Rb) + B = 0 \qquad (76)$$

from which the coefficients are found to be

$$A = -\frac{1}{abR} \qquad (77)$$

$$B = \frac{E + bR}{abR} \qquad (78)$$

Inserting in equation (73), the equation to be integrated becomes

$$\frac{\dfrac{Na}{k} + \dfrac{L'}{R}(E + bR)}{Ea - (E + Rb)e}de - \frac{\dfrac{N}{k}e}{Ea - (E + Rb)e}de - \frac{\dfrac{L'}{R}}{a - e}de = dt \qquad (79)$$

giving upon integration

$$\frac{Ne}{k(E + Rb)} + \left[\frac{NEa}{k(E + Rb)^2} - \frac{Na}{k(E + Rb)} - \frac{L'}{R}\right]\log\,[Ea - (E + Rb)e] + \frac{L'}{R}\log\,(a - e) = t + K \qquad (80)$$

Introducing the following constants in order to simplify

$$K_1 = \frac{E}{e_2(E + Rb)} \tag{81}$$

$$K_2 = \frac{E^2a}{e_2(E + Rb)^2} - \frac{Ea}{e_2(E + Rb)} - \frac{KEL'}{Ne_2R} = aK_1(e_2K_1 - 1) - K_3 \tag{82}$$

$$K_3 = \frac{KEL'}{Ne_2R} = \frac{T'}{T} \tag{83}$$

and applying the initial condition that

$$e = e_1 \qquad \text{when} \qquad t = 0$$

the final solution giving the relationship between time and voltage is obtained as

$$t = T\left[K_1(e - e_1) + K_2 \log \frac{Ea - (E + Rb)e}{Ea - (E + Rb)e_1} + K_3 \log \frac{a - e}{a - e_1}\right] \tag{84}$$

Equations (69) and (83) are equally applicable to the build-up and the build-down process. Subscript 1 designates in either case the *starting* conditions whether these happen to be the upper or the lower conditions and subscript 2 similarly the final conditions. The correct net field-circuit resistance must, of course, be used.

2. *Constant Coefficient of Dispersion.*—Substituting from the Fröhlich equation (63) in equation (56) the differential equation to be handled becomes

$$\frac{\sigma Nab}{k(b + i)^2}\frac{di}{dt} + Ri = E \tag{85}$$

This is identical in form with equation (64) with L' equal to zero. Hence the constants in the general solution may with reference to equations (65), (66), and (67) be written by inspection as

$$K_1 = \frac{Eab}{e_2(E + Rb)} \tag{86}$$

$$K_2 = K_3 = \frac{EabR}{e_2(E + Rb)^2} = \frac{RK_1}{E + Rb} \tag{87}$$

Referring to equation (69), the solution for the current-time relationship becomes

$$t = \sigma T\left[\frac{K_1}{b + i_1} - \frac{K_1}{b + i} + K_2 \log \frac{(b + i)(E - Ri_1)}{(b + i_1)(E - Ri)}\right] \tag{88}$$

To obtain a formal solution in terms of voltage, substitute from the rewritten Fröhlich equation (70) in the differential equation (56). This gives

$$\frac{\sigma N}{K}\frac{de}{dt} + \frac{Rbe}{a - e} = E \tag{89}$$

which in form is identical with equation (71) with L' equal to zero. The three constants in the general solution are obtained from equations (81), (82), and (83) by inspection

$$K_1 = \frac{E}{e_2(E + Rb)} \tag{90}$$

$$K_2 = \frac{E^2 a}{e_2(E + Rb)^2} - \frac{Ea}{e_2(E + Rb)} = aK_1(e_2K_1 - 1) \tag{91}$$

$$K_3 = 0 \tag{92}$$

By using equation (84), the solution for the voltage-time relationship may be written

$$t = \sigma T\left[K_1(e - e_1) + K_2 \log \frac{Ea - (E + Rb)e}{Ea - (E + Rb)e_1}\right] \tag{93}$$

Solution by Graphical Integration. 1. *Constant Coefficient of Dispersion.*—For the purpose of the graphical integration the differential equation (56) may be written in a more convenient form, *viz.:*

$$\frac{\sigma N}{k}\frac{de}{dt} = E - Ri = R(i_2 - i) = R\Delta i \tag{94}$$

Introduce a quantity Δe directly proportional to Δi in accordance with the relation

$$\frac{\Delta e}{e_2} = \frac{\Delta i}{i_2} \tag{95}$$

The graphical interpretation of Δe is evident from Fig. 232(a). A straight line is drawn through the origin and the final operating point on the magnetization curve corresponding to the final current i_2. It is seen that Δe at any arbitrary value of current is equal to the difference between the final voltage e_2 and the value of voltage given by the straight line. Substitution of Δe from equation (95) in equation (94) and solving for the time gives

$$t = \frac{\sigma N e_2}{kRi_2}\int_{e_1}^{e}\frac{de}{\Delta e} = \frac{\sigma N e_2}{kE}\int_{e_1}^{e}\frac{de}{\Delta e} = \sigma T\int_{e_1}^{e}\frac{de}{\Delta e} \tag{96}$$

This equation lends itself readily to graphical integration by processes similar to those already described in some detail in connection with the iron-cored-coil problem. The integral itself represents a pure numeric, the remarks already made during the discussion of the problem just mentioned being

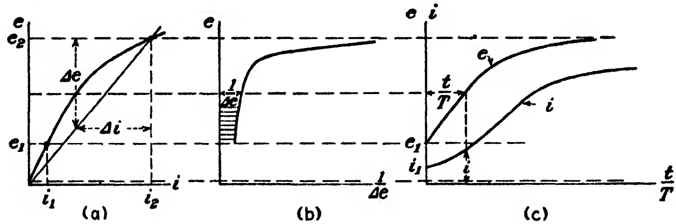


FIG. 232.—Build-up of voltage and field current of separately excited exciter. Solution by graphical integration.

pertinent also here. The integral is multiplied by a time constant calculated for the *final* conditions of flux linkages.

The graphical solution for a build-up process is shown in Fig. 232. In (*a*) is shown the magnetization curve, in (*b*) the plot of the reciprocal of Δe versus voltage where successive areas are measured between the curve and the vertical axis, and in (*c*) the time curves of voltage and current. As far as the general shape is concerned, the voltage- and current-time curves for the

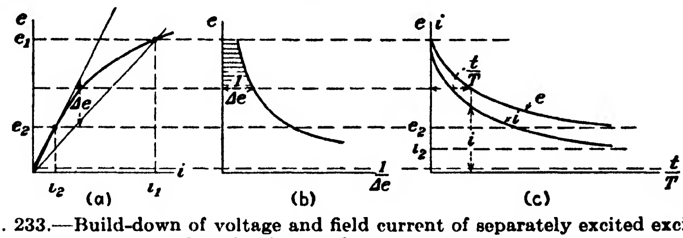


FIG. 233.—Build-down of voltage and field current of separately excited exciter. Solution by graphical integration.

separately excited machine resemble those of flux and current for the iron-cored coil.

Figure 233 shows the solution for a build-down process. This is also based on equation (96) which is valid as it stands when only the proper resistance and time constant are used. The resistance and time constant should always correspond to the *final* condition. Furthermore, Δe should always be taken as the difference between the *final* voltage e_2 and the voltage values given by the straight line through the origin and the final operating point

on the magnetization curve. For build-down Δe is obviously inherently negative.

2. *Constant Leakage Inductance.*—The differential equation (57) may be modified as follows:

$$\frac{N}{k}\frac{d\left(e + \frac{kL'}{N}i\right)}{dt} = E - Ri = R(i_2 - i) = R\Delta i \tag{97}$$

Substituting for Δi from equation (95) and solving for the time gives

$$t = \frac{Ne_2}{kE}\int_{e_1}^{e}\frac{d\left(e + \frac{kL'}{N}i\right)}{\Delta e} = T\int_{e_1}^{e}\frac{d\left(e + \frac{kL'}{N}i\right)}{\Delta e} \tag{98}$$

The numerator of the integrand represents the differential of the voltage due to the total flux referred to the armature, *i.e.*, the sum of the induced armature voltage and the voltage produced by the change in the leakage flux, the latter voltage being referred to the armature. For the purpose of integrating graphically, therefore, the reciprocal of Δe must be plotted versus the total voltage instead of against e alone. This may be done by adding the leakage term to the voltage read from the magnetization curve either analytically during the process of preparing data for the $1/\Delta e$ curve or graphically, using the principle previously illustrated in Fig. 224, by displacing the horizontal axis by an angle

$$\psi = \tan^{-1}\frac{kL'}{N} \tag{99}$$

and measuring ordinates from the so-displaced axis.

Equation (98) also holds for build-down subject to proper interpretation of the time constant and the quantity Δe as previously discussed.

Self-excited Machines. *Analytical Solutions.* 1. *Constant Coefficient of Dispersion.*—The differential equation for this case is equation (56) using, of course, the variable voltage on the right-hand side. Substituting in this for the time constant from equation (59) gives

$$\sigma T\frac{de}{dt} + Ri = e \tag{100}$$

which upon introducing the Fröhlich equation (63) becomes

$$\frac{\sigma Tab}{(b+i)^2}\frac{di}{dt} + Ri = \frac{ai}{b+i} \tag{101}$$

Separating the variables this may be written

$$\frac{\sigma Tab}{(b+i)^2\left(\frac{ai}{b+i} - Ri\right)}di = dt \tag{102}$$

which again reduces to

$$\frac{\sigma Tab}{(b+i)i(a - Rb - Ri)}di = dt \tag{103}$$

To integrate resolve into partial fractions as follows:

$$\frac{1}{(b+i)i(a - Rb - Ri)} = \frac{A}{b+i} + \frac{B}{i} + \frac{C}{a - Rb - Ri} \tag{104}$$

from which the following three simultaneous equations for the determination of the coefficients may be formed:

$$-AR - BR + C = 0 \tag{105}$$

$$A(a - Rb) + B(a - 2Rb) + Cb = 0 \tag{106}$$

$$B(a - Rb)b = 1 \tag{107}$$

Solution of these gives

$$A = -\frac{1}{ab} \tag{108}$$

$$B = \frac{1}{(a - Rb)b} \tag{109}$$

$$C = \frac{R^2}{(a - Rb)a} \tag{110}$$

If the partial fractions in equation (103) are inserted, the expression to be integrated becomes

$$\sigma T\left(-\frac{di}{b+i} + \frac{a}{a - Rb}\cdot\frac{di}{i} + \frac{bR^2}{a - Rb}\cdot\frac{di}{a - Rb - Ri}\right) = dt \tag{111}$$

giving

$$\sigma T\left[-\log(b+i) + \frac{a}{a - Rb}\log i - \frac{bR}{a - Rb}\log(a - Rb - Ri)\right] = t + K \tag{112}$$

Applying the initial condition that the current is i_1 at zero time, *viz.*,

$$i = i_1 \qquad \text{when} \qquad t = 0$$

the constant of integration K is given by

$$\sigma T\left[-\log(b + i_1) + \frac{a}{a - Rb}\log i_1 - \frac{bR}{a - Rb}\log(a - Rb - Ri_1)\right] = K \quad (113)$$

By substitution of this in equation (112) the final solution for the current-time relationship is obtained as

$$t = \sigma T\left[\frac{a}{a - Rb}\log\frac{i}{i_1} - \log\frac{b + i}{b + i_1} - \frac{Rb}{a - Rb}\log\frac{a - Rb - Ri}{a - Rb - Ri_1}\right] \quad (114)$$

To establish the formal voltage-time relation insert for the current in equation (100) by means of the rewritten Fröhlich equation [equation (70)]. This gives

$$\sigma T\frac{de}{dt} + \frac{Rbe}{a - e} = e \quad (115)$$

Separating the variables

$$\frac{\sigma T}{e - \dfrac{Rbe}{a - e}}de = dt \quad (116)$$

which again may be written

$$\frac{\sigma T a}{(a - Rb - e)e}de - \frac{\sigma T}{a - Rb - e}de = dt \quad (117)$$

The first term may be integrated by splitting it into partial fractions; thus

$$\frac{1}{(a - Rb - e)e} = \frac{A}{a - Rb - e} + \frac{B}{e} \quad (118)$$

from which are obtained the following simultaneous equations for the determination of the coefficients:

$$A - B = 0 \quad (119)$$

$$B(a - Rb) = 1 \quad (120)$$

giving

$$A = B = \frac{1}{a - Rb} \tag{121}$$

By substituting in equation (112), the expression to be integrated now becomes

$$\sigma T\left[\frac{a}{a - Rb}\left(\frac{de}{a - Rb - e} + \frac{de}{e}\right) - \frac{de}{a - Rb - e}\right] = dt \tag{122}$$

resulting in

$$\sigma T\left[\frac{a}{a - Rb}\log\frac{e}{a - Rb - e} + \log(a - Rb - e)\right] = t + K \tag{123}$$

Applying the initial condition that the voltage is e_1 at zero time, *viz.*,

$$e = e_1 \qquad \text{when} \qquad t = 0$$

the constant of integration K is given by

$$\sigma T\left[\frac{a}{a - Rb}\log\frac{e_1}{a - Rb - e_1} + \log(a - Rb - e_1)\right] = K \tag{124}$$

This inserted in equation (123) gives the desired solution for the voltage-time relationship as

$$t = \sigma T\left[\frac{a}{a - Rb}\log\frac{e(a - Rb - e_1)}{e_1(a - Rb - e)} - \log\frac{a - Rb - e_1}{a - Rb - e}\right] \tag{125}$$

The established solutions [equations (114) and (125)] are equally applicable to the build-up and the build-down process provided, of course, the proper resistance and time constant are used.

2. *Constant Leakage Inductance.*—Substituting for the time constant [equation (59)] in equation (57), the differential equation to be used becomes

$$r\left[\frac{de}{dt} + \frac{kL'}{N}\frac{di}{dt}\right] + Ri = e \tag{126}$$

which, using the Fröhlich equation [equation (63)], may be written

$$T\left[\frac{ab}{(b + i)^2} + \frac{kL'}{N}\right]\frac{di}{dt} + Ri = \frac{ai}{b + i} \tag{127}$$

Separating the variables gives

$$T\left[\frac{ab}{(b+i)^2\left(\frac{ai}{b+i} - Ri\right)} + \frac{kL'}{N\left(\frac{ai}{b+i} - Ri\right)}\right]di = dt \qquad (128)$$

which again reduces to

$$T\left[\frac{ab}{(b+i)i(a - Rb - Ri)} + \frac{kL'b}{Ni(a - Rb - Ri)} + \frac{kL'}{N(a - Rb - Ri)}\right]di = dt \qquad (129)$$

The first term may be put into a form suitable for integration by reference to the equivalent expression in equation (104) in conjunction with equations (108), (109), and (110). The second term is handled by partial fractions, *viz.*,

$$\frac{1}{i(a - Rb - Ri)} = \frac{A}{i} + \frac{B}{a - Rb - Ri} \qquad (130)$$

from which are obtained the simultaneous equations

$$-AR + B = 0 \qquad (131)$$
$$Aa + ARb = 1 \qquad (132)$$

yielding the coefficients

$$A = \frac{1}{a - Rb} \qquad (133)$$

$$B = \frac{R}{a - Rb} \qquad (134)$$

Inserting in equation (129) the expression to be integrated becomes

$$T\left[K_1\frac{di}{i} - \frac{di}{b+i} + K_2\frac{di}{a - Rb - Ri}\right] = dt \qquad (135)$$

where

$$K_1 = \frac{a + \frac{kL'}{N}b}{a - Rb} = \frac{a + \frac{T'}{T}Rb}{a - Rb} \qquad (136)$$

$$K_2 = \frac{Rb + \frac{kL'}{N}b}{a - Rb} + \frac{KL'}{NR} = \frac{\frac{T'}{T}a + Rb}{a - Rb} \qquad (137)$$

Carrying out the integrations gives

$$T[K_1 \log i - \log (b + i) - K_2 \log (a - Rb - Ri)] = t + K \tag{138}$$

Using the initial condition that the current is i_1 at zero time, *i.e.*, that

$$i = i_1 \qquad \text{when} \qquad t = 0$$

the constant of integration K becomes

$$T[K_1 \log i_1 = \log (b + i_1) - K_2 \log (a - Rb - Ri_1)] = K \tag{139}$$

and the final solution for the current-time relationship is obtained as

$$t = T\left[K_1 \log \frac{i}{i_1} - \log \frac{b + i}{b + i_1} - K_2 \log \frac{a - Rb - Ri}{a - Rb - Ri_1}\right] \tag{140}$$

In order to develop the formal voltage-time relation the Fröhlich equation [equation (70)] is used in equation (126) giving

$$T\left(1 + \frac{kL'ab}{N(a - e)^2}\right)\frac{de}{dt} + \frac{Rbe}{a - e} = e \tag{141}$$

which, separating the variables, may be written

$$T\left[\frac{1}{e - \dfrac{Rbe}{a - e}} + \frac{kL'ab}{N(a - e)^2\left(e - \dfrac{Rbe}{a - e}\right)}\right]de = dt \tag{142}$$

which in turn reduces to

$$T\left[\frac{a}{(a - Rb - e)e}de - \frac{1}{a - Rb - e}de + \frac{kL'ab}{N(a - e)e(a - Rb - e)}de\right] = dt \tag{143}$$

The first term may be handled by reference to equations (118) and (121), the third by resolution into partial fractions as follows:

$$\frac{1}{(a - e)e(a - Rb - e)} = \frac{A}{a - e} + \frac{B}{e} + \frac{C}{a - Rb - e} \tag{144}$$

From equation (144) three simultaneous equations are obtained, *viz.*,

$$-A + B - C = 0 \tag{145}$$

$$A(a - Rb) - B(2a - Rb) + Ca = 0 \tag{146}$$

$$Ba(a - Rb) = 1 \tag{147}$$

the solution of which gives the coefficients as

$$A = -\frac{1}{Rab} \tag{148}$$

$$B = \frac{1}{a(a - Rb)} \tag{149}$$

$$C = \frac{1}{Rb(a - Rb)} \tag{150}$$

Making substitutions equation (143) may now be written

$$T\left(K_1\frac{de}{e} - K_2\frac{de}{a - e} - K_3\frac{de}{a - Rb - e}\right) = dt \tag{151}$$

where the constants are given by

$$K_1 = \frac{a + \dfrac{kL'}{N}b}{a - Rb} = \frac{a + \dfrac{T'}{T}Rb}{a - Rb} \tag{152}$$

$$K_2 = \frac{kL'}{NR} = \frac{T'}{T} \tag{153}$$

$$K_3 = 1 - \frac{a\left(1 + \dfrac{kL'}{NR}\right)}{a - Rb} = -\frac{\dfrac{T'}{T}a + Rb}{a - Rb} \tag{154}$$

Equation (151) gives upon integration

$$T[K_1 \log e \; K_2 \log (a - e) + K_3 \log (a - Rb - e)] = t + K \tag{155}$$

Using the initial condition that the voltage is e_1 at zero time, *i.e.*, that

$$e = e_1 \qquad \text{when} \qquad t = 0$$

the integration constant K becomes

$$T[K_1 \log e_1 + K_2 \log (a - e_1) + K_3 \log (a - Rb - e_1)] = K \tag{156}$$

Inserting this the final solution for the voltage-time relationship is obtained as

$$t = T\left[K_1 \log \frac{e}{e_1} + K_2 \log \frac{a - e}{a - e_1} + K_3 \log \frac{a - Rb - e}{a - Rb - e_1}\right] \quad (157)$$

Solution by Graphical Integration.[1] 1. *Constant Coefficient of Dispersion.*—As for the separately excited machine the differential equation (56) may also here be written in a more convenient form for graphical integration, *viz.*,

$$\frac{\sigma N}{K}\frac{de}{dt} = e - Ri = \Delta e \quad (158)$$

The quantity Δe represents the difference between the induced voltage and the resistance drop in the field circuit. Graphically it is given by the vertical distance between the magnetization curve and the straight line drawn from the origin through the final operating point, the latter being defined by the final voltage e_2 or current i_2. With the self-excited machine, this straight line has a slope equal to the field resistance and is the *resistance line of the field.* It will be noted the Δe's do not represent identical quantities for the separately and the self-excited machine. This is important from the standpoint of rapidity of response as will be discussed in full later.

Solving equation (158) explicitly for the time gives

$$t = \frac{\sigma N}{K}\int_{e_1}^{e}\frac{de}{\Delta e} = \sigma T\int_{e_1}^{e}\frac{de}{\Delta e} \quad (159)$$

which lends itself well to graphical integration by processes already described. Also here the integral itself represents a pure numeric. The quantity Δe should here, as for the separately excited machine, always be taken with respect to the resistance line through the final operating point. The time constant, on the other hand, need not correspond to final flux conditions, since, as previously pointed out, it is definitely constant and independent of the fluxes. The equation is applicable to the build-up as well as to the build-down process.

Figure 234 illustrates the solution for build-up. In (*a*) is shown the magnetization curve, in (*b*) a plot of Δe versus voltage,[2]

[1] RÜDENBERG, *loc. cit.*

[2] This curve, superfluous for the graphical solution, is included in order to show its shape, which may be *roughly* approximated by a parabola.

and in (*c*) the reciprocal of Δe against voltage. The shaded portion in the latter indicates one of the successive areas which are measured for evaluation of the integral. The voltage- and current-time curves are shown in (*d*).

Figure 235(*a*), (*b*), (*c*), and (*d*) illustrates in a similar manner how the solution is carried out during build-down. Note how

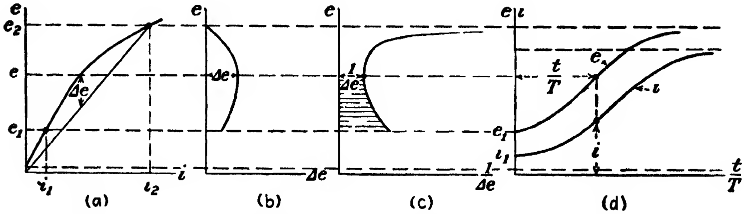


FIG. 234.—Build-up of voltage and field current of self-excited exciter. Solution by graphical integration.

Δe is measured between the magnetization curve and the resistance line through the final—now the lower—operating point. Δe is here negative.

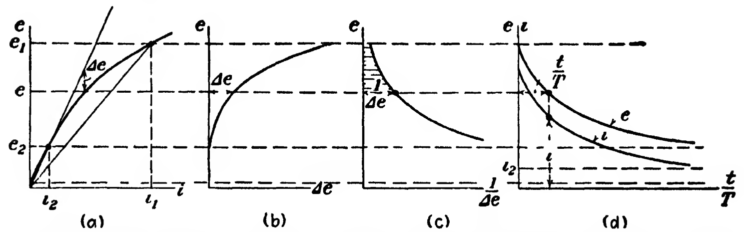


FIG. 235 —Build-down of voltage and field current of self-excited exciter Solution by graphical integration

2. *Constant Leakage Inductance.*—The differential equation (57) which now is used may be written

$$\frac{N}{k}\frac{d\left(e + \frac{kL'}{N}i\right)}{dt} = e - Ri = \Delta e \tag{160}$$

which solved for the time gives

$$t = \frac{N}{K}\int_{e_1}^{e}\frac{d\left(e + \frac{kL'}{N}i\right)}{\Delta e} = T\int_{e_1}^{e}\frac{d\left(e + \frac{L'}{T}i\right)}{\Delta e} \tag{161}$$

The integrand in this expression is identical with that in equation (98) for the separately excited machine and the same

discussion applies. Figure 236(*a*), (*b*), (*c*), and (*d*) illustrates the graphical solution for a build-up process. The effect of leakage

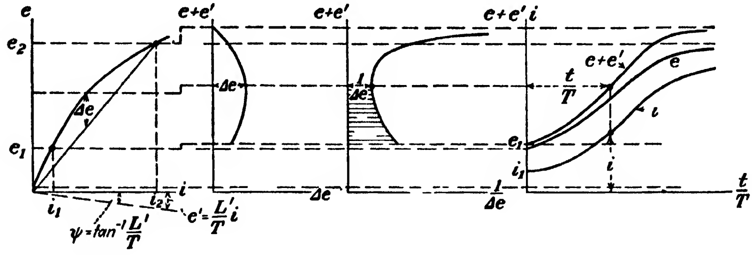


FIG. 236.—Build-up of voltage and field current of self-excited exciter. Solution by graphical integration when field leakage inductance is considered constant.

is, as seen in (*a*), taken into account by displacement of the horizontal axis for the magnetization curve.

Equation (161) also holds for build-down subject to proper interpretation of the time constant and the quantity Δe as previously pointed out.

An Approximate Analytical Solution.—Inspection of Fig. 234(*b*) shows that the curve of Δe against e bears a certain resemblance to a parabola. An *approximate solution* may be established on the assumption that the relation between the voltage difference Δe and the induced armature voltage e is represented by a *symmetrical parabola*,[1] as suggested in Fig. 237.

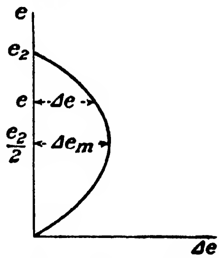


FIG. 237.—Voltage differential during build-up for self-excited exciter (see Fig. 234(*b*)) approximated by a symmetrical parabola.

The general equation for a parabola in x-y coordinates with its axis parallel to the x-axis and its concave side towards the left is

$$(y - n)^2 = -p(x - m) \tag{162}$$

For the symmetrical case at hand, Fig. 236, evidently

$$n = \frac{e_2}{2} \qquad \text{and} \qquad m = \Delta e_m \tag{163}$$

[1] RÜDENBERG, *loc. cit.*

Hence

$$\left(e - \frac{e_2}{2}\right)^2 = -p(\Delta e - \Delta e_m) \qquad (164)$$

When $\Delta e = 0$, then $e = 0$ or $e = e_2$, and consequently

$$p = \frac{e_2^2}{4\Delta e_m} \qquad (165)$$

The equation for the symmetrical parabola is therefore

$$\left(e - \frac{e_2}{2}\right)^2 = -\frac{e_2^2}{4\Delta e_m}(\Delta e - \Delta e_m) \qquad (166)$$

from which Δe is obtained as

$$\Delta e = \frac{4\Delta e_m}{e_2^2} e(e_2 - e) \qquad (167)$$

Using equation (155) and substituting for Δe by means of the above equation gives

$$t = \sigma T \int_{e_1}^{e} \frac{de}{\Delta e} = \frac{\sigma T e_2^2}{4\Delta e_m} \int_{e_1}^{e} \frac{de}{e(e_2 - e)} \qquad (168)$$

from which the following solution is obtained:

$$t = \frac{\sigma T e_2}{4\Delta e_m} \log \frac{e(e_2 - e_1)}{e_1(e_2 - e)} \qquad (169)$$

This may be solved explicitly for the voltage. To abbreviate, let

$$\alpha = \frac{4\Delta e_m}{\sigma T e_2} \qquad (170)$$

and

$$C = \frac{e_2 - e_1}{e} \qquad (171)$$

Then from equation (169)

$$\epsilon^{\alpha t} = \frac{Ce}{e_2 - e} \qquad (172)$$

from which the solution for the voltage is obtained as

$$e = \frac{e_2}{1 + Ce^{-\alpha t}} \qquad (173)$$

This expression gives, as seen, a simple explicit solution for the voltage during build-up as a function of time. It should, however, be considered as a comparatively rough approximation only, convenient perhaps in certain instances for rapid estimating purposes.

EXAMPLE 4

Statement of Problem

The sketch (Fig. 238) shows a self-excited compound-wound exciter operating with a long shunt.

The magnetization curve with separate excitation applied to the shunt field is known in terms of armature open-circuit voltage versus shunt-field current. The resistances of the shunt and series fields are R_f and R_s, respectively, and the number of turns in these fields, N_f and N_s.

This machine operates open-circuited and is driven at constant speed when the external resistance R in the shunt-field circuit is suddenly short-circuited. Outline a method of determining the build-up curve of the open-circuit terminal voltage versus time.

Neglect armature resistance as well as armature inductance and reaction. Also neglect all leakage fluxes.

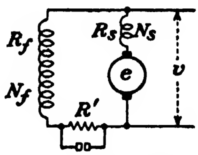


FIG. 238.—Self-excited compound-wound exciter considered in Example 4.

Solution

Since the leakage fluxes are to be neglected, all the turns of the shunt field, as well as all the turns of the series field, will link with the same flux, and the following differential equation applies:

$$N_f\frac{d\phi}{dt} + N_s\frac{d\phi}{dt} + R_f i + R_s i = e \tag{a}$$

Introducing

$$N_f + N_s = N_0 \qquad \text{and} \qquad R_f + R_s = R_0 \tag{b}$$

this differential equation reduces to

$$N_0\frac{d\phi}{dt} + R_0 i = e \tag{c}$$

which, in turn, may be written

$$T_0\frac{de}{dt} = e - R_0 i = \Delta e \tag{d}$$

where the time constant T_0 is defined by

$$T_0 = \frac{N_0\phi}{e} = \frac{N_0\phi}{k\phi} = \frac{N_0}{k} \tag{e}$$

Solving equation (*d*) explicitly for the time gives

$$t = T_0 \int_{e_1}^{e} \frac{de}{\Delta e} \tag{f}$$

which is in a convenient form for graphical integration. The original magnetization curve gives the relationship between induced armature voltage and shunt-field current. Its shape is modified by the fact that the series field, when connected, also contributes to the excitation. Before performing the graphical integration, therefore (by the standard processes described in the text), the magnetization curve must be corrected by multiplying the abscissa i_f by the ratio of the number of shunt field turns to the sum of the turns in the shunt field and the series field $N_f/(N_f + N_s)$. The graphical integration is then carried out as indicated in Fig. 234, giving the desired build-up of voltage and field current.

NOTE. Since the number of turns in the series field usually is negligible as compared with the number of turns in the shunt field, and therefore has only negligible effect when the exciter operates on open circuit, the above mentioned correction of the magnetization curve may be omitted. Also the time constant T_0 may be computed using the number of shunt turns only (*i.e.*, in the nomenclature heretofore used, T_0 becomes identical with T).

EXAMPLE 5

Statement of Problem

A certain self-excited exciter is designed to operate at moderate flux densities, the effect of saturation being very small. It is contemplated to modify this design so as to increase the flux density in the pole cores and the yokes. Assuming that the dimensions, circuits, and constants are otherwise unchanged, determine the influence of this modification in design as regards (*a*) ceiling voltage, (*b*) initial build-up rate from a voltage of, say, 20 per cent of normal, and (*c*) time of build-up from initial voltage to substantially ceiling voltage.

Solution

The differential equation (assuming constant coefficient of dispersion) is [see equation (158)]

$$\frac{\sigma N}{k} \frac{de}{dt} = \sigma T \frac{de}{dt} = e - Ri = \Delta e \tag{a}$$

The build-up rate and the time of build-up are hence given by

$$\frac{de}{dt} = \frac{\Delta e}{\sigma T} \tag{b}$$

and

$$t = \sigma T \int_{e_1}^{e} \frac{de}{\Delta e} \tag{c}$$

In the following, subscripts I and II will be used to designate quantities before and after the design has been modified so as to increase the flux density in the pole cores and yokes, respectively Since dimensions are not altered, a higher flux density will result in saturation effects being more pronounced, and hence give rise to a magnetization curve which is more curved than in the less saturated case, as indicated in Fig 239(*a*)

a The ceiling voltages are obtained at the intersection of the field-resistance line with the magnetization curves Referring to Fig 239, it is evident that the ceiling voltage decreases when the design is modified as contemplated

b The initial build-up rates are given by, and may be compared on the basis of, the following equation

$$\left(\frac{de}{dt}\right)_0 = \left(\frac{\Delta e}{\sigma T}\right)_{t=0} \tag{d}$$

Build-up takes place from an initial voltage of 20 per cent of normal It is obvious from Fig 239 that the voltage differentials are substantially equal

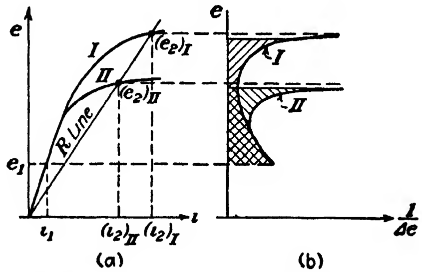


Fig 239 —(*a*) Magnetization curves and (*b*) curves of $1/\Delta e$ for self-excited exciter considered in Example 5 (I) as originally designed (II) after modification in design increasing the flux density in the pole cores and the yoke

for the two cases, since the operating point lies on the straight part of the magnetization curve Furthermore, the time constants are identical. In other words,

$$(\Delta e_I)_0 = (\Delta e_{II})_0 \qquad \text{and} \qquad T_I = T_{II} \tag{e}$$

Hence the initial build-up rates are the same before and after the design modification

c The time of build-up to substantially ceiling voltage, say 95 per cent of this voltage, is given by

$$\tau = \sigma T \int_{e_1}^{0\ 95 e_2} \frac{de}{\Delta e} \tag{f}$$

Since the time constants are identical in the two cases, conclusions may be based upon a comparison of the integral, the latter being proportional to the cross-hatched areas in Fig 239(*b*) The ceiling voltage, as already stated,

is larger for the less saturated case, and it will be noted that the power differentials are larger for this case also. In other words,

$$(e_2)_I > (e_2)_{II} \qquad \text{and} \qquad (\Delta e)_I > (\Delta e)_{II} \tag{g}$$

It is seen from the equation for build-up time that an increase in ceiling voltage and an increase in voltage differential have opposite effects and therefore cannot be definitely evaluated except for a specific numerical case. It is probable, however, that the effect of the decreased ceiling voltage is predominant, and inspection of the cross-hatched areas also seems to bring out this point. Hence it is reasonable to anticipate a slightly decreased build-up time by the design modification contemplated.

EXAMPLE 6

Statement of Problem

Discuss the effect on (*a*) the initial build-up rate and (*b*) the time of build-up of the self-excited direct-current machine of

1. increasing the field resistance;
2. increasing the speed;
3. increasing the shunt field turns;
4. carrying out both (1) and (2) in the same ratio.

In each of the above cases all other design constants are assumed to remain unchanged.

Solution

The following equations give the response rate and the time of build-up to 95 per cent of ceiling voltage:

$$\frac{de}{dt} = \frac{e - Ri}{\sigma T} = \frac{\Delta e}{\sigma T} \tag{a}$$

$$\tau = \sigma T \int_{e_1}^{0.95 e_2} \frac{de}{\Delta e} \tag{b}$$

The time constant and the proportionality factor k are given by

$$T = \frac{N\phi}{e} = \frac{N\phi}{k\phi} = \frac{N}{k} \tag{c}$$

and

$$k = \frac{e}{\phi} = \frac{p}{a} \cdot \frac{nZ}{60} \tag{d}$$

It is assumed in what follows that the ceiling voltage e_2 lies well up on the saturated part of the magnetization curve. Quantities designated by primes refer to conditions after the specified changes have taken place.

a. The effect of increase in field resistance is illustrated in Fig. 240. It is assumed that the initial operating point is kept unchanged by adjustment of the external resistance. Changing the field resistance does not affect the

time constant. It decreases, however, the ceiling voltage by a relatively small amount, since it is well up on the saturation curve, and decreases the voltage differential Δe.

Hence, referring to equation (*a*), the initial response rate is decreased. Referring to equation (*b*), the value of the integral is increased owing to the fact that the effect of the smaller Δe most probably is predominant over the effect of the decreased ceiling voltage. Hence, increasing the field resistance will increase the time of build-up.

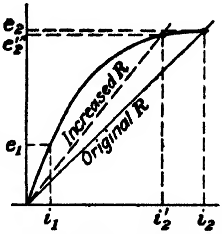


FIG. 240.—Effect of increasing field resistance of self-excited exciter (Example 6).

b. Increasing the speed increases the proportionality factor k in direct proportion, and the time constant in inverse proportion. The induced voltage for a given field current is increased in direct proportion to the increase in speed, as illustrated by the magnetization curves in Fig. 241(*a*). It is further seen that the initial operating voltage as well as the final operating voltage (the ceiling voltage) is increased.

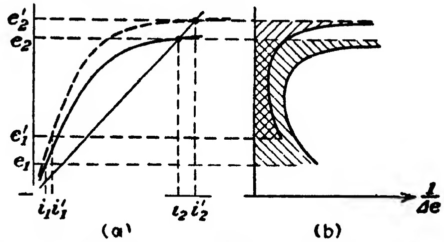


FIG. 241.—Effect of increasing the speed of rotation of self-excited exciter (Example 6). The top curves correspond to the higher speed.

Referring to equation (*a*), therefore, the initial response is increased. In equation (*b*) the effect of the increased Δe is predominant over the increase

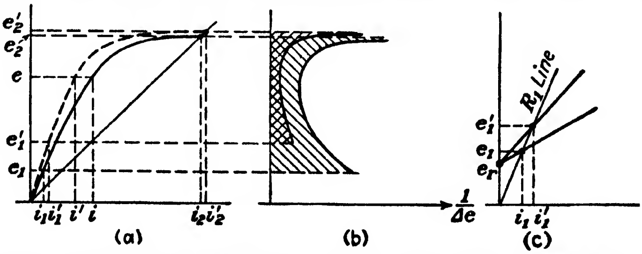


FIG. 242.—Effect of increasing the number of shunt-field turns of self-excited exciter (Example 6). In (*a*) and (*b*) the top curves correspond to the increased number of turns. In (*c*) the situation at the lower straight-line part of the magnetization curves is shown to an enlarged scale.

in ceiling voltage, and the value of the integral therefore is decreased, as illustrated by the areas in Fig. 241(*b*). This fact, combined with the decrease in time constant, will result in a decreased time of build-up.

c. Increasing the number of field turns increases the time constant in direct proportion. To produce the same induced voltage, a smaller field current (it is proportional to the inverse-turn ratio) is required, as illustrated by the magnetization curves in Fig. 242(*a*). The field resistance line remains the same, and (assuming also the same external resistance) increasing the number of turns will increase the initial voltage as well as the final voltage. The former will increase almost in the same proportion as the turn ratio, owing to the linearity of the magnetization curve in this range; and the latter (the ceiling voltage) will increase by a small amount only, owing to the effect of saturation. The voltage differential Δe increases.

Referring to equation (*a*), therefore, it is seen that the increased voltage differential tends to increase the initial response rate, while the increased time constant tends to have the opposite effect. The net result in a specific case depends upon relative values. Further light on the situation is thrown by the following analysis:

Figure 242(*c*) shows the lower straight part of the magnetization curves [*i.e.*, the lower left corner of Fig. 242(*a*)] enlarged. They start at the same residual voltage, but have different slopes, m and m'. The relationship between these is determined from the following equations written for the same voltage:

$$e = e_r + mi = e_r + m'i' = e_r + m'\frac{N}{N'}i \tag{e}$$

from which

$$m' = \frac{N'}{N}m \tag{f}$$

The voltages obtained at the points of intersection between the magnetization curves and the resistance line corresponding to the initial resistance are given by

$$e_1' = R_1 i_1' = e_r + m'i_1' \tag{g}$$
$$e_1 = R_1 i_1 = e_r + mi_1 \tag{h}$$

By subtraction is obtained

$$R_1(i_1' - i_1) = m'i_1' - mi_1 = m\left(\frac{N'}{N}i_1' - i_1\right) \tag{i}$$

from which the ratio of the two initial values of current becomes

$$\frac{i_1'}{i_1} = \frac{R_1 - m}{R_1 - \dfrac{N'}{N}m} \tag{j}$$

The ratio of the initial build-up rates [see equation (*a*)] is

$$\frac{\left(\dfrac{de}{dt}\right)_0'}{\left(\dfrac{de}{dt}\right)_0} = \frac{\dfrac{e_1' - Ri_1'}{\sigma\frac{N'}{N}T}}{\dfrac{e_1 - Ri_1}{\sigma T}} = \frac{N\,(R_1 - R)i_1'}{N'(R_1 - R)i_1} = \frac{Ni_1'}{N'i_1} \tag{k}$$

which, upon substitution from equation (i), reduces to

$$\frac{\left(\frac{de}{dt}\right)_0'}{\left(\frac{de}{dt}\right)_0} = \frac{N}{N'}\,\frac{\frac{R_1}{m} - 1}{\frac{R_1}{m} - \frac{N'}{N}} = \frac{\frac{R_1}{m} - 1}{\frac{N'}{N}\left(\frac{R_1}{m} - \frac{N'}{N}\right)} \tag{l}$$

An examination of this equation indicates that an increase in the number of field turns may result in an increase, as well as a decrease, in the initial response rate, depending upon the relative values of the quantities involved. It may be noted that the turn ratio N'/N can never be larger than R_1/m. If they were equal, an infinite response ratio would theoretically be obtained—an obviously entirely trivial case. The quantity R_1/m must always be larger than 1, as otherwise no operating point would be obtained, *i.e.*, there will be no intersection between the lower straight part of the magnetization curve and the resistance line. Its value may not be very much larger than 1, however, and probably seldom exceeds 1.5. The practical possibilities, therefore, for increasing the number of field turns (without at the same time modifying the design in many other respects) is limited. With these practical restrictions, the effect of an increase in the number of field turns will also be an increase in the initial response rate.

It may be of interest to substantiate this by a numerical example based on reasonable initial operating conditions. For instance, consider a residual voltage of 5 per cent and an initial operating voltage of 25 per cent. Then

$$e_r = 0.05 \qquad \text{and} \qquad R_1 i_1 = 0.05 + m i_1 = 0.25 \tag{m}$$

from which are obtained

$$m i_1 = 0.20 \qquad \text{and} \qquad \frac{R_1}{m} = \frac{0.25}{0.2} = 1.25 \tag{n}$$

These values may now be inserted in equation (l) giving

$$\frac{\left(\frac{de}{dt}\right)_0'}{\left(\frac{de}{dt}\right)_0} = \frac{1.25 - 1}{\frac{N'}{N}\left(1.25 - \frac{N'}{N}\right)} = \frac{0.25}{\frac{N'}{N}\left(1.25 - \frac{N'}{N}\right)} \tag{o}$$

For illustration assume increases in the number of field turns of 20, 10, 2 and 1 per cent, respectively. Using these values, the following ratios of initial build-up rates result:

N'/N	$\frac{(de'/dt)_0}{(de/dt)_0}$
1.20	4.17
1.10	1.51
1.02	1.065
1.01	1.03

These values bear out the correctness of the conclusions previously set forth.

The time constant increases directly as the number of turns, as previously pointed out, but it is reasonable to expect that the value of the integral [the area in Fig. 242(*b*)] will decrease in more than direct proportion. It has been demonstrated that the initial response rate, and consequently the voltage differential Δe, increase in more than direct proportion, and it is consequently reasonable to expect that the area depending upon Δe will do likewise. The net probable effect of an increase in the number of field turns, therefore, should be a slight decrease in the time of build-up.

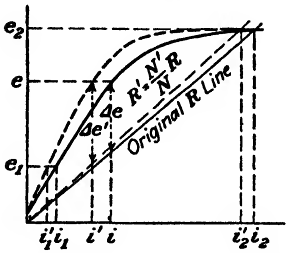


FIG. 243.—Effect on magnetization curve and field-resistance line of self-excited exciter of increasing the field resistance (including external resistance) and the shunt-field turns in this same ratio (Example 6).

d. The time constant is affected as in the previous case (*c*), and the magnetization curves are also similarly affected, as illustrated in Fig. 243. Since, however, the field resistance (including the external resistance) is increased in the same proportion as the number of turns, different resistance lines must be used in the two cases, as also indicated in Fig. 243. This results in the same initial and final operating voltages, although, of course, the initial and final currents will be different. Furthermore, the voltage differential Δe is not affected by the change.

Referring to equation (*a*), it is evident that the initial response rate will decrease in direct ratio as the number of field turns is increased. Equation (*b*) shows that the value of the integral remains the same as before, and the only factor modifying the build-up, therefore, is the time constant. Hence the time of build-up is increased in direct ratio as the number of field turns.

EXAMPLE 7

Statement of Problem

A four-pole, self-excited exciter is designed to operate at a fairly high saturation. This exciter is being driven at a constant speed and is disconnected from the alternator field circuit which it usually supplies. The field switch is closed and the voltage across the armature terminals starts to build up. It requires a certain time to build up from the residual value to, say, 90 per cent of its final value. Discuss the effect on this time when the following changes are made:

a. The length of the air gap is increased by a small amount, say 20 per cent.

b. The armature winding is changed from simplex lap to simplex wave, and the cross section of the armature conductors is halved.

All the other factors are assumed to remain the same. A qualitative discussion is all that is needed in (*a*), but a numerical answer should be given for (*b*). Neglect armature resistance and armature reaction.

Solution

The time of build-up from residual voltage to 90 per cent of the final voltage is given by

$$\tau = \sigma T \int_{e_1}^{0.9 e_2} \frac{de}{\Delta e} \tag{a}$$

a It is assumed that increasing the air gap by about 20 per cent does not materially change the field leakage (coefficient of dispersion). The reluctance in the air gap is increased in proportion to its lengthening, and hence the magnetization curve will be about 20 per cent lower over the straight portion, the difference decreasing as saturation is approached. This is illustrated in Fig. 244. It is apparent from

$$T = \frac{N\phi}{e} = \frac{N}{k} = \frac{60aN}{pZn} \tag{b}$$

that the time constant remains unaltered. It will be noted, however, that the voltage differential Δe is a good deal smaller when the air gap is lengthened. At the same time, however, the upper voltage limit—the ceiling—is decreased. Thus there are two counteracting effects present, but it is probable that the effect of the decreased voltage differential will predominate. Accordingly, the time of build-up is increased by lengthening of the air gap. It should be borne in mind, however, that a completely definite conclusion would have to be based on a specific numerical case.

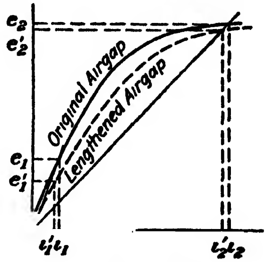


FIG. 244.—Effect of lengthening the air gap of self-excited exciter (Example 7).

b The following table relates to number of poles, number of parallel paths in the armature winding, and induced armature voltage for simplex lap and simplex wave windings, asuming the same flux.

	Simplex lap	Simplex wave
Number of poles, p	4	4
Number of parallel paths, a	4	2
Ratio, p/a	1	2
Voltage	e	$2e$

The shape of the magnetization curve is not affected by a change in winding type, but the voltage scale would be doubled for the wave winding. Referring to equation (*b*), it is obvious that the time constant for the wave winding would be one-half that for the lap winding, *i.e.*,

$$T_{\text{wave}} = \tfrac{1}{2} T_{\text{lap}} \tag{c}$$

The integral part of the equation for time of build-up is equal for the two winding types, since

$$\left[\int_{2e_1}^{0.9(2e_2)} \frac{de}{2\Delta e}\right]_{\text{wave}} = \left[\int_{e_1}^{0.9e_2} \frac{de}{\Delta e}\right]_{\text{lap}} \tag{d}$$

Hence the time of build-up is halved when the winding is changed from simplex lap to simplex wave.

Exciter Response.—It is evident from the preceding that, when an exciter is called upon to change its voltage by virtue of a sudden change of resistance in its field circuit, the build-up or build-down takes place gradually. Furthermore, during the change the *rate of change* of voltage with respect to time is *not constant*. In other words, the *response rate*, or *exciter speed* as it is also termed, expressed in volts per second is a variable.

The response rate of an exciter is an important quantity which has considerable bearing on its effectiveness as a regulating agent, and particularly so in connection with the question of stability.

The American Institute of Electrical Engineers is contemplating standardization of definitions regarding the response of excitation systems. Final formulations are not yet available. As far as essential ideas go, however, the following statements may serve the purpose:

1. *Exciter Response.*—Exciter response is the rate of build-up or build-down in volts per second of the main exciter voltage when a short circuit across the regulating rheostat in the exciter field is suddenly applied or removed by the action of the voltage regulator.

NOTE: The response, which, in general, varies appreciably during the build-up or build-down process, depends on the characteristics of the excitation system as a unit, on the value of exciter voltage, and on the characteristics of the alternator field and armature circuits.

2. *Nominal Exciter Response.*—*a.* The nominal exciter response is the constant rate of build-up in volts per second of the main exciter voltage which, beginning at nominal slip-ring voltage, will give the same area in volt-seconds as the actual build-up curve starting at the same initial voltage taken over a period of one-half second with the exciter operating at rated speed and at no load when its shunt-field regulating resistance is suddenly short-circuited, leaving in the circuit the fixed resistance, if any, required to limit the maximum exciter voltage or field current. If the exciter is separately excited the voltage of the pilot exciter, battery, or other source used, shall be the same as will be used in actual operation.

b. Nominal slip-ring voltage is the voltage across the alternator slip rings required to generate rated kilovolt-amperes at rated voltage, frequency and power factor with the field winding at a temperature of 75 degrees C.

NOTE: The nominal response defined above equals eight (8) times the volt-second area between the build-up curve, starting at nominal slip-ring voltage, and a line representing this voltage, taken over a period of one-half second. This fact may conveniently be made use of

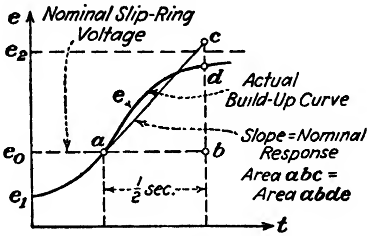


FIG. 245.—Determination of nominal exciter response when the experimental or computed build-up curve is available.

in evaluating the response rate from test data. Figure 245 illustrates how the nominal exciter response is determined.

The first definition of exciter response given above is obviously the *fundamental and general one*. It takes proper cognizance of the fact that the response rate is a variable and of the several factors which influence it. The nominal response given by the second definition, on the other hand, is a *definite quantity* readily determinable by calculation or by test with the exciter operating at no load. A definition of this nature is necessary in order that confusion shall not arise when mentioning numerical values of response rates in volts per second. The nominal exciter response, therefore, is suitable for the purpose of specifications, and as a basis for guarantees. It is also useful as a basis for *approximately* comparing the effectiveness of excitation systems.

In addition to being expressed in volts per second, response rates may also be stated as *ratios with respect to nominal slip-ring voltage* $\left(\frac{de/dt}{e_0}\right)$ or rated terminal voltage $\left(\frac{de/dt}{e_n}\right)$.

EXAMPLE 8

Statement of Problem

The following data relate to a self-excited shunt-wound exciter:

Rating.	200 kw.
Voltage (normal).	250 volts

Armature resistance	0.0113 ohm
Armature inductance	0.0063 henry
Field resistance (hot)	23.3 ohms
Nominal field inductance	26.6 henrys
Total resistance available for use in series with shunt field	66.7 ohms
100 per cent field current	6.6 amp.
Coefficient of dispersion	1.15

NOTE: The nominal field inductance may be defined as the number of flux linkages per field ampere at rated armature voltage with the armature winding open-circuited.

DATA FOR MAGNETIZATION CURVE.
OBTAINED WITH THE MACHINE SELF-EXCITED

Percentage Voltage	Percentage Field Current
4	0
40	34
60	54
80	75
90	86
100	100
110	117.5
120	139
130	170
135	192
140	219
143	240

a. Determine the nominal response of this exciter when the nominal slip-ring voltage of the alternator which it will supply is 188 volts.

b. Calculate an approximate value for the nominal response, using an analytical method based on the representation of the Δe versus e curve by a symmetrical parabola.

c. What time is required for build-up from nominal slip-ring voltage to 95 per cent of the ceiling voltage when all external resistance is suddenly short-circuited?

d. What is the build-up time specified in (*c*) when based on the nominal response as a constant build-up rate?

Solution

a. Figure 246 shows the magnetization curve and the field-resistance line with field rheostat short-circuited. From these two curves the third curve giving $1/\Delta e$ versus e is plotted.

The differential equation for build-up is [see equations (158) and (159)]

$$\frac{\sigma N}{k}\frac{de}{dt} = e - Ri = \Delta e \qquad (a)$$

or

$$t = \frac{\sigma N}{k}\int_1^e \frac{de}{\Delta e} = \sigma T \int_1^e \frac{de}{\Delta e} \tag{b}$$

In this case $T = N/k = N\phi/Ri = L_0/\sigma R = 26.6/1.15 \times 37.9 = 0.610$ sec., since nominal field inductance is 26.6 henrys and normal field resistance is 250/6.6 or 37.9 ohms. Consequently,

$$t = 1.15 \times 0.610 \int_{e_1}^{e} \frac{de}{\Delta e} = 0.702 \int_{0.75}^{e} \frac{de}{\Delta e} \tag{c}$$

Evaluation of this integral by taking successive areas between the $1/\Delta e$ curve and the voltage axis gives the build-up curve in Fig. 247. From this curve

$$\begin{aligned}\text{Nominal response} = 0.55 \text{ per unit per sec.} &= 0.53 \times 250 \\ &= 132.5 \text{ volts per sec.}\end{aligned}$$

b. By referring to equation (173), the approximate expression for the voltage during build-up is

$$e = \frac{e_2}{1 + C\epsilon^{-\alpha t}} = \frac{e_2}{1 + \dfrac{e_2 - e_1}{e_1}\epsilon^{-\alpha t}} \tag{d}$$

where

$$\alpha = \frac{4\Delta e_m}{\sigma T e_2} \tag{e}$$

From magnetization curve and field-resistance line,

$$\Delta e_m = 38.7 \text{ per cent} = 0.387 \text{ per unit}$$

Hence

$$\alpha = \frac{4 \times 0.387}{1.15 \times 0.610 \times 1.41} = 1.56$$

and

$$e = \frac{1.41}{1 + \dfrac{1.41 - 0.75}{0.75}\epsilon^{-1.56t}} = \frac{1.41}{1 + 0.88\epsilon^{-1.56t}}$$

This build-up curve is also plotted in Fig. 247. From this curve

Nominal response = 0.52 per unit per sec. = 0.52 × 250 = 130 volts per sec.

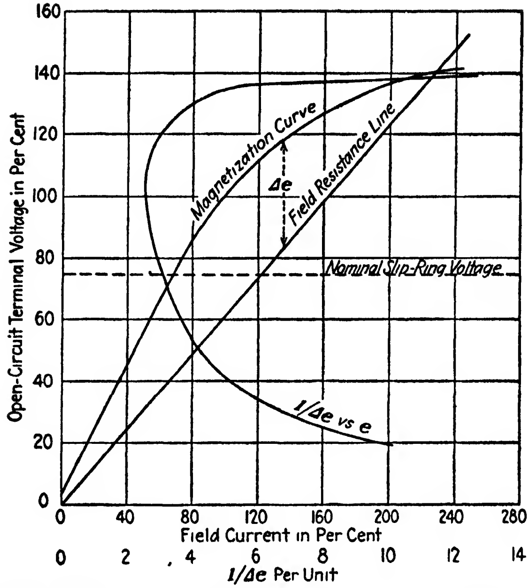


FIG. 246.—Magnetization curve, field-resistance line, and curve of $1/\Delta e$ for the self-excited exciter in Example 8.

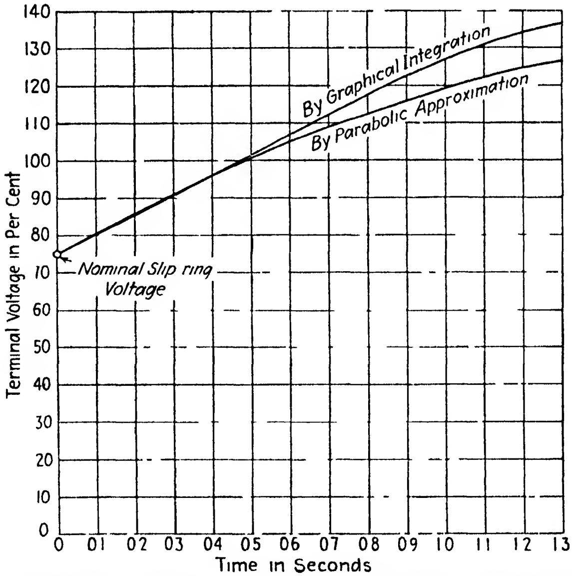


FIG. 247.—Calculated build-up curves for self-excited exciter (Example 8)

An alternative (and in fact more logical) method in this case is to calculate the nominal response rate directly. Using equation (*d*) the area under the build-up curve up to any time τ is

$$A = \int_0^\tau \frac{e_2}{1 + C\epsilon^{-\alpha t}} dt = e_2 \left[\tau - \frac{1}{\alpha} \log \frac{1 + C}{1 + C\epsilon^{-\alpha\tau}} \right]$$

With $\tau = 0.5$ sec. the nominal response rate is given by

$$\text{Nominal response} = 8 \left[(e_2 - e_0)\tau - \frac{e_2}{\alpha} \log \frac{1 + C}{1 + C\epsilon^{-\alpha\tau}} \right]$$

$$= 8 \left[(1.41 - 0.75)0.5 - \frac{1.41}{1.56} \log \frac{1 + 0.88}{1 + 0.88\epsilon^{-0.78}} \right]$$

$$= 8[0.33 - 0.903 \log 1.338]$$

$$= 8[0.33 - 0.264] = 0.528 \text{ per unit/sec.}$$

c. The ceiling voltage is 141 per cent. Hence

$$e_2 = 0.95 \times 141 = 134 \text{ per cent} = 335 \text{ volts}$$

From the (exact) curve in Fig. 247 the time required to build up from nominal slip-ring voltage to this value is 1.20 sec.

d. Based on nominal response rate the build-up time becomes

$$\tau = \frac{335 - 188}{132.5} = \frac{147}{132.5} = 1.11 \text{ sec.}$$

Comparison of Response of Separately- and Self-excited Machines.—In order to discuss the relative behavior in regard to response rates and build-up and build-down times for separately excited and self-excited machines consider a suitable form of the differential equation, for instance

$$\frac{de}{dt} = \frac{\Delta e}{\sigma T} \qquad \text{or} \qquad t = \sigma T \int_{e_1}^{e} \frac{de}{\Delta e} \tag{174}$$

This has been previously given as equations (96) and (152) for the two types of excitation, respectively.

The final voltage to which the machine may build up, *i.e.*, the upper value of e_2, is called the *ceiling voltage*. With the separately excited machine, the ceiling voltage can be varied between wide limits by suitable change of pilot-exciter voltage, external shunt-field resistance, or both. With self-excitation, on the other hand, the optimum ceiling voltage is definitely limited by the resistance of the shunt-field winding itself. It

cannot be brought beyond the value corresponding to zero external resistance in the field circuit.

The time constants appearing in the above equation have the following make-up [see equations (58) and (59)] for separately excited and self-excited machines, respectively:

$$T = \frac{Ne_2}{kE} \tag{175}$$

$$T = \frac{N}{k} \tag{176}$$

With separate excitation it is possible to control the value of the time constant by varying the voltage E of the pilot exciter. If it is desired that the ceiling voltage remain unaffected by the change in time constant, E and R (the total resistance in the field circuit) may be varied in the same proportion. This is obviously practicable since the ceiling voltage e_2 depends only on the upper limit of field current $i_2 = E/R$ and the magnetization curve. The time constant of a separately excited machine, therefore, may be made small by suitable selection of pilot-exciter voltage and external field resistance. A low time constant [see equation (174)] increases the response rates and expedites the build-up and build-down processes.

With self-excitation the time constant is inherently fixed by the design and cannot be controlled by external means. Hence in this respect the self-excited machine is at no small disadvantage relative to the separately excited machine.

Equation (174) shows that the response rate as well as the integral itself, whose numerical value governs the build-up and build-down times, depends on the quantity Δe. This quantity for the two types of excitation is given by

Separate excitation:

$$\Delta e = e_2 - R_e i = e_2 - e_2\frac{i}{i_2} = e_2\left(i - \frac{i}{i_2}\right) \tag{177}$$

Self-excitation:

$$\Delta e = e - Ri = e - e_2\frac{i}{i_2} = e_2\left(\frac{e}{e_2} - \frac{i}{i_2}\right) \tag{178}$$

For the purpose of comparison, assume the same ceiling voltage in the two cases. It is evident that on this basis Δe is larger for

separate excitation than for self-excitation during the entire process of build-up or build-down, except at the final operating point where the two Δe's are the same and necessarily equal to zero. It follows from this that the separately excited machine will have a higher response rate over practically the entire range. The difference is particularly pronounced during the first part of the transient. Maximum deviation in favor of the separately excited machine will obviously be found in the *initial* response rate, *i.e.*, in the volt-per-second value at the instant build-up or build-down begins.

Theoretically, the value of the integral itself in equation (174) is infinity if taken between the ultimate limits. Practically, however, the final conditions are reached in a finite time. As a reasonable basis for comparison of ultimate values of the integral for the two types of excitation, it may be evaluated over, say, 95 per cent of its total range. If this be done, the value of the integral for the separately excited machine, owing to the considerably larger Δe's, will be much smaller than for the machine with self-excitation. This fact contributes to the reduction of the time required for build-up or build-down to ultimate conditions.

Most frequently, except under abnormal operating conditions on the power system, an exciter will, under the action of the automatic voltage regulators, be called upon to build up or down only *partially*. Differently expressed, this means that the limits of integration in equation (174) will not be the steady-state voltage values e_1 and e_2 as previously defined, but certain voltages intermediate between the steady-state limits. Also for this mode of operation, it is obvious that the integral will have a much lower value when separate excitation is employed.

Summing up the essence of the above discussion it may be said that from the standpoint of response and time of build-up and build-down, and hence from the standpoint of general effectiveness in an excitation system, the separately excited machine has distinct advantages over the machine with self-excitation. Furthermore, the former possesses additional flexibility of application in that pertinent factors, such as ceiling voltage and time constant, may be controlled by external means independent of the design of the machine itself.

If the effectiveness of an excitation system is to be improved by changing from self-excitation to separate excitation, using a

pilot exciter, it is well to bear in mind that the improvement which can be obtained is always subject to certain practical limitations. For instance, the amount by which the ceiling voltage may be increased (if an increase is desired) depends upon the saturation. When this is pronounced the amount of additional field current required to raise the ceiling becomes large and may have to be limited by considerations of heating, even though otherwise attainable by means of a very high pilot-exciter voltage. Furthermore, it should be recalled that an interrelationship exists between ceiling voltage e_2, response rate de/dt, and amount of external resistance in the field circuit. A new ceiling voltage and a desired rate of response cannot, in general, be arbitrarily specified, since it may be impossible to satisfy the combination even though no restrictions be imposed on the pilot-exciter voltage and the external field resistance. Thus, if in the self-excited case all external resistance is cut out at the ceiling, it is obvious that in changing to separate excitation the pilot-exciter voltage must be at least equal to the ceiling voltage self-excited, if the change from self-excitation to separate excitation is not going to result in a decreased ceiling. The rate of response during build-up is governed by the voltage of the pilot-exciter and the ceiling voltage. Certain combinations of response rate and ceiling voltage may lead to the paradox of negative external resistance under separately excited conditions, as borne out in Example 9.

EXAMPLE 9

Statement of Problem

It is desired to provide separate excitation by means of a suitable pilot exciter to the self-excited, shunt-wound exciter on which design and operating data are given in Example 8. What should the pilot-exciter voltage be to double the build-up rate at nominal slip-ring voltage? The ceiling voltage is to remain the same as for self-excited operation.

Solution

Referring to equations (174) to (178) inclusive, the build-up rates at nominal slip-ring voltage for operation self-excited and separately excited are given below:

Self-excited:

$$\left(\frac{de}{dt}\right)_{e=e_0} = \frac{e_0 - e_2\dfrac{i_0}{i_2}}{\sigma\dfrac{N}{k}} \tag{a}$$

Separately Excited:

$$\left(\frac{de}{dt}\right)_{e=e_0} = \frac{e_2 - e_2\dfrac{i_0}{i_2}}{\sigma\dfrac{Ne_2}{kE}} = \frac{1 - \dfrac{i_0}{i_2}}{\sigma\dfrac{N}{k}}E \tag{b}$$

Since the build-up rate is to be doubled, it follows from these equations that

$$2 = \frac{E\left(1 - \dfrac{i_0}{i_2}\right)}{e_0 = e_2\dfrac{i_0}{i_2}} \tag{c}$$

from which

$$E = \frac{2\left(e_0 - e_2\dfrac{i_0}{i_2}\right)}{- \dfrac{i_0}{i_2}} \tag{d}$$

The necessary numerical values are obtained from the data in Example 8:

$e_0 = 188$ volts $i_0 = 0.68 \times 6.6 = 4.49$ amp.
$e_2 = 1.406 \times 250 = 351.5$ volts $i_2 = 2.28 \times 6.6 = 15.05$ amp.

$$\frac{i_0}{i_2} = \frac{4.49}{15.05} = 0.2975$$

Hence

$$E = \frac{2(188 - 351.5 \times 0.2975)}{1 - 0.2975} = \frac{2 \times 83.5}{0.7025} = 238 \text{ volts}$$

It is thus seen that a pilot-exciter voltage of 238 volts apparently would double the build-up rate at nominal slip-ring voltage. However, since this voltage is lower than the ceiling voltage (351.5 volts), the latter cannot actually be reached. The inconsistency lies in the requirement which specifies double build-up rate at the previous ceiling voltage. In order not to change the ceiling, the minimum permissible pilot-exciter voltage must be 351.5 volts (equal to the ceiling voltage). With this pilot-exciter voltage, the response rate when operating self-excited is given by the calculation below.

When $E = e_2 = 351.5$ volts it follows from equations (*a*) and (*b*) that

$$\left[\frac{\left(\dfrac{de}{dt}\right)_{\text{sep}}}{\left(\dfrac{de}{dt}\right)_{\text{self}}}\right]_{e=e_0} = \frac{e_2\left(1 - \dfrac{i_0}{i_2}\right)}{e_0 - e_2\dfrac{i_0}{i_2}} = \frac{351.5 \times 0.7025}{83.5} = 2.95$$

EXAMPLE 10

Statement of Problem

A self-exicted exciter is supplying an alternator whose nominal slip-ring voltage is e_0. The resistance of its field winding is R.

In order to increase the response, it is contemplated to separately excite the exciter from a pilot exciter of suitable voltage.

a. Determine the voltage of the latter (E) which will increase the initial rate of build-up six times when operating at nominal slip-ring voltage and the regulator contacts suddenly close. The lowest possible voltage (e_1) obtained with the full amount of external resistance in the circuit is to remain the same. The ceiling voltage (e_2), on the other hand, is to be increased by 10 per cent. Give the answer in terms of the nominal slip-ring voltage.

The ceiling voltage self-excited with no external resistance in the field circuit is twice the nominal slip-ring voltage ($e_2 = 2e_0$), and the current at this ceiling is three times the current at nominal slip-ring voltage ($i_2 = 3i_0$). The lowest value of voltage is one-third of nominal slip-ring voltage ($e_1 = e_0/3$). The 10 per cent increase in ceiling voltage is accompanied by 33.3 per cent increase in ceiling current. The magnetization curve may be assumed to be linear below nominal slip-ring voltage.

b. In terms of the field resistance R, determine the amount of external resistance required under conditions of self-excitation and separate excitation, indicating in the latter case also the amount which permanently must be left in the circuit in order that the specified ceiling shall not be exceeded.

c. Formulate an expression for the ratio separately excited to self-excited of the build-up time from nominal slip-ring voltage to 95 per cent of ceiling voltage at the above-determined pilot-exciter voltage. Discuss this ratio and indicate its order of magnitude.

Solution

The exciter, separately and self-excited, is shown in Fig. 248(a) and (b), respectively. Figure 249(a) shows the magnetization curve on which are

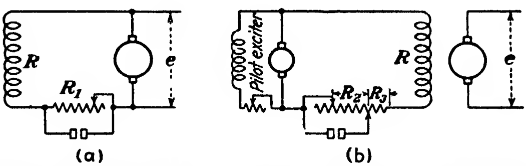


FIG. 248.—Schematic diagrams of exciter arrangements considered in Example 10. (a) Self-excited. (b) Separately excited.

located the lowest operating voltage, the nominal slip-ring voltage, and the ceiling voltages in the two cases.

a. The following equations give the response rates at nominal slip-ring voltage:

Self-excited:

$$\left(\frac{de}{dt}\right)_{e=e_0} = \frac{e_0 - Ri_0}{\sigma T_{self}} = \frac{e_0 - Ri_0}{\sigma N/k} \qquad (a)$$

Separately Excited:

$$\left(\frac{de}{dt}\right)_{e=e_0} = \frac{e_2' - e_2'\frac{i_0}{i_2'}}{\sigma T_{sep}} = \frac{e_2'\left(1 - \frac{i_0}{i_2'}\right)}{\sigma N e_2'/kE} = \frac{1 - \frac{i_0}{i_2'}}{\sigma N/kE} \qquad (b)$$

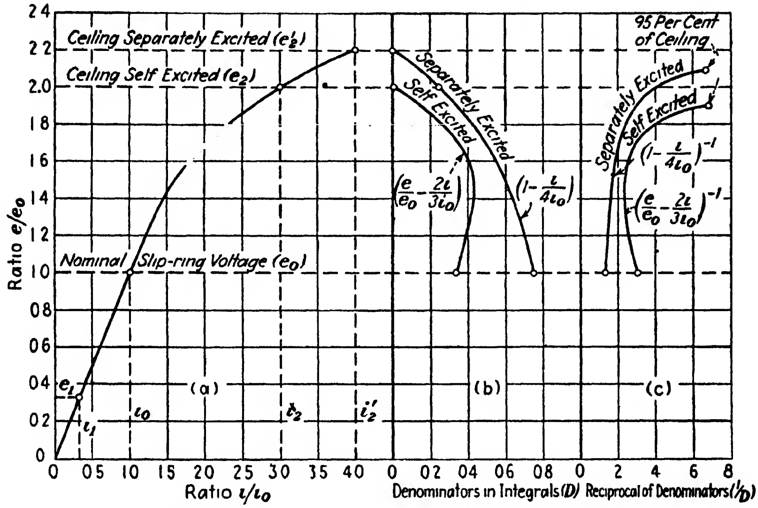


FIG. 249.—(*a*) Magnetization curve of exciter in Fig. 248; (*b*) and (*c*) curves of quantities representative of comparative build-up characteristics when the machine is self-excited and separately excited. (For complete explanation see Example 10.)

This build-up rate will be increased *m* times by separate excitation if

$$\left(1 - \frac{i_0}{i_2'}\right)E = m(e_0 - Ri_0) \qquad (c)$$

From the given data,

$$\frac{i_0}{i_2'} = \frac{i_0}{\frac{4}{3}i_2} = \frac{1}{4} \quad \text{and} \quad R = \frac{e_2}{i_2} = \frac{2e_0}{3i_0}$$

Hence

$$\left(1 - \frac{1}{4}\right)E = \frac{3}{4}E = me_0\left(1 - \frac{2}{3}\right) = \frac{me_0}{3}$$

giving

$$E = \frac{4me_0}{9}$$

In order to increase the response at nominal slip-ring voltage six times (*i.e.*, $m = 6$), the voltage of the pilot exciter must consequently be

$$E = {}^{24}\!/_{9}e_0 = {}^{8}\!/_{3}e_0 = 2.67e_0$$

b. The lowest operating point fixes the value of external resistance in the exciter field circuit. For the self-excited machine this resistance is

$$R_1 = \frac{e_1}{i_1} - R = \frac{e_0}{i_0} - R = \frac{3}{2}R - R = \frac{R}{2}$$

The external resistance R_3, permanently in the field circuit when separate excitation is used, is found from the conditions at ceiling voltage. At this operating point,

$$R + R_3 = \frac{E}{i_2'} = \frac{4me_0}{9 \times 4i_0} = \frac{3mR}{18}$$

from which

$$R_3 = \left(\frac{3m}{18} - 1\right)R$$

The total external resistance when separate excitation is used, and from this the part R_2, which can be short-circuited out, is determined from conditions at minimum operating voltage. At this point

$$R + R_2 + R_3 = \frac{E}{i_1} = \frac{4me_0}{9i_0/3} = \frac{4me_0}{3i_0} = 2mR$$

from which

$$R_2 = \left(2m - 1 - \frac{3m}{18} + 1\right)R = \frac{11m}{6}R$$

When the build-up rate at nominal slip-ring voltage is increased six times ($m = 6$) the external resistances become

$$R_2 = 11R \qquad \text{and} \qquad R_3 = 0$$

c. The time required for build-up from nominal slip-ring voltage to 95 per cent of ceiling voltage is given by

Self-excited:

$$\tau_{\text{self}} = \sigma T_{\text{self}} \int_{e_0}^{0.95e_2} \frac{de}{\Delta e_{\text{self}}} = \frac{\sigma N}{k} \int_{e_0}^{1.90e_0} \frac{de}{e - Ri} = \frac{\sigma N}{k} \int_{e_0}^{1.90e_0} \frac{de}{e_0\left(\dfrac{e}{e_0} - \dfrac{2}{3}\dfrac{i}{i_0}\right)}$$

Separately Excited:

$$\tau_{sep} = \sigma T_{sep} \int_{e_0}^{0.95 e_2'} \frac{de}{\Delta e_{sep}} = \frac{\sigma N e_2'}{kE} \int_{e_0}^{2.09 e_0} \frac{de}{e_2'\left(1 - \frac{i}{i_2'}\right)}$$

$$= \frac{9\sigma N}{4mk} \int_{e_0}^{2.09 e_0} \frac{de}{e_0\left(1 - \frac{i}{4i_0}\right)}$$

The ratio of the build-up times thus becomes

$$\frac{\tau_{sep}}{\tau_{self}} = \frac{9}{4m} \frac{\displaystyle\int_{e_0}^{2.09 e_0} \frac{de}{1 - \frac{1}{4}\frac{i}{i_0}}}{\displaystyle\int_{e_0}^{1.90 e_0} \frac{de}{\frac{e}{e_0} - \frac{2}{3}\frac{i}{i_0}}}$$

which for $m = 6$ reduces to

$$\frac{\tau_{sep}}{\tau_{self}} = \frac{3}{8} \frac{\displaystyle\int_{e_0}^{2.09 e_0} \frac{de}{1 - \frac{1}{4}\frac{i}{i_0}}}{\displaystyle\int_{e_0}^{1.90 e_0} \frac{de}{\frac{e}{e_0} - \frac{2}{3}\frac{i}{i_0}}}$$

In order to determine the ratio numerically, it is necessary to evaluate the integrals. This can be done graphically but exact evaluation requires that the actual magnetization curve be available. However, sufficient support for an approximate quantitative solution may be had from the given data. Such figures as can be obtained are collected in the following table:

e	i	$1 - \frac{1}{4}\frac{i}{i_0}$	$\frac{e}{e_0} - \frac{2}{3}\frac{i}{i_0}$	$\left(1 - \frac{1}{4}\frac{i}{i_0}\right)^{-1}$	$\left(\frac{e}{e_0} - \frac{2}{3}\frac{i}{i_0}\right)^{-1}$
e_0	i_0	0.75	0.333	1.33	3.00
$1.5e_0$	$1.6i_0$	0.60	0.435	1.66	2.30
$1.9e_0$	$2.6i_0$	0.34	0.147	2.92	6.80
$2.0e_0$	$3.0i_0$	0.25	0	4.00	∞
$2.09e_0$	$3.4i_0$	0.15	.....	6.67	
$2.2e_0$	$4.0i_0$	0	.....	∞	

The values of the denominators in each integral are plotted in Fig. 249(*b*) and their reciprocals in (*c*). These curves are approximate only, since there are too few actual points to define them completely. A rough evaluation of the areas between the curves and the voltage axis taken from nominal slip-ring voltage to 95 per cent of the respective ceilings indicates a ratio of

$$\frac{(\text{Area})_{\text{sep}}}{(\text{Area})_{\text{self}}} \cong 0.90$$

The approximate ratio of the build-up times therefore becomes

$$\frac{\tau_{\text{sep}}}{\tau_{\text{self}}} \cong \frac{3 \times 0.90}{8} = 0.34$$

It is reasonable to expect, therefore, that the build-up time separately excited will be approximately one-third of the build-up time for self-excited operation.

CHAPTER XVI

ANALYSIS OF LOADED EXCITERS

While in the preceding chapter open-circuit operation only was considered, the present chapter will concern itself with the transient analysis of exciters carrying loads. Although the usual exciter load, *i.e.*, the field winding of the alternator, represents a combination of resistance and variable inductance, the discussion will, for the sake of generality, also include certain other types of load. In other words, the treatment will cover transients in direct-current machines, in general, without limiting itself to such machines applied as exciters.

When the machine is loaded, the constants of the armature circuit enter into the problem in addition to those of the field circuit. Besides the constants of the load the armature resistance and inductance as well as the armature reaction may have to be considered. It has previously been mentioned that with the self-excited machine the latter three factors should, strictly speaking, be included even at no load. However, under no-load conditions their effect is negligibly small. With load on the machine and large values of armature current, their effect may evidently be materially increased. In spite of this they are frequently omitted also when load is carried in order that the analysis may be facilitated.

The effect of the armature resistance is usually the most important. Hence the resistance is the armature constant which first ought to be taken into account. The leakage inductance of the armature is ordinarily quite small and the reactance voltage during the transients, therefore, as a rule rather insignificant even at comparatively rapid current changes. The magnitude of the armature reaction depends on the position of the brushes. If these be in the neutral zone, as they would be with machines having compensating poles, the armature reaction is zero. As the angle of displacement of the brushes from the neutral plane never should be very large, the armature reaction should usually be small.

Although the inclusion of the armature constants in the transient analysis may be refinements which in a practical case would be omitted, complete methods are desirable and should be available. Such methods are at least necessary so that the magnitude of the effect of the armature constants may be studied. Only in this way can the degree of approximation and hence the applicability of the simpler methods of analysis be established. In the discussion which is to follow, therefore, the armature constants will be ignored as well as taken into account. The load conditions to be covered are

A. Open circuit
B. Resistance load
C. Resistance and constant-inductance load
D. Resistance and variable-inductance load

For each type of loading armature constants will be included according to the following classification of cases:

Case 1.—Armature resistance, inductance and reaction neglected ($R_a = 0$, $L_a = 0$, $A = 0$).

Case 2.—Armature resistance considered, inductance and reaction neglected (R_a, $L_a = 0$, $A = 0$).

Case 3.—Armature resistance and inductance considered, reaction neglected (R_a, L_a, $A = 0$).

Case 4.—Armature resistance and reaction considered, inductance neglected (R_a, $L_a = 0$, A).

Case 5.—Armature resistance, inductance, and reaction considered (R_a, L_a, A).

The transient analysis under load conditions will be discussed with particular reference to the self-excited machine. The separately excited machine, however, can be handled by the same general methods. As a matter of fact, the analysis of problems involving the latter is often somewhat simpler. Furthermore, the discussion as well as the specific formulations are in general confined to the build-up process. The same procedure, however, is applicable when build-down takes place.

No Load. *Case* 1.—The analysis neglecting both armature resistance and inductance as well as armature reaction has been covered in detail in the preceding chapter.

Case 2.—Referring to Fig. 250, the differential equation [see Chap. XV, equation (100)] when armature resistance is included becomes

$$\sigma T\frac{de}{dt} + Ri = e - R_a i_a \tag{1}$$

Since at no load the armature and field carry the same current, this may be written

$$\sigma T\frac{de}{dt} = e - (R + R_a)i = e - R_0 i = \Delta e \tag{2}$$

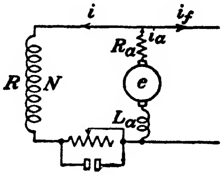


FIG. 250.—Schematic diagram of self-excited exciter.

which solved for the time gives

$$t = \sigma T\int_{e_1}^{e}\frac{de}{\Delta e} \tag{3}$$

This is of exactly the same form as previously handled analytically and graphically. If the latter method is employed, the resistance line is merely drawn to correspond to the total resistance, *i.e.*, the sum of the armature resistance and the resistance in the field circuit proper as indicated in Fig. 251. The armature resistance decreases the voltage difference Δe during build-up and increases it during build-down. Hence it decreases the response rate and increases the time of build-up. For build-down the effects are opposite. The influence of armature resistance, however, is practically negligible at no load.

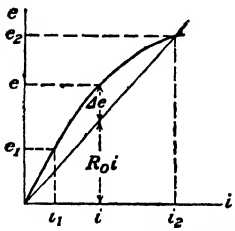


FIG. 251.—Magnetization curve and equivalent field-resistance line of self-excited exciter. The equivalent field resistance is the sum of the actual field resistance and the armature resistance.

Case 3.—At no load, armature inductance merely acts as if additional field leakage were present. Referring to equation (161) in Chap. XV, the solution for time is given by

$$t = \sigma T\int_{e_1}^{e}\frac{d\left(e + \frac{L_a}{T}i\right)}{\Delta e} \tag{4}$$

which can readily be handled by methods previously discussed. Since armature resistance is also included Δe should be calculated as indicated in equation (2).

Case 4.—The demagnetizing armature reaction which is directly proportional to the armature current is expressible by

$$A = k_a i_a = \frac{\alpha Z}{180aN} \tag{5}$$

where α = the displacement of the brushes from the neutral zone, in electrical degrees

a = number of parallel paths in the armature winding

Z = number of armature conductors per pole

N = number of field turns

The magnetization curve gives the relationship between induced *open-circuit* armature voltage and field current. During operation, it is the *actually* induced armature voltage which is of interest and which should be used in the various differential equations. When armature reaction is present, a larger field current is required to generate a certain armature voltage than on open circuit, the difference being exactly equal to the armature reaction expressed in terms of equivalent field current. Hence before the magnetization curve can be used in the solution of problems involving armature reaction, it must be corrected for the effect of the latter. This can readily be done in the present case where the field and armature currents are the same.

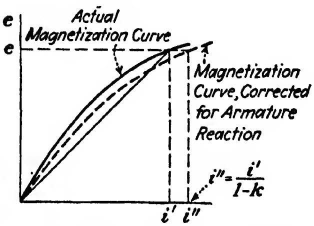


FIG. 252.—Correction of magnetization curve for effect of armature reaction in self-excited exciter. No external load.

Referring to Fig. 252, let i' correspond to a point on the original magnetization curve and i'' correspond to the point on the corrected curve for the same voltage. Then

$$i'' = \frac{i'}{1 - k_a} \tag{5a}$$

Hence the corrected curve is obtained by simply multiplying the abscissae of the original curve by $1/(1 - k_a)$.

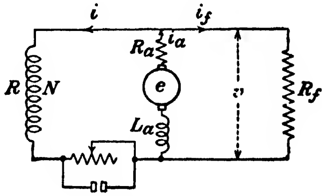


FIG. 253.—Schematic diagram of self-excited machine supplying an external resistance load.

Having the magnetization curve corrected in this manner for armature reaction, the formulas and the general solution are as for case 2. It will be noted that the general effect of the armature reaction is of the same character as that of the armature resistance.

Case 5.—When armature resistance and inductance, as well as armature reaction, are present, the general formulation and

solution are as for case 3 in connection with use of the magnetization curve corrected for armature reaction as in case 4.

Resistance Load.—The circuit to be considered is shown in Fig. 253. Since the machines when used as exciters supply alternator fields, all quantities referring to the load circuit will consistently be designated by the subscript f.

Case 1.—In this case the induced voltage and the terminal voltage are identical. The voltage-time solution (or curve) may therefore be determined as for no load. Having this, the load current is given by

$$i_f = \frac{e}{R_f} \tag{6}$$

Case 2.—The differential equation for this case becomes

$$\sigma T\frac{de}{dt} + Ri = v = e - R_a i_a = R_f i_f \tag{7}$$

also

$$i_a = i + i_f \tag{8}$$

Combining this with the last equality from equation (7) gives

$$i_f = \frac{e - R_a i}{R_a + R_f} \tag{9}$$

Substituting for the load current in the differential equation, this may be written

$$\sigma T\frac{de}{dt} = R_f i_f - Ri = \frac{R_f}{R_a + R_f}e - \frac{RR_a + R_a R_f + R_f R}{R_a + R_f}i \tag{10}$$

from which by rearrangement

$$\sigma T\frac{R_a + R_f}{R_f}\frac{de}{dt} = e - (RR_a + R_a R_f + R_f R)i \tag{11}$$

This may be reduced to

$$T_0\frac{de}{dt} = e - R_0 i = \Delta i \tag{12}$$

where

$$T_0 = \sigma\frac{R_a + R_f}{R_f}T \tag{13}$$

$$R_0 = RR_a + R_a R_f + R_f R \tag{14}$$

From equation (12) the solution for the time is obtained as

$$t = T_0 \int_{e_1}^{e} \frac{de}{\Delta e} \tag{15}$$

which is identical in form with the solution for open-circuit conditions. The expression merely involves a modified time constant and voltage difference. The modified time constant is larger than the actual (the difference usually being negligible), while the voltage difference is decreased for build-up and increased for build-down as compared with the open-circuit case. Hence the response rate is decreased and the time increased during the build-up process. The relative effect during build-down cannot be definitely predicted; most probably, however, the response will be increased and the time decreased since the change in Δe presumably is predominant over the apparent increase in time constant. The actual solution of equation (15) is obtained by the same methods as for no load, case 2.

Since the voltage-time relation has been determined, the load current is calculated by equation (9).

Case 3.—A point-by-point solution is the only practicable one in this case, and the equations must be arranged with this in view. The general differential equation becomes

$$\sigma T \frac{de}{dt} + Ri = v = e - R_a i_a - L_a \frac{di_a}{dt} = R_f i_f \tag{16}$$

Using the first and last term of this gives

$$\frac{de}{dt} = \frac{R_f i_f - Ri}{\sigma T} \tag{17}$$

while the last two terms in connection with equation (8) results in

$$L_a \frac{di}{dt} + L_a \frac{di_f}{dt} = e - R_a i - (R + R_f) i_f = \Delta e' - (R_a + R_f) i_f \tag{18}$$

which again may be written

$$\frac{di_f}{dt} = \frac{\Delta e' - (R_a + R_f) i_f - L_a \dfrac{i_n - i_{n-1}}{\Delta t}}{L_a} \tag{19}$$

Equations (17) and (19) are in a form suitable for point-by-point calculations. The former gives the rate of change of induced voltage and the latter the rate of change of load current, both to be assumed constant during the time interval. The interpretation of $\Delta e'$ is evident from Fig. 254. It will be noted that the rate of change of field current required in the numerator of equation (19) is taken as the difference between the field current at the end and the beginning of an interval divided by its length. The field-current values are obviously obtained by reference to the magnetization curve as the solution for voltage by equation (17) progresses.

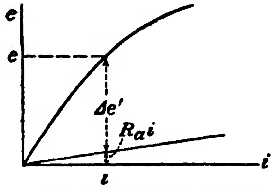


FIG. 254.—Magnetization curve and armature-resistance line for self-excited machine illustrating interpretation of the quantity $\Delta e'$.

A suitable schedule for the point-by-point solution of this problem is suggested in Table 61.

TABLE 61.—POINT-BY-POINT CALCULATION OF EXCITER RESPONSE. RESISTANCE LOAD. ARMATURE CONSTANTS INCLUDED

(1) Int. No.	(2) t	(3) e	(4) i	(5) i_f	(6) Ri	(7) $R_a i$	(8) $R_f i_f$	(9) $(R_a + R_f)i_f$	(10) $L_a \frac{i_n - i_{n-1}}{\Delta t}$	(11) $\frac{de}{dt}$	(12) $\frac{di_f}{dt}$
0	0										
1	Δt										
2	$2\Delta t$										
etc.	etc.										

$$e_n\text{: Col. } (3)_n = (3)_{n-1} + (11)_{n-1} \cdot \Delta t = (3)_{n-1} + \frac{[(8) - (6)]_{n-1}}{T}\Delta t$$

$$i_n\text{: Col. } (4)_n = \text{value from mag. curve for } e_n$$

$$i_{f(n)}\text{: Col. } (5)_n = (5)_{n-1} + (12)_{n-1} \cdot \Delta t = (5)_{n-1} + \frac{[(3) - (7) - (9) - (10)]_{n-1}}{L_a}\Delta t$$

Case 4.—This case can be handled as case 2 and the same equations are applicable. Instead of using the open-circuit magnetization curve, however, a magnetization curve corrected for armature reaction must be used. By referring to equation (5), the armature reaction may be written

$$A = k_a i_a = k_a\left(i + \frac{e - R_a i}{R_a + R_f}\right) = \frac{k}{R_a + R_f}(e + R_f i) \tag{20}$$

The relationship between a point on the open-circuit magnetization curve and the point corresponding to the same voltage on the corrected curve is then given by

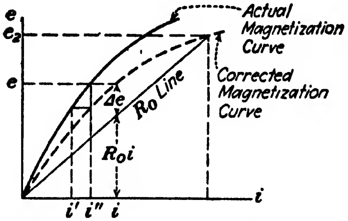


FIG. 255.—Correction of the magnetization curve for effect of armature reaction in self-excited machine. External resistance load.

$$i'' = i' + \frac{k_a}{R_a + R_f}(e + R_f i') \tag{21}$$

from which

$$i'' = \frac{(R_a + R_f)i' + k_a e}{R_a + (1 - k_a)R_f} \cong \frac{1}{1 - k_a}\left(i' + \frac{k_a e}{R_f}\right) \tag{22}$$

This allows the corrected magnetization curve to be drawn as indicated in Fig. 255.

Case 5.—When armature resistance, inductance, and reaction all are considered, the solution is obtained by application of the point-by-point method outlined under case 3 in connection with the use of a corrected magnetization curve as discussed for the preceding case.

Resistance and Constant-inductance Load.—The circuit to be analyzed is shown in Fig. 256.

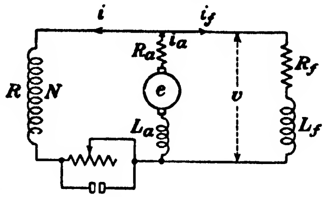


FIG. 256.—Schematic diagram of self-excited machine supplying external resistance and inductance load.

Case 1.—Since none of the armature constants are to be considered the induced voltage and the terminal voltage are identical, and the voltage-time relation may be established as for no load. The equation for the load circuit is then

$$L_f \frac{di_f}{dt} + R_f i_f = e = E(t) \tag{23}$$

The load current can always be obtained from this equation by means of a point-by-point solution. This will evidently be quite simple and no further comment regarding it is necessary. Also, the point-by-point method is applicable whether the voltage-time relationship is formal or represented by a curve.

If an explicit expression for voltage in terms of time is available, the load current may be found using the *superposition*

theorem.[1] The possibility of establishing a final analytical solution in this manner depends on whether the expressions so obtained actually are integrable by formal methods. If not, they may yield to graphical methods of evaluation or they may be handled on the integraph.

Case 2.—A point-by-point solution must be applied. The general differential equation is

$$\sigma T\frac{de}{dt} + Ri = e - R_a i_a = L_f\frac{di_f}{dt} + R_f i_f \qquad (24)$$

Using the first two terms and substituting from equation (8) gives

$$\sigma T\frac{de}{dt} = e - (R + R_a)i - R_a i_f = \Delta e - R_a i_f \qquad (25)$$

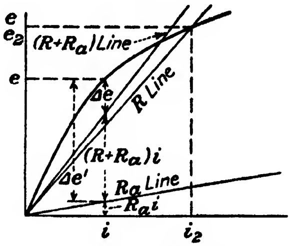


FIG. 257.—Magnetization curve and field- and armature-resistance lines for self-excited machine illustrating interpretation of the quantities Δe and $\Delta e'$.

while the last two terms with the armature current eliminated may be written

$$L_f\frac{di_f}{dt} = e - R_a i - (R_a + R_f)i_f = \Delta e' - (R_a + R_f)i_f \qquad (26)$$

From equations (25) and (26), are obtained

$$\frac{de}{dt} = \frac{\Delta e - R_a i_f}{\sigma T} \qquad (27)$$

$$\frac{di_f}{dt} = \frac{\Delta e' - (R_a + R_f)i_f}{L_f} \qquad (28)$$

A point-by-point solution is now readily carried through based on the above equations. The graphical interpretation of Δe and $\Delta e'$ is shown in Fig. 257. The calculations may conveniently be handled as suggested in Table 62.

Case 3.—The differential equation for this case becomes

$$\sigma T\frac{de}{dt} + Ri = e - R_a i_a - L_a\frac{di_a}{dt} = R_f i_f + L_f\frac{di_f}{dt} \qquad (29)$$

By using the first and the last term and eliminating the armature

[1] See, for instance, BUSH, V., "Operational Circuit Analysis," pp. 56, *et seq.*, John Wiley & Sons, Inc., New York, 1929.

current by means of equation (8), the rate of change of induced voltage is given by

$$\frac{de}{dt} = \frac{R_f i_f - Ri + L_f \frac{di_f}{dt}}{\sigma T} \tag{30}$$

Similarly is obtained from the second and third term of equation (29)

$$(L_a + L_f)\frac{di_f}{dt} = e - R_a i - (R_a + R_f)i_f - L_a\frac{di}{dt}$$

$$= \Delta e' - (R_a + R_f)i_f - L_a\frac{di}{dt} \tag{31}$$

from which the rate of change of load current becomes

$$\frac{di_f}{dt} = \frac{\Delta e' - (R_a + R_f)i_f - L_a\frac{i_n - i_{n-1}}{\Delta t}}{L_a + L_f} \tag{32}$$

Equations (30) and (32) lend themselves to point-by-point solution. The rate of change of voltage during an interval and

TABLE 62.—POINT-BY-POINT CALCULATION OF EXCITER RESPONSE. RESISTANCE AND INDUCTANCE LOAD. ARMATURE RESISTANCE AND REACTION INCLUDED

(1) Int. No.	(2) t	(3) e	(4) i	(5) i_f	(6) $R_a i$	(7) $(R + R_a)i$	(8) $R_a i_f$	(9) $(R_a + R_f)i_f$	(10) de/dt	(11) di_f/dt
0	0									
1	Δt									
2	$2\Delta t$									
etc.	etc.									

e_n:—Col. $(3)_n = (3)_{n-1} + (10)_{n-1} \cdot \Delta t = (3)_{n-1} + \frac{[(3) - (7) - (8)]_{n-1}}{\sigma T}\Delta t$

i_n:—Col. $(4)_n$ = value from mag. curve for e_n

$i_{f(n)}$:—Col. $(5)_n = (5)_{n-1} + (11)_{n-1} \cdot \Delta t = (5)_{n-1} +$

$$\frac{[(3) - (6) - (9)]_{n-1}}{L_f}\Delta t$$

thereby the voltage and field current at the end of the interval, is obtained from the former, the rate of change of load current and thereby its value at the end of the interval from the latter. The

rate of change of load current required in equation (30) may for the purpose of calculating the initial rate of change of voltage be taken equal to zero. For subsequent intervals the value from equation (32) is used.

The calculations may be arranged as suggested in Table 63.

TABLE 63.—POINT-BY-POINT CALCULATION OF EXCITER RESPONSE. RESISTANCE AND INDUCTANCE LOAD. ARMATURE CONSTANTS INCLUDED

(1) Int. No.	(2) t	(3) e	(4) i	(5) i_f	(6) Ri	(7) R_ai	(8) R_fi_f	(9) $(R_a + R_f)i_f$	(10) $L_a\frac{i_n - i_{n-1}}{\Delta t}$	(11) $\frac{de}{dt}$	(12) $\frac{di_f}{dt}$	(13) $L_f\frac{di_f}{dt}$
0	0											
1	Δt											
2	$2\Delta t$											
etc.	etc.											

e_n:—Col. $(3)_n = (3)_{n-1} + (11)_{n-1} \cdot \Delta t = (3)_{n-1} + \frac{[(8) - (6) + (13)]_{n-1}}{\sigma T}\Delta t$

i_n:—Col. $(4)_n$ = value from mag. curve for e_n

$i_{f(n)}$:—Col. $(5)_n = (5)_{n-1} + (12)_{n-1} \cdot \Delta t = (5)_{n-1} + \frac{[(3) - (7) - (9) - (10)]_{n-1}}{L_a + L_f}\Delta t$

Case 4.—The magnetization curve should be corrected for armature reaction as discussed for the resistance load, case 4. If the corrected curve is used, a point-by-point solution is applied as described under case 2.

Case 5.—Upon having corrected the magnetization curve for armature reaction a point-by-point solution is applied as described for case 3.

Resistance and Variable-inductance Load.—The equations, as well as the general methods of (point-by-point) solution, are the same as for the load with strictly constant inductance. In using the equations and in carrying out the solutions, however, cognizance must be taken of the fact that the load inductance is *variable*, *i.e.*, a function of the load current. Hence this relationship between inductance and current must be known so that the proper inductance can be used for each interval in the point-by-point calculations. In general, the inductance corresponding to the value of current at the beginning of an interval would be considered constant during the same interval. It should be noted that in applying the equations for the constant-inductance-load condition to that of variable inductance, the load inductance L_f must be interpreted to signify the *total* inductance.

In order to obtain the inductance as a function of current, the relation between flux and current must be known; in other words a magnetization curve or its equivalent must, in general, be available. The inductance at a particular value of current is then proportional to the slope of the curve at that point, being given by

$$L_f = \frac{d\lambda_f}{di_f} = N\frac{d\phi_f}{di_f} \tag{33}$$

Figure 258(a) indicates how the inductance is determined by means of tangents to the flux-current curve, while (b) is a plot of the so-obtained inductance versus current.

If the magnetization curve includes the total flux, the total inductance is at once obtained. If, as more often is the case,

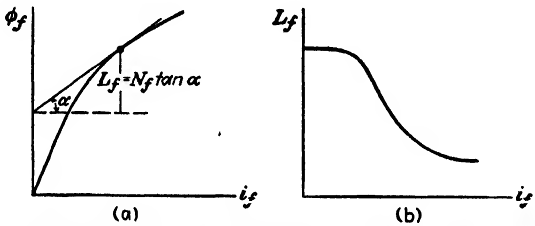


FIG. 258.—(a) Determination of inductance by slope of magnetization curve; (b) variable inductance as function of magnetizing current.

it merely involves the iron flux proper, the inductance due to this flux only is found, and the leakage inductance must be added.[1]

[1] The nominal inductance of the field circuit of an exciter or alternator is defined as the number of flux linkages per field ampere at rated armature voltage with the armature winding open-circuited. The relationship between nominal inductance L_0 and actual inductance L_f for any particular value of flux (voltage) is given by

$$L_f = L_0\left(\frac{i_0}{e_0}\right)\frac{de}{di} \tag{a}$$

where e_0 and i_0 represent the rated armature voltage and the corresponding field current expressed in actual volts and amperes, respectively, and the derivative de/di the slope of the magnetization curve when plotted volts versus amperes.

If percentage or per-unit values are used for armature voltage and field current [with 100 per cent (unit) field current corresponding to 100 per cent (unit) voltage], the relationship between nominal inductance and actual inductance simply becomes as

Case 1.—By making use of a variable load inductance as outlined above, the method of solution discussed for the preceding load condition, case 1, equation (23), applies.

Case 2.—By making use of a variable load inductance as outlined above, the method of solution discussed for the preceding load condition, case 2, equations (24) to (28) inclusive, and Table 62 apply.

Case 3.—By making use of a variable load inductance as outlined above, the method of solution discussed for the preceding load condition, case 3, equations (29) to (32), inclusive, and Table 63 apply.

Case 4.—By making use of a variable load inductance as outlined above, the method of solution discussed for the preceding load condition, case 4, applies.

Case 5.—By making use of a variable load inductance as outlined above, the method of solution discussed for the preceding load condition, case 5, applies.

Applicable Methods of Solution.—Table 64 represents a schedule of methods for the solution of problems involving electrical transients in direct-current machines when the effect of saturation is considered. The table covers no load as well as the more common load conditions with various degrees of refinement included in the solutions.

It will be noted that the point-by-point method is universally applicable, as, of course, would be expected. This method can always be resorted to and should give excellent results when properly handled.

The most elegant solutions are those involving graphical integration. They are based on compact formulations of the differential equations and are usually distinguished by their directness and ease of manipulation. Unfortunately, their application is almost entirely limited to the cases where the effects of armature constants are ignored.

The analytical solutions, although interesting *per se*, are of lesser practical importance. Since they are based on the Fröhlich

---

$$L_f = L_0 \frac{de}{di} \qquad (b)$$

where the derivative de/di represents the slope of the magnetization curve plotted on a percentage or per-unit basis.

equation (or other empirical representation of the nonlinear characteristics), they are fundamentally approximate. Those that properly apply the Fröhlich equation, however, may nevertheless be relied upon to yield results to engineering accuracy.

TABLE 64.—SCHEDULE OF METHODS FOR THE SOLUTION OF ELECTRICAL TRANSIENTS IN DIRECT-CURRENT MACHINES
Applicable to Exciter Problems

Case number	No load	Resistance load	Resistance and constant-inductance load	Resistance and variable-inductance load
Case 1.......	(*a*) (*b*) (*c*) (*d*)	(*a*) (*b*) (*c*) (*d*)	(*a*) (*a*) + (*d*) (*b*) + (*d*) (*c*) + (*d*) (*c*) + (*e*) (*d*)	(*a*) (*a*) + (*d*) (*b*) + (*d*) (*c*) + (*d*) (*d*)
Case 2.......	(*a*) (*b*) (*c*) (*d*)	(*a*) (*b*) (*c*) (*d*)	(*d*)	(*d*)
Case 3.......	(*a*) (*b*) (*c*) (*d*)	(*d*)	(*d*)	(*d*)
Case 4.......	(*a*) (*b*) (*c*) (*d*)	(*a*) (*b*) (*c*) (*d*)	(*d*)	(*d*)
Case 5.......	(*a*) (*b*) (*c*) (*d*)	(*d*)	(*d*)	(*d*)

(*a*) Graphical integration.
(*b*) Analytical solution based on Fröhlich's equation.
(*c*) Analytical solution based on parabolic representation of Δe.
(*d*) Point-by-point solution.
(*e*) Solution by use of superposition theorem.

Solution for Load Current in Case C(1), by Means of the Superposition Theorem.—For the self-excited machine an approximate solution for the open-circuit voltage build-up

was obtained by representing the voltage difference Δe by a symmetrical parabola. The result, although usually not very accurate, is interesting because it constitutes the only analytical solution giving the voltage explicitly in terms of time. Referring to Chap. XV, equation (173), the expression obtained is

$$e = f(t) = \frac{e_2}{1 + \frac{e_2 - e_1}{e_1}\epsilon^{-\frac{4\Delta e_m}{\sigma T e_1}}} = \frac{e_2}{1 + C\epsilon^{-\alpha t}} \tag{34}$$

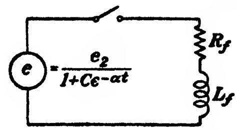


FIG. 259.—Voltage of special time characteristics applied to circuit consisting of resistance and constant inductance.

When a voltage of this form is impressed on a load circuit containing resistance and constant inductance (Fig. 259), it is possible to determine the load current analytically by means of the *superposition theorem*. Use the formulation

$$i = A(0)\cdot f(t) + \int_0^t f(\lambda)\cdot A'(t-\lambda)\cdot d\lambda \tag{35}$$

Obviously, in the present problem

$$A(t) = \frac{1}{R_f}\left(1 - \epsilon^{-\frac{R_f t}{L_f}}\right) = \frac{1}{R_f}(1 - \epsilon^{-\beta t}) \tag{36}$$

$$A(0) = 0 \tag{37}$$

$$A'(t) = \frac{1}{L_f}\epsilon^{-\beta t} \tag{38}$$

$$A'(t-\lambda) = \frac{1}{L_f}\epsilon^{-\beta(t-\lambda)} \tag{39}$$

and hence by substitution in equation (35)

$$i = \int_0^t \frac{e_2}{1 + C\epsilon^{-\alpha\lambda}}\cdot\frac{1}{L_f}\epsilon^{-\beta(t-\lambda)}\cdot d\lambda = \frac{e_2}{L_f}\epsilon^{-\beta t}\int_0^t \frac{\epsilon^{\beta\lambda}}{1 + C\epsilon^{-\alpha\lambda}}d\lambda \tag{40}$$

To integrate, write

$$\int \frac{\epsilon^{\beta\lambda}}{1 + C\epsilon^{-\alpha\lambda}}d\lambda = \frac{1}{\beta}\int\frac{dZ}{1 + CZ^n} \tag{41}$$

where

$$Z = \epsilon^{\beta\lambda} \qquad \text{and} \qquad n = -\frac{\alpha}{\beta} \tag{42}$$

Expanding the integrand in equation (41) by means of the binomial theorem this becomes

$$\int \frac{dZ}{1 + CZ^n} = \int (1 - CZ^n + C^2Z^{2n} - C^3Z^{3n} + \cdots)dZ$$
$$= Z\left(1 - \frac{C}{n+1}Z^n + \frac{C^2}{2n+1}Z^{2n} - \frac{C^3}{3n+1}Z^{3n} + \cdots\right) \quad (43)$$

Substituting back in equation (40), the solution for current is obtained as

$$i_f = \frac{e_2}{L_f}\epsilon^{-\beta t}\left[\frac{1}{\beta}\epsilon^{\beta\lambda}\left(1 - \frac{C}{1+\alpha\beta}\epsilon^{-\alpha\lambda} + \frac{C^2}{1-2\alpha\beta}\epsilon^{-2\alpha\lambda} - \frac{C^3}{1-3\alpha\beta}\epsilon^{-3\alpha\lambda} + \cdots\right)\right]_{\lambda=0}^{\lambda=t} \quad (44)$$

which inserting limits may be written

$$i_f = \frac{e_2}{R_f}\left[1 - \frac{C}{1-\alpha\beta}\epsilon^{-\alpha t} + \frac{C^2}{1-2\alpha\beta}\epsilon^{-2\alpha t} - \frac{C^3}{1-3\alpha\beta}\epsilon^{-3\alpha t} + \cdots - \left(1 - \frac{C}{1-\alpha\beta} + \frac{C^2}{1-2\alpha\beta} - \frac{C^3}{1-3\alpha\beta} + \cdots\right)\epsilon^{-\beta t}\right] \quad (45)$$

Exciter Supplying the Field Current of an Unloaded Alternator.—The simplest exciter-alternator problem results when the alternator is open-circuited. This eliminates completely the effect of the alternator armature circuit and the system which the alternator supplies. In fact the problem reduces to that of an exciter supplying a resistance and variable-inductance load, this load now being represented by the field winding of the alternator. Two methods of handling this problem will be briefly discussed, *viz.*:

1. Analytical solution considering alternator as resistance and variable-inductance load on exciter.
2. Solution by a graphical process.

The first method requires a point-by point solution but permits, if desired, the inclusion of the effect of the armature constants of the exciter. The graphical method is more direct but is limited to the case where all exciter armature constants are ignored. This method, in other words, requires that the exciter voltage-time variation be determinable independent of the load current which the exciter supplies, *i.e.*, from considerations of the exciter operating open-circuited.

1. *Analytical Solution Considering Alternator as Resistance and Variable-inductance Load on Exciter.*—Procedures for solving

this problem with various degrees of refinement have been previously discussed in this chapter under a similar heading. These methods all yield the alternator field current as a function of time. Since the alternator is unloaded the corresponding armature voltage may be read directly from the open-circuit magnetization curve. No additional comments on the above-mentioned methods seem necessary.

The alternator voltage may also be solved for directly by a minor modification of the attack which will be developed below. Using the following notation:

e = terminal voltage of exciter
i_f = field current of generator
R_f = total resistance in generator field
N_g = number of turns in generator field $\times 10^{-8}$
σ_g = coefficient of dispersion
ϕ = air-gap flux per pole
E = induced armature voltage (or open-circuit terminal voltage) of generator
T_g = generator-time constant

the equation for generator field circuit may be written

$$\sigma_g N_g \frac{d\phi}{dt} + R_f i_f = e \tag{46}$$

The proportionality between air-gap flux and induced voltage may be expressed by

$$E = k_g \phi \tag{47}$$

It is also convenient to introduce a time constant (formulated as previously done for the exciter) as follows:

$$T_g = \frac{N_g \phi}{E} = \frac{N_g}{k_g} \tag{48}$$

It will be noted that this represents an absolute constant of the machine. Substituting equations (47) and (48) in (46) gives

$$\sigma_g T_g \frac{dE}{dt} + R_f i_f = e \tag{49}$$

This may also be written

$$\frac{dE}{dt} = \frac{e - R_f I_f}{\sigma_g T_g} \tag{50}$$

which represents a more convenient form for the point-by-point solution.

Consider first the case where the exciter voltage-time relation is independently available. In other words, e in equation (50) is known as a function of time. The relationship between generator induced voltage and field current is given by the open-circuit magnetization curve. Under these conditions, the point-by-point solution of equation (50) becomes quite simple and may be carried out in accordance with the schedule suggested in Table 65.

When the effect of one or more of the armature constants of the exciter are to be included, the differential equations of exciter and alternator must be solved simultaneously. Thus, for the general case, they may be written

$$\sigma T \frac{de}{dt} + Ri = e - R_a i_a - L_a \frac{di_a}{dt} = R_f i_f + \sigma_g T_g \frac{dE}{dt} \tag{51}$$

It may be helpful to compare this with equation (29) which is identical except for the last term. By using the first and the last term and eliminating the exciter armature current by means of equation (8), the rate of change of exciter induced voltage becomes

$$\frac{de}{dt} = \frac{R_f i_f - Ri + \sigma_g T_g \dfrac{dE}{dt}}{\sigma T} \tag{52}$$

From the second and third term of equation (51) is obtained

$$\frac{dE}{dt} = \frac{e - R_a i - (R_a + R_f) i_f - L_a\left(\dfrac{di}{dt} + \dfrac{di_f}{dt}\right)}{\sigma_g T_g}$$

$$= \frac{e - R_a i - (R_a + R_f) i_f - L_a \dfrac{(i_n - i_{n-1}) + (i_{fn} - i_{f(n-1)})}{\Delta t}}{\sigma_g T_g} \tag{53}$$

Equations (52) and (53) may be compared with equations (30) and (32). They are solved simultaneously by a point-by-point solution which may be arranged as in Table 66.

2. *Solution by a Graphical Process.*—This particular method has been suggested and described by Rüdenberg.[1] As already stated, this method assumes that the exciter build-up curve is available, being determined, as previously described in detail, with no load on the exciter. The graphical solution will be based on equation (50).

The relationship between generator-induced voltage and field current is given by the open-circuit magnetization curve.

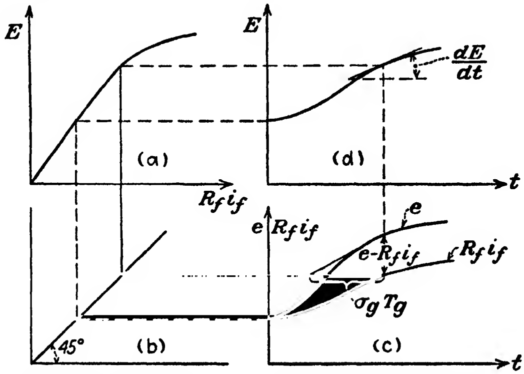


Fig. 260.—Graphical determination of build-up of terminal voltage of unloaded alternator as a result of exciter action. (For complete explanation see text.)

For convenience in the graphical solution, this may be replotted versus field-resistance drop (*i.e.*, versus $R_f i_f$) as indicated in (*a*) of Fig. 260, which illustrates the graphical process. The independently determined exciter-voltage time curve is drawn in (*c*). The purpose of the straight line in (*b*) is merely to facilitate the transfer of values by means of T square and triangle. These curves can be drawn prior to and independent of the determination of the voltage build-up of the generator. In (*c*) and (*d*) will be found the curves of generator field-resistance drop and generator induced voltage versus time, which are determined and constructed as the analysis progresses.

The procedure becomes as follows: The value of $R_f i_f$ corresponding to a point on the magnetization curve in (*a*) is transferred via the 45-deg. line in (*b*) to the corresponding curve in (*c*). The rate of change of generator voltage in accordance with

[1] Rüdenberg, Reinhold, "Die Spannungsregelung grosser Drehstromgeneratoren nach plötzlicher Entlastung," *Wiss. Veröffentlich. Siemens-Konzern*, Vol. IV, No. 2, 1925.

equation (50) is introduced graphically by the construction shown in (*c*). A horizontal distance equal to the total generator time constant ($\sigma_g T_g$) is measured off to the left, as indicated. Since the vertical distance between the two curves represents $e - R_f i_f$, the direction of the hypotenuse of the triangle obviously indicates the slope of the generator-voltage-time curve. By parallel transfer of the direction of the hypotenuse in (*c*) to (*d*) the voltage build-up curve may be mapped out step by step.

TABLE 65.—POINT-BY-POINT CALCULATION OF VARIATIONS IN ALTERNATOR OPEN-CIRCUIT VOLTAGE DUE TO EXCITER RESPONSE
Effect of Exciter Armature Constants Neglected

(1) Int. No.	(2) t	(3) e	(4) i_f	(5) $R_f i_f$	(6) dE/dt	(7) E

e_n:—Col. $(3)_n$ = value from independently determined exciter build-up curve.

i_{fn}:—Col. $(4)_n$ = value from alternator magnetization curve corresponding to E_n.

$$E_n\text{:—Col. } (7)_n = (7)_{n-1} + (6)_{n-1}\Delta t = (7)_{n-1} + \frac{[(3) - (5)]_{n-1}}{\sigma_g T_g}\Delta t.$$

EXAMPLE 1

Statement of Problem

Figure 261 shows a direct-current, self-excited, shunt-wound exciter supplying the field current of a large synchronous alternator. The alternator carries no load, and its voltage is adjusted to 80 per cent of its rated value. Under these conditions, the external resistance in the exciter field circuit is suddenly short-circuited, allowing the exciter and alternator voltages to build up to their ultimate (ceiling) values. Determine and plot curves of exciter voltage and alternator field current versus time.

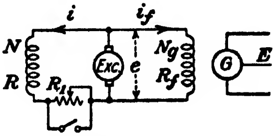


FIG. 261.—Schematic diagram of self-excited exciter supplying the field current of a large unloaded alternator (Example 1).

In the solution take the nonlinear characteristic of the exciter and the variable inductance of the alternator field circuit into account. Neglect resistance and leakage inductance in the exciter armature and assume that the reaction of

TABLE 66.—POINT-BY-POINT CALCULATION OF VARIATIONS IN ALTERNATOR OPEN-CIRCUIT VOLTAGE DUE TO EXCITER RESPONSE
Effect of Exciter Armature Constants Included

(1) Int. No.	(2) t	(3) e	(4) i	(5) i_f	(6) Ri	(7) $R_a i$	(8) $R_f i_f$	(9) $(R_a + R_f)i_f$	(10) $L_a \dfrac{i_n - i_{n-1}}{\Delta t}$	(11) $L_a \dfrac{i_{fn} - i_{f(n-1)}}{\Delta t}$	(12) $\dfrac{de}{dt}$	(13) $\dfrac{dE}{dt}$	(14) $\sigma_g T_g \dfrac{dE}{dt}$	(15) E

e_n:—Col. $(3)_n = (3)_{n-1} + (11)_{n-1}\Delta t = (3)_{n-1} + \dfrac{[(8) - (6) + (14)]_{n-1}}{\sigma T}$.

i_n:—Col. $(4)_n$ = value from exciter magnetization curve corresponding to e_n.
i_{fn}:—Col. $(5)_n$ = value from alternator magnetization curve corresponding to E_n.

E_n:—Col. $(15)_n = (15)_{n-1} + (13)_{n-1}\Delta t = (15)_{n-1} + \dfrac{[(3) - (7) - (9) - (10) - (11)]_{n-1}}{\sigma_g T_g}\Delta t$

the exciter armature current on its field is so small that it does not have to be taken into account

The following data relate to the exciter (same as used in Example 8 of Chap XV):

Rating	200 kw
Voltage (normal)	250 volts
Armature resistance	0 0113 ohm
Armature inductance	0 0063 henry
Field resistance (hot)	23 3 ohms
Nominal field inductance	26 6 henrys[1]
Total resistance available for use in series with the shunt field	66 7 ohms
100 per cent field current	6 6 amp
Coefficient of dispersion	1 15

EXCITER MAGNETIZATION CURVE (FIG 262) OBTAINED WITH THE MACHINE SELF-EXCITED

Percentage Voltage	Percentage Field Current
4	0
40	34
60	54
80	75
90	86
100	100
110	117 5
120	139
130	170
135	192
140	219
143	240

The following data relate to the generator

Type	Water wheel
Rating	28 000 kva
Voltage	14 000 volts
Phases	Three
Frequency	60 cycles
Speed	75 r p m
Number of poles	96
Short-circuit ratio	1
Armature resistance	0 4 per cent
Armature leakage reactance	23 per cent
Transient reactance	30 per cent
Synchronous reactance	100 per cent
Field voltage	250 volts
Field resistance at 100 deg C	0 25 ohm
Nominal field inductance	2 00 henrys[1]

[1] The nominal field inductance may be defined as the number of flux linkages per field ampere at rated armature voltage with the armature winding open-circuited.

Field excitation at full load 80 per cent power factor	190 kw.
Normal field range	90 to 210 per cent
Normal (100 per cent) field current	450 amp.

The magnetization curve of this alternator is given in Fig. 263 with additional values as follows:

Percentage Voltage	Percentage Field Current
133.5	260
136.0	300
137.0	320

Solution

In view of the stipulated assumptions, the problem breaks up into two parts: (*a*) to determine the build-up of exciter voltage as a function of time,

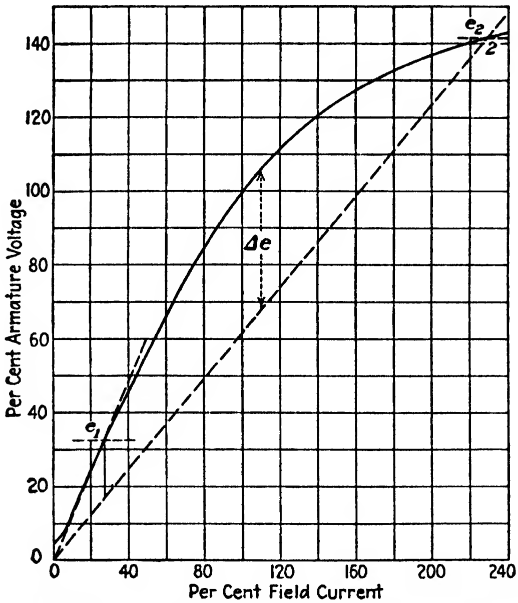


FIG. 262.—Magnetization curve and field-resistance line of exciter in Fig. 261 (Example 1).

and (*b*) to determine the alternator field current due to the variable exciter voltage being impressed on the alternator field.

a. From the magnetization curve of the alternator, it is seen that, for 80 per cent armature voltage, the field current is 73 per cent or, in amperes,

$$i_f = 0.73 \times 450 = 328.8 \text{ amp.}$$

The initial value of exciter terminal voltage is then

$$v = e = 328.8 \times 0.25 = 82.2 \text{ volts} = 32.9 \text{ per cent}$$

This value spotted on the exciter magnetization curve gives the initial operating point with a corresponding exciter field current of

$$i = 26.8 \text{ per cent} = 0.268 \times 6.6 = 1.77 \text{ amp.}$$

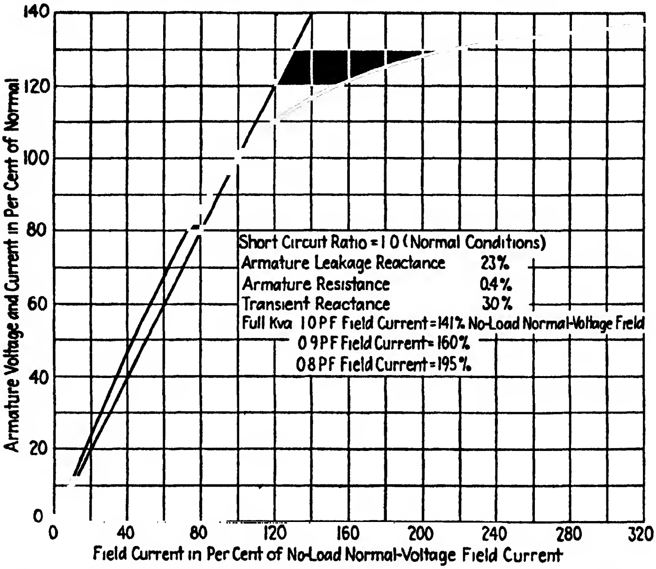


FIG. 263.—Typical open-circuit and short-circuit characteristics of large water-wheel-driven, 60-cycle, 80 per cent power-factor generator (Example 1).

The external resistance in series with the exciter field is consequently

$$R_1 = \frac{82.2}{1.77} - 23.3 = 46.5 - 23.3 = 23.2 \text{ ohms}$$

all of which is short-circuited out during the build-up process.

To locate the resistance line (with no external resistance), the value of current at 100 per cent voltage (250 volts) may be used. This current is

$$i = \frac{250}{23.3} = 10.73 \text{ amp.} = 162.5 \text{ per cent}$$

The resistance line is hence a straight line drawn through the origin and the point $e = 100$ per cent, $i = 162.5$ per cent. The intersection of this line with the magnetization curve gives the final operating point indicating a ceiling voltage of 141.5 per cent.

The build-up of the exciter voltage will be found by graphical integration as outlined in Chap. XV. Referring to equation (159), Chap. XV, the expression on which this is based is

$$t = \sigma T \int \frac{de}{\Delta e} \qquad (a)$$

The time constant [equation (59), Chap. XV] becomes

$$\sigma T = \frac{\sigma N}{k} = \frac{\sigma N \phi_n}{e_n} = \frac{L_n i_{fn}}{e_n} = \frac{26.6 \times 6.6}{250} = 0.702 \text{ sec.}$$

From the magnetization curve and the resistance line (Fig. 262), the voltage differentials Δe are obtained. The data for and the results of the graphical

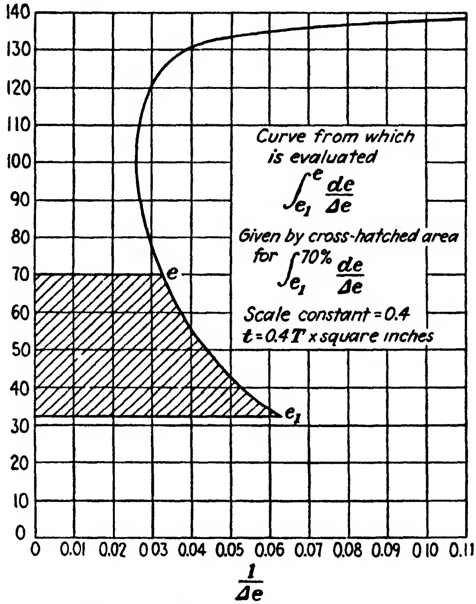


FIG. 264.—Curve of $1/\Delta e$ versus percentage voltage for build-up of exciter in Fig. 261 (Example 1).

evaluation are included in Table 67, while the actual process is executed in Fig. 264 by measuring the appropriate successive areas. The build-up curve of exciter voltage is plotted in Fig. 265.

b. The relationship between the nominal inductance L_0, the field flux linkages (times 10^{-8}) corresponding to 100 per cent armature voltage $N_g\phi_0$, and the normal field current i_{f0} is given by

$$L_0 = \frac{\sigma_g N_g \phi_0}{i_{f0}} \tag{b}$$

Inserting numerical values, the flux linkages at 100 per cent voltage are thus:

$$\sigma_g N_g \phi_0 = 2.00 \times 450 = 900$$

from which the initial flux linkages $N_g\phi_1$ at 80 per cent armature voltage may be found to be

$$\sigma_g N_g \phi_1 = 0.8 \times 900 = 720$$

The field current required to produce this flux is

$$i_{f1} = 0.73 \times 450 = 328.8 \text{ amp.}$$

The alternator field current corresponding to the exciter ceiling voltage becomes

$$i_{f2} = \frac{1.415 \times 250}{0.25} = 1{,}416 \text{ amp.} = 314 \text{ per cent}$$

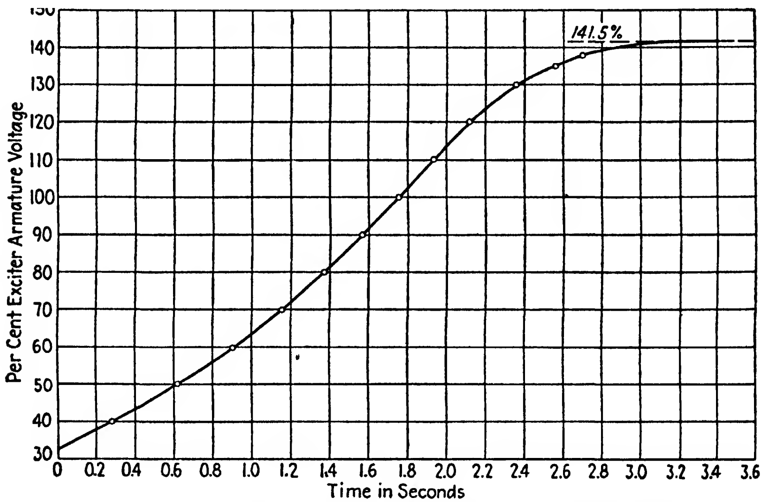


FIG. 265.—Calculated voltage build-up curve for exciter in Fig. 261 (Example 1).

the corresponding armature voltage (from the magnetization curve) being 137.0 per cent, and consequently the final value of flux linkages $N_g\phi_2$

$$\sigma_g N_g \phi_2 = 1.37 \times 900 = 1{,}230$$

The build-up of alternator flux and field current is governed by the following equation [see equation (46)]

$$\sigma_g N_g \frac{d\phi}{dt} + R_f i_f = e \tag{c}$$

where e = exciter voltage (already plotted as a function of time)
R_f = alternator field resistance

Although transcribable into other forms, this equation will in the present problem be solved as it stands by a point-by-point analysis. The calculations are carried out as indicated in Table 68. Figure 266 shows the desired curve of alternator field current versus time. Note that approximately 5 sec. are required to complete the build-up. [For comparison a dotted curve is shown in the same figure giving build-up in approximately 3 sec. This is obtained using a nominal field inductance of the generator of 0.38

henry, but the same characteristics otherwise. This inductance is much below normal for a 28,000-kva. alternator of ordinary design. The addi-

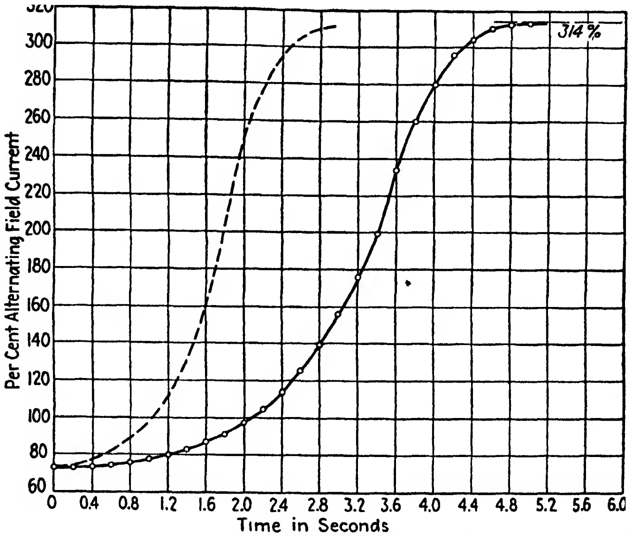


FIG. 266.—Calculated build-up of the field current of the alternator in Fig. 261 as a result of exciter response (Example 1). Note: The dotted curve drawn in for comparison is for a nominal generator field inductance of 0.38 henry.

tional curve, however, serves to indicate the general influence of field inductance (time constant) on the build-up process.]

TABLE 67

$e\%$	$\Delta e\ \%$	$\dfrac{1}{\Delta e}$	Area from Fig. 264	$\int_{e_1}^{e} \frac{de}{\Delta e} =$ 0.1 × area	$t = \sigma T \int_{e_1}^{e} \frac{de}{\Delta e}$ sec.
32.88	15.9	0.0628	0	0	0
40	19.2	0.0521	4.11	0.411	0.289
50	22.9	0.0437	8.89	0.889	0.623
60	26.8	0.0373	12.92	1.292	0.907
70	30.9	0.0323	16.42	1.642	1.152
80	33.9	0.0295	19.49	1.949	1.368
90	37.3	0.0268	22.26	2.226	1.565
100	38.5	0.0260	24.86	2.486	1.745
110	37.6	0.0266	27.46	2.746	1.930
120	34.2	0.0292	30.22	3.022	2.120
130	26.0	0.0385	33.50	3.350	2.350
135	16.8	0.0595	35.81	3.581	2.560
138.5	9.0	0.111	38.35	3.835	2.690

TABLE 68.—POINT-BY-POINT CALCULATION OF BUILD-UP-CURVE OF ALTERNATOR FIELD CURRENT
(Example 1)

t sec.	e volts	i_f amp.	$R_f i_f$ volts	$e - R_f i_f$ volts	$\sigma_g N_g \frac{d\phi}{dt}$	$\sigma_g N_g \phi$	$\frac{\sigma_g N_g \phi}{\sigma_g N_g \phi_0}$ %	i_f %
0 0	82 2	328 8	82 2	0 0	0 0	720 00	80 00	73 0
0 1	87 7	328 8	82 2	5 5	5 50	720 00	80 00	73 0
0 2	94 3	328 95	82 24	12 06	12 06	720 55	80 06	73 1
0 3	101 3	329 4	82 35	18 95	18 95	721 76	80 22	73 2
0 4	108 8	329 9	82 48	26 32	26 32	723 66	80 41	73 3
0 5	115 9	331 2	82 80	33 10	33 10	726 29	80 70	73 6
0 6	123 8	333 0	83 25	40 55	40 55	729 60	81 07	74 0
0 7	131 2	335 3	83 83	47 37	47 37	733 66	81 52	74 5
0 8	140 2	337 5	84 38	55 82	55 82	738 40	82 04	75 0
0 9	149 0	342 00	85 50	63 50	63 50	743 98	82 67	76 0
1 0	158 8	344 70	86 18	72 62	72 62	750 33	83 37	76 6
1 1	169 2	350 10	87 53	81 67	81 67	757 59	84 18	77 8
1 2	181 0	355 05	88 76	92 24	92 24	765 76	85 08	78 9
1 3	192 5	360 00	90 00	102 50	102 50	774 98	86 11	80 0
1 4	203 6	366 75	91 7	111 91	111 91	785 28	87 25	81 5
1 5	216 5	373 50	93 4	123 12	123 12	796 47	88 49	83 0
1 6	229 0	382 50	95 6	133 37	133 37	808 78	89 86	85 0
1 7	242 7	389 25	97 31	145 39	145 39	822 12	91 35	86 5
1 8	256 5	396 90	99 2	157 27	157 27	836 66	92 96	88 2
1 9	271 0	409 50	102 4	168 62	168 62	852 39	94 71	91 0
2 0	284 0	423 00	105 7	178 25	178 25	869 25	96 48	94 0
2 1	298 0	438 30	109 6	188 42	188 42	887 08	98 56	97 4
2 2	309 0	454 50	113 6	195 37	195 37	905 92	100 65	101 0
2 3	318 8	472 50	118 1	200 67	200 67	925 48	102 83	105 0
2 4	328 0	493 20	123 3	204 70	204 70	945 53	105 06	109 6
2 5	336 8	513 00	128 3	208 55	208 55	966 00	107 33	114 0
2 6	340 2	535.50	133 9	206 32	206 32	986 86	109 65	119 0
2 8	347 5	564 7	141 2	206 3	206 3	1007 49	111 94	125 5
3 0	355 0	625 5	156 4	198 6	198 6	1048 8	116 5	139 0
3 2	355 0	702 0	175 5	179 5	179 5	1088 5	120 9	156 0
3 4	355 0	792 0	198 0	157 0	157 0	1124 4	124 9	176 0
3 6	355 0	895 5	223 9	131 1	131 1	1155 8	128 4	199 0
3 8	355 0	1053 0	263 3	91 7	91 7	1182 0	131 3	234 0
4 0	355 0	1170 0	292 5	62 5	62 5	1200 3	133 4	260 0
4 2	355 0	1260 0	315 0	40 0	40 0	1212 8	134 8	280 0
4 4	355 0	1332 0	333 0	22 0	22 0	1220 8	137 5	296 0
4 6	355 0	1359 0	339 8	15 2	15 2	1225 2	136 1	302 0
4 8	355 0	1395 0	348 8	6 2	6 2	1228 2	136 5	310 0
5 0	355 0	1404 0	351 0	4 0	4 0	1229 4	136 6	312 0

Exciter Supplying the Field Current of a Loaded Alternator.—The problem involving an exciter connected to a loaded alternator, for instance, an alternator supplying power to a system, is considerably more complicated than the one previously discussed with the alternator on open circuit. With load on the alternator, there will be a reaction back from the armature circuit on the field circuit which varies as the armature current changes, and the effect of which should be taken into account in the solution. Furthermore, the nonlinear characteristics of alternator as well as exciter should be included. However, unless simplifications are made, the problem becomes quite unwieldy and hence a rigorous solution is seldom attempted.

The most complete methods, at least in a number of respects, are those originally described by Bush and Booth.[1] Their analysis particularly relates to the problem of sudden load changes (the load being balanced) but can be extended to other types of discontinuities. Their methods which allow the influence of varying armature current and nonlinearity to be considered involve point-by-point computations utilizing primarily rather elaborate graphical methods.

As already stated, however, this general problem is now almost exclusively handled on an approximate basis. It is customary to introduce an *equivalent* time constant[2] for the alternator field circuit, and to consider this time constant fixed. This obviously corresponds to representing the field by an equivalent constant inductance. The time constant actually varies with saturation, and it would theoretically be possible to allow for changes in it as the point-by-point analysis progresses. If this refinement is not introduced, as it seldom is, an average value of the time constant over the range of saturation encountered may be used.[3]

[1] BUSH, V. and R. D. BOOTH, "Power-system Transients," *Trans. A.I.E.E.*, 1925, p. 80.

[2] PARK, R. H., and B. L. ROBERTSON, "The Reactances of Synchronous Machines," *Trans. A.I.E.E.*, Vol. 47, p. 514, April, 1928.

[3] Knowledge of the variation with saturation of time constants is limited. Reference in this connection may be made to the following papers: L. A. Kilgore, "Calculation of Synchronous-machine Constants. Reactances and Time Constants Affecting Transient Characteristics," *Trans. A.I.E.E.*, p. 1201, 1931; S. H. Wright, "Determination of Synchronous-machine Constants by Test. Reactances, Resistances, and Time Constants," *Trans. A.I.E.E.*, p. 1331, 1931.

The time constant of particular interest in problems involving sudden load changes, sudden faults, etc. is the *direct-axis transient time constant.* Two values of this (both changing with saturation), *viz.*, the direct-axis transient, open-circuit time constant T'_{d0}, and the direct-axis transient, short-circuit time constant T'_d are readily determinable by test. The former relates to the case where the alternator is on open circuit (*i.e.*, unloaded) and the second to the case of a symmetrical three-phase short circuit directly at the terminals. This latter is then strictly an equivalent time constant for symmetrical short-circuit conditions. If there is reactance between the machine and the point of symmetrical three-phase short circuit, the short-circuit time constant may be corrected for the effect of the external reactance,[1] the approximate equivalent time constant under such conditions being

$$T_d \cong T'_{d0}\frac{X'_d + X_e}{X_d + X_e} \tag{54}$$

where X_d and X'_d are the direct-axis synchronous and transient reactances, respectively, and X_e the external reactance. If the short circuit is dissymmetrical, the same formula may be employed using for the external reactance the total positive-sequence system reactance (driving-point reactance) as viewed from the terminals of the generator. The conception of equivalent time constant applies also under normal (unfaulted) conditions of operation as when the generator supplies an individual load or is connected to a system. The distinction between unfaulted and faulted conditions under otherwise identical circuit arrangements is reflected in the value of the "external" reactance (X_e) in equation (54). Hence even in a complicated system an *approximate equivalent time constant* may be determined.

The direct-axis transient open-circuit time constant equals the ratio of actual field inductance to field resistance, although it is possible that a test value may differ slightly from this ratio owing to the effect of eddy currents. The relationship between the direct-axis transient, open-circuit time constant and other previously defined field and armature constants (including the

[1] Park and Robertson, *loc. cit.*

nominal inductance and the time constant T_g)[1] is of interest and may be expressed by the following relations:

$$T'_{d0} \cong T_f = \frac{L_f}{R_f} = \frac{\sigma_g N_g}{R_f}\frac{d\phi}{di_f} = \frac{\sigma_g N_g}{R_f k_g}\frac{dE}{di_f}$$
$$= \frac{L_0}{R_f}\left(\frac{i_{f0}}{E_0}\right)\frac{dE}{di_f} = \frac{\sigma_g T_g}{R_f}\frac{dE}{di_f} \tag{55}$$

Hence the time constants T'_{d0} and T_f vary with saturation and depend upon the operating point on the magnetization curve. The time constant T_g on the other hand is definitely a constant.

Typical values of direct-axis transient, open-circuit time constants for various classes of synchronous machines are as follows:

Turbogenerators	4–7 sec.
Waterwheel generators..................	3–6 sec.
Synchronous motors....................	2–4 sec.
Synchronous condensers	5–7 sec.

The general methods of handling exciter action in connection with alternators connected to power systems is discussed in more detail in Chap. XVII. It is there shown how regulator action and exciter response may be included in point-by-point solutions of problems involving sudden load changes and faults, and consequently oscillations of the synchronous machines.

EXAMPLE 2

Statement of Problem

For the alternator on which data are given in Example 1 calculate for open-circuit conditions,

a. The field inductance at 100 and 50 per cent of normal armature voltage.

b. The time constant of the field winding at these operating points.

c. The time constant as defined by equation (48).

Solution

a. By referring to equation (*b*) in the footnote on page 552, the field inductance is given in terms of the nominal inductance by

$$L_f = L_0\frac{dE}{di_f}$$

when both E and i_f are expressed in percentage of per unit. Obtaining the necessary slopes of the magnetization curve from Fig. 263, and using a nominal inductance of 2.00 henrys, the required inductances become

[1] See pages 552 and 557.

$$E = 100 \text{ per cent: } L_f = \frac{2.0(100 - 43)}{100} = 1.14 \text{ henrys}$$

$$E = 50 \text{ per cent: } L_f = \frac{2.0 \times 50}{43} = 2.32 \text{ henrys.}$$

b. Referring to equation (55), the corresponding time constants (essentially equal to the direct-axis, transient, open-circuit time constants) become

$$E = 100 \text{ per cent: } T'_{d0} \cong T_f = \frac{1.14}{0.25} = 4.56 \text{ sec.}$$

$$E = 50 \text{ per cent: } T'_{d0} \cong T_f = \frac{2.32}{0.25} = 9.28 \text{ sec.}$$

c. Again using equation (55), the time constant as defined by equation (48) is

$$\sigma_g T_g = \frac{L_0 i_{f0}}{E_0} = \frac{2.0 \times 450 \times \sqrt{3}}{14{,}000} = 0.1115 \text{ sec.}$$

CHAPTER XVII

ACTION OF VOLTAGE REGULATORS AND EXCITATION SYSTEMS

Function of Automatic Voltage Regulators.—The purpose of a voltage regulator is to control the voltage which an alternator (or synchronous condenser) supplies to the system. The voltage in question may be the terminal voltage or the voltage at some other point, for instance, the high-tension voltage on the line side of the step-up transformers. In most instances it is desired to maintain constant voltage at the regulated point, although sometimes it may be desired automatically to increase the voltage as the load comes on. This calls for compounding features in the regulator. In either case the regulator acts as the agent which detects the voltage variation and initiates the proper corrective steps to bring the voltage back to the desired magnitude. This may be done by adjusting an external series resistance in the alternator field circuit. In modern excitation systems, however, the adjustment is invariably brought about by changing external series resistance in the field circuit of the exciter. It is immaterial in this connection whether the latter is separately or self-excited.

A number of different regulator types are available. Even a brief description is beyond the scope of this treatise, and the reader is referred to the electrical-engineering handbooks and to the various manufacturers' publications which deal with regulators. Suffice it to say that the regulators used in this country today may be subdivided into two general classifications: (1) regulators of the vibrating-contact type, and (2) regulators of the rheostatic type. With either class the general result of regulator action is the same although achieved in different manners. The newer designs of either type may involve a "high-speed" feature which becomes effective when the voltage deviates from normal by an appreciable amount, such as is caused by system faults, the sudden application or dropping of large blocks of load, power surges among generating stations,

etc. The high-speed feature usually involves the complete short-circuiting of the entire external resistance in the exciter field circuit or its complete introduction into the field circuit, depending upon whether or not the regulator is called upon to correct for an excessive voltage drop or an excessive voltage rise.

In the vibrating-contact regulator the contacts are continuously vibrating, alternately short-circuiting and reinserting a definite part of the external field resistance. With this regulator the different excitation requirements to maintain normal alternator voltage are met by changing the relative lengths of the time intervals during which the regulating resistance is short-circuited and inserted in the circuit. The old Tirrill regulator, which has been so extensively used in the past, belongs to this class. The action of this type of regulator is analyzed in some detail below.

The rheostatic types of regulator have the advantage that they do not involve mechanisms which are continuously in motion. During normal operation, they adjust the external resistance in the exciter field to the value required to maintain the proper alternator voltage. Hence under steady conditions such a regulator may remain inactive and at rest for long periods of time. For this reason and in view of the fact that also these regulators can be associated with high-speed features when necessary, this type is now gradually superseding the vibrating-contact type for modern applications.

Originally in three-phase systems, the voltage of only one of the phases was relied upon to actuate the regulator. During conditions of normal operation, this method gives entirely satisfactory results. However, under conditions of dissymmetrical faults resulting in more or less unbalanced voltages the single-phase scheme is not always reliable and may under certain conditions even give rise to incorrect action. It is now customary therefore to provide an arrangement whereby the regulator responds to the positive-sequence voltage. This may be done by feeding the operating relays or solenoids of the regulator through a network (sequence filter) which passes only the positive-sequence component. It may also be done by letting the actuating element of the regulator be a three-phase torque motor of the squirrel-cage induction type. The net

torque produced by such a device is predominantly proportional to the square of the positive-sequence voltage even under severe degrees of unbalance. The filter network mentioned above, as well as the three-phase torque motor, is in general supplied through a bank of two V-connected potential transformers.

Analysis of Transients with the Vibrating-contact Regulator.—In order to establish some fundamental conceptions regarding the regulatory effects of the vibrating-contact regulator when used on excitation systems, it is useful to examine the transients in a simple circuit, containing resistance and inductance, in which a part of the resistance is periodically short-circuited.[1] For the sake of simplicity, consider the circuit linear and assume that a strictly constant direct-current voltage is applied, as shown in Fig. 267. Except for the omission of saturation, this represents the field circuit of a separately excited machine when operating at no load (see Fig. 230). The results from the present analysis, although approximate, correctly illustrate the character of the transients and bring out certain pertinent points relative to the characteristics of such regulators.

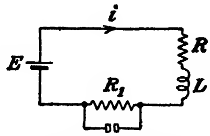


Fig. 267.—Schematic diagram showing control of the current in an inductive circuit by means of a vibrating-contact regulator. The regulator alternately short-circuits and reinserts the resistance R_1.

With the contacts closed, the differential equation for the circuit becomes

$$L\frac{di}{dt} + Ri = E \tag{1}$$

the solution of which is

$$i = A\epsilon^{-\frac{R}{L}t} + \frac{E}{R} \tag{2}$$

where A is a constant of integration. Let the contacts close at time $t = t_1$ and let the current at this instant be i_1. Then

$$A = \left(i_1 - \frac{E}{R}\right)\epsilon^{\frac{R}{L}t_1} \tag{3}$$

Inserting this in equation (2) the solution for current may be written

[1] Nickle, C. A., and R. M. Carothers, "Automatic Voltage Regulators. Application to Power-transmission Systems," *Trans. A.I.E.E.*, p. 957, 1928.

$$i = \frac{E}{R} - \left(\frac{E}{R} - i_1\right)\epsilon^{-\alpha(t-t_1)} \tag{4}$$

where

$$\alpha = \frac{R}{L} \tag{5}$$

With the contacts open the differential equation becomes

$$L\frac{di}{dt} + R_2 i = E \tag{6}$$

whose solution is

$$i = \frac{E}{R_2} - \left(\frac{E}{R_2} - i_2\right)\epsilon^{-\beta(t-t_2)} \tag{7}$$

assuming that the contacts open at a time $t = t_2$ when the current is i_2 and that

$$\beta = \frac{R_2}{L} \tag{8}$$

When the contacts operate under sustained conditions, *i.e.*, when they periodically short-circuit and reinsert the regulating resistance, the final value of current from equation (4) must equal the initial current in equation (7). Furthermore, the final value of current from equation (7), at, say, time $t = t_3$, must in turn equal the initial current in equation (4). Hence

$$i_{2s} = \frac{E}{R} - \left(\frac{E}{R} - i_{1s}\right)\epsilon^{-\alpha(t_2-t_1)} \tag{9}$$

$$i_{1s} = \frac{E}{R_2} - \left(\frac{E}{R_2} - i_{2s}\right)\epsilon^{-\beta(t_3-t_2)} \tag{10}$$

In these equations i_{1s} = the initial current with the contacts closed and the final current with the contacts open, under sustained conditions of operation

i_{2s} = the final current with the contacts closed and the initial current with the contacts open, under sustained conditions of operation

Now introduce

$$t_2 - t_1 = t_c \tag{11}$$

$$t_3 - t_2 = t_0 \tag{12}$$

where t_c is the time during which the contacts are closed and t_0 the time during which the contacts are open. Their sum, $t_c + t_0$ obviously represents the period of vibration of the

contacts. Substituting these symbols, equations (9) and (10) may be written

$$i_{2s} = \frac{E}{R} - \left(\frac{E}{R} - i_{1s}\right)\epsilon^{-\alpha t_c} \tag{13}$$

$$i_{1s} = \frac{E}{R_2} - \left(\frac{E}{R_2} - i_{2s}\right)\epsilon^{-\beta t_0} \tag{14}$$

from which i_{1s} and i_{2s} are obtained by simultaneous solution as

$$i_{1s} = E\frac{\frac{1}{R_2} + \left(\frac{1}{R} - \frac{1}{R_2}\right)\epsilon^{-\beta t_0} - \frac{1}{R}\epsilon^{-(\alpha t_c+\beta t_0)}}{1 - \epsilon^{-(\alpha t_c+\beta t_0)}} \tag{15}$$

$$i_{2s} = E\frac{\frac{1}{R} - \left(\frac{1}{R} - \frac{1}{R_2}\right)\epsilon^{-\alpha t_c} - \frac{1}{R_2}\epsilon^{-(\alpha t_c+\beta t_0)}}{1 - \epsilon^{-(\alpha t_c+\beta t_0)}} \tag{16}$$

These equations determine the minimum value i_{1s} and the maximum value i_{2s} of the current under sustained operating conditions. The variation of current between these limits is exponential and is given by equations (4) and (7). The current rises during the interval of time t_c while the contacts are closed, and decays during the interval t_0 while the contacts are open. Figure 268 illustrates the current-time relationship under sustained conditions. It follows from this that the exciter voltage will undergo corresponding fluctuations.

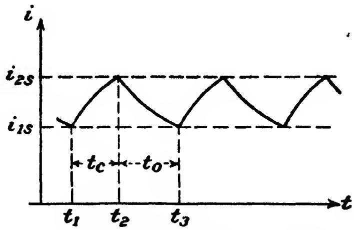


FIG. 268.—Current-time relationship for the circuit in Fig. 267 when controlled by a vibrating-contact regulator under sustained conditions of operation.

The *average* value of current is of interest. Using equations (4) and (7), the average current under sustained conditions becomes

$$\begin{aligned} i_{av} &= \frac{1}{t_0 + t_c}\left\{\int_{t_1}^{t_2}\left[\frac{E}{R} - \left(\frac{E}{R} - i_{1s}\right)\epsilon^{-\alpha(t-t_1)}\right]dt \right. \\ &\qquad \left. + \int_{t_2}^{t_3}\left[\frac{E}{R_2} - \left(\frac{E}{R_2} - i_{2s}\right)\epsilon^{-\beta(t-t_2)}\right]dt\right\} \\ &= \frac{1}{t_0 + t_c}\left\{E\left[\frac{t_c}{R} + \frac{t_0}{R_2} - \frac{1}{\alpha R}(1 - \epsilon^{-\alpha t_c}) - \frac{1}{\beta R_2}(1 - \epsilon^{-\beta t_0})\right] \right. \\ &\qquad \left. + \frac{i_{1s}}{\alpha}(1 - \epsilon^{-\alpha t_c}) + \frac{i_{2s}}{\beta}(1 - \epsilon^{-\beta t_0})\right\} \end{aligned} \tag{17}$$

Substituting for i_{1s} and i_{2s} from equations (15) and (16) and reducing, the final solution for the average current is

$$i_{av} = E\left[\frac{\frac{t_c}{R} + \frac{t_0}{R_2}}{t_0 + t_c} - \frac{L}{t_0 + t_c}\left(\frac{R_2 - R}{RR_2}\right)^2 \frac{(1 - \epsilon^{-\alpha t_c})(1 - \epsilon^{-\beta t_0})}{1 - \epsilon^{-(\alpha t_c + \beta t_0)}}\right] \tag{18}$$

It may be noted that, since $1/R$ equals the conductance of the circuit with the contacts closed and $1/R_2$ the conductance with the contacts open, the first term in the bracket of the above expression represents the *average conductance.* The average current naturally depends on the impressed voltage and the circuit constants. It also depends, as seen, on the value of the time intervals during which the contacts are open and closed, respectively. More conveniently expressed, it depends on the period (or frequency) of vibration in conjunction with the ratio of "time open" to "time closed." For the purpose of gaining some idea of the relative influence of these factors, the average current will be examined for a few special cases.

1. *Zero Inductance.*—Substitution of $L = 0$ in equation (18) gives

$$i_{av} = E\frac{\frac{t_c}{R} + \frac{t_0}{R_2}}{t_0 + t_c} = E\,G_{av} \tag{19}$$

Hence in the absence of inductance, the average current is proportional to the average conductance. This same condition will naturally be approached when the inductance is *very small.* The current-time relation for this case is shown in Fig. 269(a).

2. *Infinite Inductance.*—This condition makes equation (18) indeterminate. Evaluating the part in question by standard methods gives

$$\mathop{\mathcal{L}}_{L\to\infty} \frac{L\left(1 - \epsilon^{-\frac{R}{L}t_c}\right)\left(1 - \epsilon^{-\frac{R_2}{L}t_0}\right)}{1 - \epsilon^{-\frac{1}{L}(Rt_c + R_2t_0)}} = \frac{RR_2t_ct_0}{Rt_c + R_2t_0} \tag{20}$$

Inserting the above result in equation (18) the average current becomes

$$i_{av} = E\left[\frac{\frac{t_c}{R} + \frac{t_0}{R_2}}{t_0 + t_c} - \left(\frac{R_2 - R}{RR_2}\right)^2 \frac{RR_2t_ct_0}{(t_0 + t_c)(Rt_c + R_2t_0)}\right] \tag{21}$$

which upon contraction reduces to

$$i_{av} = E\frac{t_0 + t_c}{Rt_c + R_2t_0} = \frac{E}{\dfrac{Rt_c + R_2t_0}{t_0 + t_c}} = \frac{E}{R_{av}} \tag{22}$$

Hence when the inductance is infinite, the average current is governed by the *average resistance* of the circuit. For large values of inductance, this condition is evidently approached. As a matter of fact, even for moderate values of inductance, equation (22) is applicable as a close approximation provided

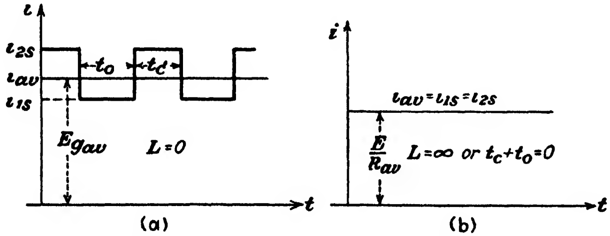


FIG 269 —Current-time relationship for the circuit in Fig 267 when controlled by a vibrating-contact regulator under sustained conditions of operation (*a*) When the inductance is zero. (*b*) When the inductance or the frequency of vibration is infinite.

the frequency of vibration of the contacts is not too low. The current-time curve for this case is shown in Fig. 269(*b*).

3. *Infinite Frequency*.—As were the former, which involved infinite inductance, the present is also a hypothetical case which can only be approached in practice. Infinite frequency of vibration implies zero period. Consequently the "time open" and the "time closed" would both be zero, and any distinction between these two time intervals is lost track of. In order to obtain a solution, it will be assumed that the condition of zero period is approached as a limit with the retention of a definite ratio of "time open" to "time closed." Let

$$\frac{t_0}{t_c} = m \tag{23}$$

Then from equation (18)

$$i_{av} = E\left[\frac{\dfrac{t_c}{R} + \dfrac{mt_c}{R_2}}{(1+m)t_c} - \frac{L}{(1+m)t_c}\left(\frac{R_2 - R}{RR_2}\right)^2\frac{(1 - \epsilon^{-\alpha t_c})(1 - \epsilon^{-\beta m t_c})}{1 - \epsilon^{-(\alpha+\beta m)t_c}}\right] \tag{24}$$

For infinite frequency, *i.e.*, $t_c = 0$, the last term becomes indeterminate. Again evaluating the part in question by standard methods gives

$$\mathop{\mathcal{L}}_{t_c \to 0} \frac{(1 - \epsilon^{-\alpha t_c})(1 - \epsilon^{-\beta m t_c})}{t_c(1 - \epsilon^{-(\alpha + \beta m)t_c})} = \frac{m\alpha\beta}{\alpha + m\beta} = \frac{mRR_2}{(R + mR_2)L} \quad (25)$$

Substituting this expression in equation (24), the average current is obtained as

$$i_{av} = \frac{E}{1 + m}\left[\frac{mR + R_2}{RR_2} - \frac{(R_2 + R)^2 m}{RR_2(R + mR_2)}\right] \quad (26)$$

which reduces to

$$i_{av} = \frac{E(1 + m)}{R + mR_2} = \frac{E}{\dfrac{R + mR_2}{1 + m}} = \frac{E}{R_{av}} \quad (27)$$

This shows that, when the frequency of vibration of the contacts is infinite, the average current depends on the *average* resistance of the circuit. Hence the effect of infinite frequency is the same as that of infinite inductance, a result which is not at all startling. For very high frequencies the "infinite" condition is approached and equation (27) is applicable as an approximation.

In most practical cases the combined effects of inductance and contact frequency will be sufficiently great to make the solution for average current by equation (22) or (27) a good working approximation. This is illustrated quantitatively in Example 1.

EXAMPLE 1

Statement of Problem

The shunt-wound exciter, for which detailed data are given in Example 8, Chap. XV, is to operate separately excited at a pilot-exciter voltage of 500 volts and with a ceiling of 350 volts. It is controlled by a Tirrill regulator which short-circuits and reinserts the entire external field resistance. This resistance has such a value that 20 per cent of normal voltage would be obtained on the main exciter for steady operation with this resistance continuously in the circuit.

When the Tirrill regulator operates in such a manner that the contacts are closed for 0.04 sec. and open for 0.16 sec., calculate the average value of field current on the assumption of (*a*) zero inductance, (*b*) infinite inductance, and (*c*) infinite frequency of vibration, and compare the values so obtained

with (*d*) the correctly computed average field current, using in this calculation a constant inductance equal to the nominal field inductance of the machine.

Solution

With reference to the magnetization curve (Fig. 246, the field current at the lower operating point (20 per cent voltage) is

$$i_1 = 0.151 \times 6.6 = 1.00 \text{ amp.}$$

and at the ceiling (350 volts = 140 per cent)

$$i_2 = 2.19 \times 6.6 = 14.45 \text{ amp.}$$

The maximum total resistance in the field circuit [external and internal, symbol R_2 in equation (18)] is consequently

$$R_2 = \frac{E}{i_1} = \frac{500}{1.00} = 500 \text{ ohms}$$

and the minimum total resistance [symbol R in equation (18)]

$$R = \frac{E}{i_2} = \frac{500}{14.45} = 34.6 \text{ ohms}$$

NOTE: Since the resistance of the field winding is 23.3 ohms there is an external resistance of 500 − 23.3 = 476.7 ohms of which 34.6 − 23.3 = 11.3 ohms is permanently in the circuit.

a. Zero Inductance.—In this case equation (19) applies. The average conductance becomes:

$$G_{av} = \frac{\dfrac{t_c}{R} + \dfrac{t_0}{R_2}}{t_c + t_0} = \frac{\dfrac{0.04}{34.6} + \dfrac{0.16}{500}}{0.2}$$
$$= \frac{0.001155 + 0.00032}{0.2} = 0.007375 \text{ mho}$$

giving the average current as

$$i_{av} = EG_{av} = 500 \times 0.007375 = 3.69 \text{ amp.}$$

b. Infinite Inductance.—In this case equation (22) applies. The average resistance becomes:

$$R_{av} = \frac{Rt_c + R_2t_0}{t_c + t_0} = \frac{34.6 \times 0.04 + 500 \times 0.16}{0.2}$$
$$= \frac{1.384 + 80.0}{0.2} = 406.9 \text{ ohms}$$

resulting in an average current of

$$i_{av} = \frac{E}{R_{av}} = \frac{500}{406.9} = 1.23 \text{ amp.}$$

c. Infinite Frequency of Vibration.—Equation (27) applies. This is the same as for the previous case, resulting in the same value of average current.

d. Finite Inductance and Frequency of Vibration.—Equation (18) gives the correct average current. Using nominal inductance, 26.6 henrys, the exponential decrements become:

$$\alpha t_c = \frac{34.6 \times 0.04}{26.6} = 0.0520$$

$$\beta t_0 = \frac{500 \times 0.16}{26.6} = 3.00$$

$$\alpha t_c + \beta t_0 = 3.052$$

giving for the exponentials

$$\begin{aligned} \epsilon^{-0.052} &= 0.9493 \qquad & 1 - \epsilon^{-0.052} &= 0.0507 \\ \epsilon^{-3.00} &= 0.0498 \qquad & 1 - \epsilon^{-3.00} &= 0.9502 \\ \epsilon^{-3.052} &= 0.0473 \qquad & 1 - \epsilon^{-3.052} &= 0.9527 \end{aligned}$$

Substituting the proper values in equation (18), the average current is found to be

$$\begin{aligned} i_{av} &= 500\left[0.007375 - \frac{26.6}{0.2}\left(\frac{500 - 34.6}{500 \times 34.6}\right)^2 \frac{0.0507 \times 0.950}{0.953}\right] \\ &= 500\left[0.007375 - \frac{133 \times 0.0505}{1380}\right] \\ &= 500[0.007375 - 0.004875] = 500 \times 0.0025 = 1.25 \text{ amp.} \end{aligned}$$

Comments.—Comparison of the results indicates that neglecting the inductance gives a value of average current which is far too high, as might be expected. On the other hand, the result based on infinite inductance or infinite frequency of vibration is almost identical with that obtained from the complete formula. The difference is only 1.6 per cent and would be even smaller for higher values of average current (*i.e.*, with a higher ratio of "time closed" to "time open" of the regulator contacts).

Current Variation When the Ratio "Time Open" to "Time Closed" Is Suddenly Changed.—When the contacts are closed, the current builds up exponentially in accordance with equation (4), and, when they are open, it builds down in a similar manner as given by equation (7). During the build-up with the contacts closed, the current increases from an initial value i_1 to a value i_2 given by

$$i_2 = \frac{E}{R} - \left(\frac{E}{R} - i_1\right)\epsilon^{-\alpha t_c} \tag{28}$$

When this current is reached, the contacts open and the current then decreases to a value i_3 given by

$$i_3 = \frac{E}{R_2} - \left(\frac{E}{R_2} - i_2\right)\epsilon^{-\beta t_0} \tag{29}$$

When this current is reached, the contacts again close, and the current again builds up reaching, say, the value i_4. Under sustained conditions, *i.e.*, the contacts vibrating at a definite frequency and with a fixed ratio of t_c/t_0, $i_3 = i_1$ and $i_4 = i_2$, as previously discussed, and equations (13) and (14) take the place of equations (28) and (29). When, on the other hand, the ratio t_c/t_0 is suddenly changed (assuming, however, the frequency unaltered), $i_3 \neq i_1$ and $i_4 \neq i_2$, etc., until *new sustained values* are reached. In other words, the *entire current level changes* apart from the fluctuations due to the position of the contacts.

During the transient period caused by a change in time ratio, the difference between the sustained current i_{1s} and the current i_1 for a given (arbitrary) closing of the contacts is [see equation (15)]

$$i_{1s} - i_1 = E\frac{\frac{1}{R_2} + \left(\frac{1}{R} - \frac{1}{R_2}\right)\epsilon^{-\beta t_0} - \frac{1}{R}\epsilon^{-(\alpha t_c+\beta t_0)}}{1 - \epsilon^{-(\alpha t_c+\beta t_0)}} - i_1 \tag{30}$$

Substituting for i_2 in equation (29) by means of equation (28) gives

$$i_3 = \frac{E}{R_2} - \left(\frac{E}{R_2} - \frac{E}{R}\right)\epsilon^{-\beta t_0} - \left(\frac{E}{R} - i_1\right)\epsilon^{-(\alpha t_c+\beta t_0)} \tag{31}$$

This expresses, as seen, the current i_3 in terms of i_1, the value of current at the preceding closure of the contacts. Establishing as above the difference between i_{1s} and i_3, this becomes

$$i_{1s} - i_3 = \left[E\frac{\frac{1}{R_2} + \left(\frac{1}{R} - \frac{1}{R_2}\right)\epsilon^{-\beta t_0} - \frac{1}{R}\epsilon^{-(\alpha t_c+\beta t_0)}}{1 - \epsilon^{-(\alpha t_c+\beta t_0)}} - i_1\right]\epsilon^{-(\alpha t_c+\beta t_0)} \tag{32}$$

The ratio obtained by dividing equation (32) by equation (30) is

$$\frac{i_{1s} - i_3}{i_{1s} - i_1} = \epsilon^{-(\alpha t_c+\beta t_0)} \tag{33}$$

This result shows that the envelope of the minimum values of current is exponential in form, and equation (33) directly gives the decrement per cycle of vibration, *i.e.*, in a time equal to the period. Hence the current, for any instant at which the contacts close, is given by

$$i_1 = i_{1s} - (i_{1s} - i_{10})\epsilon^{-\frac{\alpha t_c + \beta t_0}{t_c + t_0}t} = i_{1s} - (i_{1s} - i_{10})\epsilon^{-\sigma t} \quad (34)$$

where the decrement factor σ may be written

$$\sigma = \frac{\alpha t_c + \beta t_0}{t_c + t_0} = \frac{Rt_c + R_2 t_0}{L(t_c + t_0)} = \frac{R_{av}}{L} \quad (35)$$

It is interesting to note that the decrement factor depends on the inductance in connection with the *average* resistance.

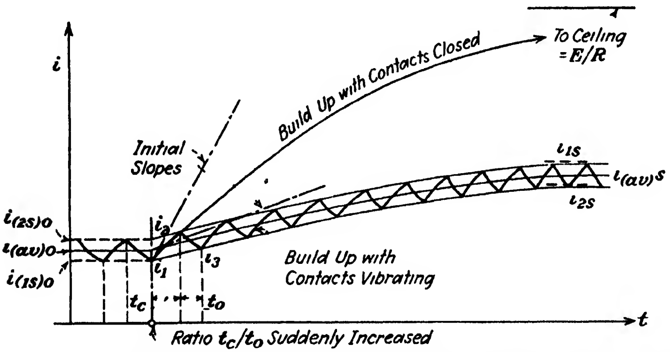


FIG. 270.—Build-up of current in an inductive circuit (Fig 267) controlled by a vibrating-contact regulator when the ratio of *time closed* to *time open* is suddenly increased

Although equation (34) applies to the envelope of minima for continuous values of time, the times t must obviously be integer multiples of the period to give *actual* currents.

It may now be immediately inferred, and this can, of course, readily be definitely shown by a mathematical process identical with the one just employed, that the envelope of maxima, *i.e.*, the current values at the opening of the contacts, as well as the average current, follows the same exponential law. Hence when the ratio of "time closed" to "time open" is suddenly changed the maximum currents and the average current change from their initial to their final sustained values as follows:

$$i_2 = i_{2s} - (i_{2s} - i_{20})\epsilon^{-\sigma t} \quad (36)$$

$$i_{av} = i_{(av)s} - (i_{(av)s} - i_{(av)0})\epsilon^{-\sigma t} \quad (37)$$

Figure 270 illustrates the build-up of current when the ratio of "time closed" to "time open" is suddenly increased.

It will be noted that the new steady-state conditions are slowly approached while the contacts go through a series of vibrations. In the figure is also shown the build-up curve when the regulator contacts remain closed, as the case would be with a high-speed regulator. It is obvious that with the contacts definitely closed the build-up process is much speedier and a given level of steady-state operation will be reached in a much shorter time than when the contacts continue vibrating.

EXAMPLE 2

Statement of Problem

The following data relate to a standard exciter (essentially the same as used in Example 8, Chap. XV).

Type: Self-excited or separately-excited shunt-wound machine.

Voltage (normal)	250 volts
Armature resistance	0.0113 ohm
Armature inductance	0.0063 henry
Field resistance (hot)	23.3 ohms
Nominal field inductance	26.6 henrys[1]
100 per cent field current	6.6 amp.

Data for Magnetization Curve.
Obtained with the Machine Self-excited

Percentage Voltage	Percentage Field Current
4	0
40	34
60	54
80	75
90	86
100	100
110	117.5
120	139
130	170
135	192
140	219
143	240

Calculate the nominal response rate of this exciter for the conditions of separate and self-excitation specified below. The nominal slip-ring voltage may be taken as 80 per cent of normal.

[1] The nominal field inductance may be defined as the number of flux linkages per field ampere at rated armature voltage with the armature winding open-circuited.

It is further desired to examine and compare the build-up rates which will be obtained with this exciter when controlled by a vibrating-contact regulator of the standard Tirrill type and by a vibrating-contact regulator (or its equivalent) of the high-speed type. For this purpose calculate the initial build-up and build-down rates expressed in volts per second, and the time of build-up or build-down over 95 per cent of the range involved, assuming initial operation at slip-ring voltages of 60, 80, and 100 per cent, respectively. It is assumed that the build-up is caused by a sudden load increment on the alternator requiring a final slip-ring voltage of 120 per cent in order to restore normal alternating-current voltage, and that build-down is caused by a sudden decrease in alternator load requiring build-down to 40 per cent slip-ring voltage in order to restore normal alternating-current voltage.

Assume that the Tirrill regulator, when called upon to build up or build down, suddenly changes its ratio of time closed to time open to that corresponding to the final conditions. Also assume that the high-speed regulator, when substantially 95 per cent of the build-up or build-down has taken place with definitely closed or open contacts, resumes its vibrations with the appropriate closed- to open-time ratio for the final voltage conditions. The frequency of vibration of the regulator contacts may be taken as 5 cycles per second.

Values should be obtained with the exciter self-excited as well as separately excited by a pilot exciter of suitable size. Assume the terminal voltage of the pilot exciter to be strictly constant and equal to 355 and 500 volts, respectively. Using two values of this voltage will give an idea of the effect which the pilot-exciter voltage exerts on the build-up and build-down rates of the main exciter. In each case let the external resistance in the exciter field circuit have such a value that 20 per cent of normal voltage would be obtained on the main exciter for steady operation with this resistance continuously in the circuit. When self-excitation is used, the entire external resistance is short-circuited by the action of the regulator. The ceiling voltage is to be considered the same in all cases and equal to that obtained under self-excited conditions with only the resistance of the field winding proper in the circuit.

Calculations are to be performed assuming constant inductance in the exciter field and also taking into account the variation in this inductance due to saturation.

Solution

The solution of this example, which can be handled by methods outlined in this and preceding chapters, is lengthy and involves a number of repetitions in order to cover the desired range of variables. For this reason the details of the solution are not reproduced here. The results are collected in Table 69.

A comparison of the tabulated figures will throw light on the relative effect of separate excitation and self-excitation and will permit a comparison of nominal and initial response rates as well as lengths of time required for build-up and build-down between given limits. They also illustrate the difference in action of a Tirrill regulator and a regulator of the high-speed type.

TABLE 69 — RESULTS FROM EXCITER RESPONSE CALCULATIONS
(Example 2)

	Self-excited		Separately Excited Pilot voltage E = 355 volts		Separately Excited Pilot voltage E = 500 volts	
	Constant inductance (= nominal)	Variable inductance	Constant inductance (= nominal)	Variable inductance	Constant inductance (= nominal)	Variable inductance
Nominal response volts per sec	100	96	205	205	273	267
High-speed regulator						
1 Initial response rates volts per sec						
Build-up from (*a*) $e = 60\%$	96	86	388	346	576	515
(*b*) $e = 80\%$	117.5	98.5	340	286	545	457
(*c*) $e = 100\%$	132	89	281	190	487	329
Build-down from (*d*) $e = 60\%$	−57	−51	−1420	−1270	−2000	−1790
(*e*) $e = 80\%$	−92.5	−77.5	−2175	−1830	−3060	−2570
(*f*) $e = 100\%$	−153	−103	−3110	−2100	−4390	−2960
2 Times of build-up and build-down[1] sec						
Build-up (*a*) 60 − 120%	1.68	1.71	0.690	0.70	0.490	0.50
(*b*) 80 − 120%	1.21	1.19	0.560	0.55	0.395	0.395
(*c*) 100 − 120%	0.73	0.69	0.375	0.355	0.270	0.255
Build-down (*d*) 60 − 40%	1.15	1.29	0.045	0.051	0.032	0.036
(*e*) 80 − 40%	1.75	1.935	0.071	0.079	0.0505	0.056
(*f*) 100 − 40%	2.14	2.23	0.090	0.0985	0.0640	0.070
Tirrill regulator						
1 Initial response rates, volts per sec						
Build-up from (*a*) $e = 60\%$	50	45	310	276	435	388
(*b*) $e = 80\%$	53.5	45	233	196	328	276
(*c*) $e = 100\%$	46.3	31.3	141	95	199	134
Build-down from (*d*) $e = 60\%$	−14.2	−12.7	−298	−266	−420	−374
(*e*) $e = 80\%$	−28.5	−24.0	−611	−514	−862	−724
(*f*) $e = 100\%$	−64.1	−43.3	−985	−665	−1390	−940
2 Times of build-up and build-down[1] sec						
Build-up (*a*) 60 − 120%	6.4	6.4	2.06	1.96	1.46	1.40
(*b*) 80 − 120%	6.2	5.5	2.06	1.94	1.46	1.38
(*c*) 100 − 120%	5.4	5.3	2.06	1.90	1.46	1.34
Build-down (*d*) 60 − 40%	9.0	11.6	0.504	0.58	0.357	0.41
(*e*) 80 − 40%	9.7	11.9	0.504	0.58	0.357	0.41
(*f*) 100 − 40%	9.9	11.5	0.504	0.55	0.357	0.39

[1] Based on 95 per cent of current range rather than 95 per cent of voltage range. On the latter basis, the time figures would be slightly lower in the ranges affected by the nonlinearity of the magnetisation curve.

Voltage Fluctuations When Alternator Load Is Suddenly Dropped.—Consider a loaded three-phase alternator operating in the steady state at normal terminal voltages (it may, for instance, be feeding into an external system). The field current of the alternator is supplied by a regulator-controlled exciter either separately or self-excited as the case may be. Assume next that the load is suddenly dropped by opening the circuit breakers and that the time element involved in reducing the armature current of the alternator to zero is negligible. The alternator is now on open circuit, and the general methods of determining its voltage variation as a result of exciter action are therefore as described on pages 556 *et seq.*, Chap. XVI. However, before one of these can be applied, it is necessary to determine the conditions in the system (exciter-alternator) immediately after the dropping of the load.

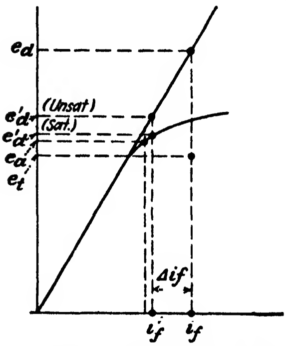


Fig. 271.—Open-circuit saturation curve and air-gap line for alternator illustrating the change in field current which occurs when the load is suddenly dropped.

Since there can be no sudden change in field flux linkages, the disappearance of the armature current must be compensated by a corresponding decrease in field current. A further consequence of the initial constancy of flux linkages is that the voltage corresponding to these, *i.e.*, e'_d, remains the same directly after as just before the dropping of the load. Since the machine, upon disconnecting the load, is open-circuited, the terminal voltage will immediately rise to this value. This fact may be made use of in determining the value to which the field current suddenly drops. The procedure, therefore, would be to compute e'_d and to obtain the new field current from the open-circuit magnetization curve corresponding to the value of e'_d. This is illustrated in Fig. 271. It may be mentioned in this connection that the voltage behind direct-axis transient reactance e_i, which is more readily computed than e'_d and which for ordinary load conditions is almost identical in magnitude, may in most instances be used with sufficient approximation instead of the latter.

The decrease in alternator field current will upset the previously existing voltage balance in the alternator field circuit,

since the exciter voltage is so far unchanged, and there will be a tendency to increase the flux of the generator, resulting in a still further increase in terminal voltage. How quickly exciter action will get into the picture depends primarily upon the voltage regulator. In this connection there are two factors to be considered: (1) the time lag of the regulator contacts (including the relays, etc.) and (2) the width of the region of insensitivity. Since the latter is, say, of the order of 1 or 2 per cent of the alternator voltage, and this voltage would change by a consider-

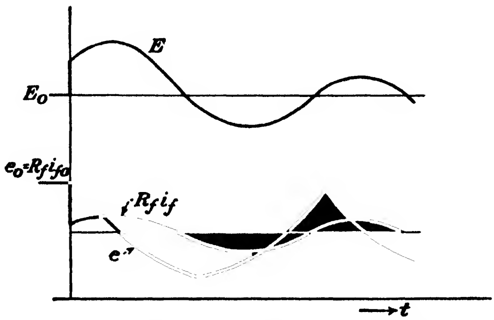


FIG. 272.—Variation in alternator voltage, exciter voltage and alternator field current when the load on the alternator is suddenly dropped. High-speed regulation is assumed.

ably larger amount when an appreciable load is suddenly disconnected, this factor may initially be disregarded. Hence the initiation of exciter action depends upon the time lag only, which usually should not exceed, say, 0.05 sec. Assuming that the regulator is a high-speed regulator, there will be an immediate tendency as soon as it acts, for a relatively rapid decrease in exciter voltage. In spite of this, the field current, flux, and voltage of the generator will continue to rise until the exciter voltage equals the resistance drop in the alternator field circuit. A further reduction in exciter voltage will decrease the alternator field current, flux, and voltage.

The variations in alternator voltage, exciter voltage, and alternator field current (resistance drop) are illustrated in Fig. 272. Although it would be desirable to have exciter build-up begin before the alternator voltage is fully restored to normal, the latter must actually drop to the lower limit of the region of insensitivity and in addition the time lag factor must be added

before the high-speed regulator reverses the exciter action. It is obvious from these considerations that the alternator voltage and the other quantities considered will pass through a series of fluctuations with decreasing amplitudes until new steady-state conditions are attained, and an appreciable number of seconds may actually elapse before such conditions are reached.

In the above discussion any tendency of the regulator to depart from its operation as a distinctly high-speed regulator has not been considered. If the regulator is of the vibrating-contact type, it is possible that as the alternator voltage approaches normal the regulator may have a tendency to resume its vibrations and hence to delay somewhat the process of bringing about new steady-state conditions.

Treatment of the same problem from the quantitative viewpoint will be found in Example 3. In this example the formal relationships are reviewed and established. Furthermore, for a specific case all pertinent initial values are computed, both before and immediately after the load is disconnected.

The complete solution of the problem in Example 3[1] was obtained on the M.I.T. differential analyzer, injecting the proper voltage-current relationships by following the magnetization curves for the alternator and the exciter. In this manner the desired curves of alternator and exciter voltage were obtained directly on the output tables. The results are given in Fig. 276. The curves represent alternator voltage (in volts to neutral) and exciter induced voltage (in volts) obtained with and without considering the effect of exciter armature resistance. Control by a high-speed regulator was assumed, having no time lag but a relatively wide period of insensitivity, *viz.*, 2 per cent on each side of normal alternator voltage. These curves serve to substantiate the statements made in the above qualitative discussion. They indicate that a relatively large number of seconds will elapse before new steady-state conditions are obtained. As regards the effect of exciter armature resistance, it is interesting

[1] In this solution a nominal inductance of 0.38 henry was used for the alternator field (instead of the value given in Example 3). The former value is too small for large turboalternators of normal design. The period of the oscillations in Fig. 276, therefore, is somewhat shorter than it would have been had a more representative inductance value been used.

to note that its influence on amplitudes is practically negligible. It does, on the other hand, somewhat change the time relationships. It will be noted that the presence of resistance actually has a beneficial effect in that it reduces the time required to reach steady-state conditions.

EXAMPLE 3

Statement of Problem

A three-phase turboalternator supplies rated kilovolt-amperes to a load at 90 per cent power factor (lagging). It receives its field current from a self-excited exciter of suitable rating, as indicated in Fig. 273. The terminal voltage of the alternator is controlled by a vibrating-contact regulator of

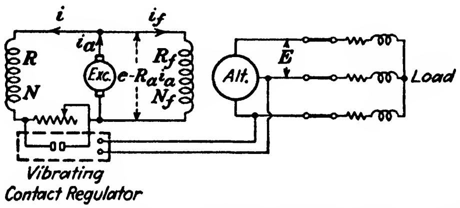


FIG. 273.—Self-excited exciter supplying the field current of a large alternator. The excitation system is controlled by a high-speed vibrating-contact regulator (Example 3).

the high-speed type acting upon a resistance in the shunt-field circuit of the exciter.

a. Establish the necessary equations and outline in sufficient detail the methods to be used in a point-by-point solution for determining the fluctuations in the alternator voltage when the total three-phase load is suddenly disconnected.

b. Determine numerically all pertinent initial conditions immediately before and immediately after the circuit breaker is opened (in the latter case also include values of the rate of change of the alternator and the exciter voltage).

In the solution take the nonlinear characteristics of the alternator and the exciter into account. Assume that the armature inductance and the armature reaction of the exciter are so small that they definitely need not be taken into account. On the other hand, include the effect of armature resistance.

In regard to the operation of the regulator, it may be assumed that its vibratory action ceases when the alternator voltage deviates from normal by 2 per cent. When this voltage is more than 2 per cent above normal, the contacts short-circuiting the exciter shunt-field resistance are definitely open, and when it is more than 2 per cent below normal, they are definitely closed.

Generator Data

Type	Turbo-alternator
Rating	37,500 kva
Line voltage (normal)	14,000 volts
Transient reactance	1 25 times the leakage (Potier) reactance
Field resistance at 75°C	0 5 ohm
Nominal field inductance	3 22 henrys[1]
Lowest possible value of field current with regulator in the circuit and open contacts	50 per cent of normal no-load value

Characteristic curves are given in Fig 274

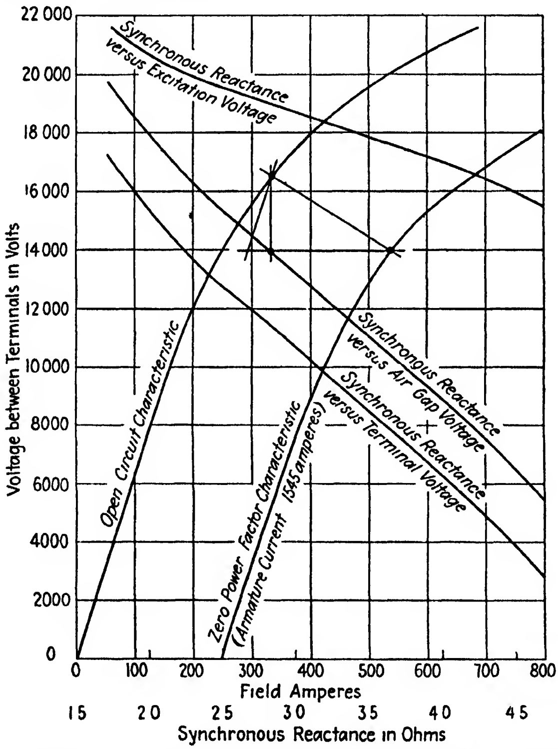


FIG 274 —Characteristic curves for the turbogenerator in Fig 273 (Example 3)

[1] The nominal field inductance may be defined as the number of flux linkages per field ampere at rated armature voltage with the armature winding open-circuited

Exciter Data (same as in Example 8, Chap. XV):

Type....................	self-excited, shunt-wound machine
Voltage (normal)........	250 volts
Armature resistance......	0.0113 ohm
Armature inductance.....	0.0063 henry
Field resistance (hot).....	23.3 ohms
Nominal field inductance.	26.6 henrys[1]
100 per cent field current	6.6 amp.

DATA FOR MAGNETIZATION CURVE.
OBTAINED WITH THE MACHINE SELF-EXCITED

Percentage Voltage	Percentage Field Current
4	0
40	34
60	54
80	75
90	86
100	100
110	117.5
120	139
130	170
135	192
140	219
143	240

Solution

a. Establishment of Equations

The differential equations determining the fluctuations are [see equation (51), Chap. XVI]:

Exciter:

$$\frac{\sigma N}{k}\frac{de}{dt} + iR + i_a R_a = e$$

or, since $\quad i_a = i + i_f \quad$ and $\quad \dfrac{N}{k} = T,$

$$\sigma T\frac{de}{dt} + i(R + R_a) + i_f R_a = e \tag{a}$$

Alternator:

$$\frac{\sigma_f N_f}{k_g}\frac{dE}{dt} + R_f i_f + i_a R_a = e$$

or, since $\quad i_a = i + i_f \quad$ and $\quad N_f/k_g = T_g,$

$$\sigma_f T_g \frac{dE}{dt} + i_f(R_f + R_a) + iR_a = e \tag{b}$$

[1] The nominal field inductance may be defined as the number of flux linkages per field ampere at rated armature voltage with the armature winding open-circuited.

For purposes of a point-by-point solution, these equations may be written

$$\Delta e = \frac{\Delta t}{\sigma T}[e - (R + R_a)i - R_a i_f] \tag{c}$$

and

$$\Delta E = \frac{\Delta t}{\sigma_f T_g}[e - (R_f + R_a)i_f - R_a i] \tag{d}$$

The circuit breaker suddenly opens at $t = 0$. Values of e, i, and i_f at this instant (e_0, i_0, and i_{f0}, respectively) can be computed under certain assumptions from the initial conditions, as illustrated in (*b*). Substitution of these values in (*c*) and (*d*) then gives the increments Δe and ΔE for the first interval Δt. At the end of this interval,

$$e_1 = e_0 + (\Delta e)_1$$

and

$$E_1 = E_0 + (\Delta E)_1$$

The currents i_1 and i_{f1} can then be found from the exciter and alternator magnetization curves corresponding to e_1 and E_1. The second and other intervals can then be handled similarly. When E is larger than 102 per cent, the value of R to be used is that corresponding to regulator contacts open. When E is lower than 98 per cent, R should be that value with contacts closed. For values between 98 and 102 per cent, in the absence of specific information as to behavior in this region, it will be assumed that the contacts remain either closed or open, depending upon whether the region is entered at the lower or upper boundary.

Summarizing, for the nth interval,

$$(\Delta e)_n = \frac{\Delta t}{\sigma T}[e_{(n-1)} - (R + R_a)i_{(n-1)} - R_a i_{f(n-1)}] \tag{e}$$

$$(\Delta E)_n = \frac{\Delta t}{\sigma_f T_g}[e_{(n-1)} - (R_f + R_a)i_{f(n-1)} - R_a i_{(n-1)}] \tag{f}$$

$$e_n = e_{(n-1)} + (\Delta e)_n \tag{g}$$

$$E_n = E_{(n-1)} + (\Delta E)_n \tag{h}$$

e_n and i_n are related by exciter magnetization curve.

E_n and i_{fn} are related by alternator magnetization curve.

Proper choice of R and inclusion of regulator action are as noted in the preceding paragraph.

These computations can readily be carried out in tabular form. Equations (*e*) and (*f*) with numerical coefficients are given at the end of Part (*b*).

b. Numerical Calculations

Transient Reactance.—Rated alternator line current is

$$I = \frac{37{,}500}{\sqrt{3} \times 14} = 1{,}545 \text{ amp.}$$

From Fig. 274,

$$\text{Potier reactance} = \frac{2{,}500}{\sqrt{3} \times 1{,}545} = 0.934 \text{ ohm per phase.}$$

Therefore

$$X'_d = 1.25 \times 0.934 = 1.169 \text{ ohms per phase}$$

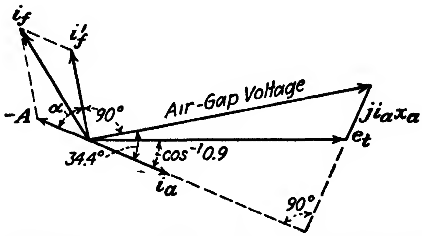


FIG. 275.—Vector diagram for turbogenerator in Fig. 273 under full-load conditions (Example 3).

Full-load Alternator Field Current.—Referring to the vector diagram, Fig. 275,

Air-gap voltage =

$$\sqrt{3}\sqrt{\left(\frac{14{,}000}{\sqrt{3}} + 1{,}545 \times 0.934 \times 0.436\right)^2 + (1{,}545 \times 0.934 \times 0.9)^2}$$
$$= 15{,}270 \text{ volts}$$

This voltage is at an angle of 34.4 deg. with i_a. Entering the magnetization curve with this voltage,

$$i_f = 289 \text{ field amp.}$$

From the Potier triangle, the equivalent armature reaction

$$A = 205 \text{ amp.}$$

Also

$$\alpha = 180° - 34.4° - 90° = 55.6°$$

The full-load alternator field current is then

$$i_f = \sqrt{(205 \sin 55.6°)^2 + (289 + 205 \cos 55.6°)^2}$$
$$= 440 \text{ amp.}$$

This corresponds to an excitation voltage (as read from the magnetization curve) of 18,630 volts in the alternator and a terminal voltage of 440 × 0.5 = 220 volts for the exciter.

Initial Value of e.—Since the exciter terminal voltage is 220 volts, we have, neglecting iR_a,

$$e = 220 + 440 \times 0.0113 = 225 \text{ volts}$$

Initial Value of i.—From magnetization curve for exciter, $i = 5.67$ amp

Field Resistance of Exciter:

Contacts closed, $R = 23.3$ ohms
Contacts open, $i_f = 0.50 \times 249 = 124.5$ amp.
$e = 0.5 \times 124.5 + 125.8 \times 0.0113 = 63.7$ volts

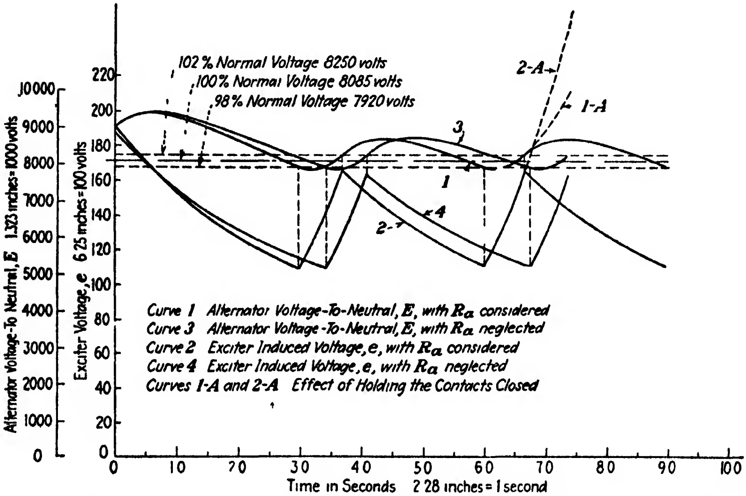


FIG. 276.—Variation in alternator voltage and exciter voltage of the system in Fig. 273 when the load is suddenly dropped. The curves were obtained on the M.I.T. differential analyzer. (For circuit constants, operating conditions, and other details, see Example 3.)

From magnetization curve for exciter,

$$i = 1.34 \text{ amp.}$$

Hence

$$R = \frac{0.5 \times 124.5}{1.34} = 46.5 \text{ ohms}$$

Value of E Immediately after Breaker Opening.—The assumption of constant voltage behind transient reactance will be made. Before breaker opening,

$$e_t = \sqrt{3}\sqrt{\left(\frac{14,000}{3} + 1,545 \times 1.169 \times 0.436\right)^2 + (1,545 \times 1.169 \times 0.9)^2}$$
$$= 15,600 \text{ volts}$$

Hence, the terminal voltage suddenly rises to $E = 15,600$ volts.

NOTE: If desired, e_d' can be found and assumed constant. This should lead to better results. In view of the other approximations being made,

however, the additional precision hardly seems warranted. Knowing the position of the i_f vector from the vector diagram, Fig. 275, the direct and quadrature axes can be located and e'_d found to be 14,600 volts.

Value of i_f Immediately after Breaker Opening.—From magnetization curve corresponding to an excitation voltage of 15,600 volts, i_f' suddenly drops to

$$i_f = 301 \text{ amp.}$$

Values of $\sigma N/k$ and $\sigma_f N_f/k_g$.

Exciter:

$$\frac{\sigma N}{k} = \sigma \frac{N\phi}{e} = \frac{N(\sigma\phi_0)}{R_0 i_0} = \frac{L_0}{R_0} = \frac{26.6}{27.9} = 0.702 \text{ sec.} = \sigma T$$

($R_0 = 250/6.6 = 37.9$ ohms = normal field resistance.)

Alternator:

$$\frac{\sigma_f N_f}{k_g} = \sigma_f \frac{N_f \phi_g}{E} = \sigma_f \frac{N_f \phi_g}{i_f}\frac{i_f}{E} = \frac{N_f(\sigma_f \phi_g)}{i_f}\frac{i_f}{E}$$

But $[N_f(\sigma_f\phi_g)]/i_f =$ nominal field inductance if i_f is field current for a terminal voltage of 100 per cent on open circuit. From alternator magnetization curve,

$$i_f = 249 \text{ amp.}$$

Hence, when used with line voltage

$$\frac{\sigma_f N_f}{k_g} = \frac{3.22 \times 249}{14{,}000} = 0.0572 \text{ sec.} = \sigma_f T_g$$

Substitution of the above numerical values in equations (*e*) and (*f*) gives

$$(\Delta e)_n = \frac{\Delta t}{0.702}\left[e_{(n-1)} - \begin{Bmatrix} 46.5 \\ \text{or} \\ 23.3 \end{Bmatrix} i_{(n-1)} - 0.0113 i_{f(n-1)}\right] \text{volts} \qquad (i)$$

and

$$(\Delta E)_n = \frac{\Delta t}{0.0572}\left[e_{(n-1)} - 0.0113 i_{(n-1)} - 0.511 i_{f(n-1)}\right] \text{volts} \qquad (j)$$

the 46.5 or 23.3 in equation (*i*) being used for $E > 102$ per cent and $E > 98$ per cent, respectively. For 102 per cent $> E > 98$ per cent, use 46.5 if alternator terminal voltage decreases and 23.3 if it increases in accordance with the previously stated assumption. Initial values for equations (*i*) and (*j*) are

$$e_0 = 225 \text{ volts} \qquad i_{f0} = 301 \text{ amp.}$$
$$i_0 = 5.67 \text{ amp.} \qquad E_0 = 15{,}600 \text{ volts}$$

Exciter Response under Conditions of Sudden Changes in Alternator Load.—When there is a sudden change in alternator load owing, for instance, to the switching on of a load increment or the occurrence of a fault, there will, in general, be a sudden change in the field current. This is a consequence of the constant flux-linkage theorem exactly as previously discussed in connection with the specific problem of suddenly dropping *all* load from the alternator. Whether the field current increases or decreases depends upon the direction of the load change and especially upon the change occurring in the reactive power. Thus, if a short circuit occurs, compelling the machine to supply a large amount of lagging reactive power, the field current will suddenly increase. Conversely, when a fault is disconnected there is usually a sudden decrease in field current due to the machine being relieved of the heavy lagging reactive-power requirement.

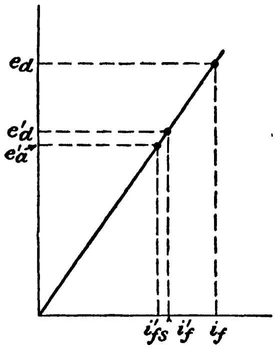


FIG. 277.—Idealized straight-line magnetization curve (or air-gap line) for an alternator, illustrating the change in field current which occurs when the armature current suddenly changes.

The sudden change in field current is primarily a result of the sudden change in the direct-axis component of armature current and is affected relatively little by any simultaneous change in quadrature-axis component. On the assumption of constant flux linkages in the direct axis (constant e_d'), the change in direct-axis current becomes the only influence and only this factor will be considered below. Furthermore, it is customary in most practical cases to omit any direct consideration of saturation (as was done in the previously discussed case involving complete loss of load and subsequent operation on open circuit), except as its effect may be reflected indirectly (and usually only approximately) in the values of reactance used.

Under *steady-state* conditions the relationship is that of a demagnetizing armature reaction. Hence if the armature-reaction constant is known, the increment in field current corresponding to a given increment in direct-axis armature current can be computed. Under transient conditions (when the armature current *suddenly* changes), a slightly modified

proportionality factor must be used, based upon the consideration of initially constant flux linkages. From the vector diagram, Fig. 161, Chap. XII, and the idealized straight-line magnetization curve, Fig. 277 (or the air-gap line in cases where saturation is present), and letting e'_a represent the projection of the air-gap voltage on the quadrature axis the following equations may be written:

$$\left.\begin{aligned} e_d &= e_t \cos \delta + i_d X_d \\ e'_d &= e_t \cos \delta + i_d X'_d \\ e'_a &= e_t \cos \delta + i_d X_a \end{aligned}\right\} \tag{38}$$

$$\frac{i_f}{e_d} = \frac{i'_f}{e'_d} = \frac{i'_{fs}}{e'_a} \tag{39}$$

The additional field current required on account of armature reaction in the steady state then becomes:

$$\Delta i_{fs} = i_f - i'_{fs} = \left(\frac{i_f}{e_d}\right)(e_d - e'_a) = \frac{i_f}{e_d}(X_d - X_a)i_d = k_d i_d \tag{40}$$

hence the proportionality factor under steady-state conditions, *i.e.*, the coefficient of demagnetizing armature reaction, is given by

$$k_d = \frac{X_d - X_a}{e_d/i_f} = \frac{X_d - X_a}{\text{slope of air-gap line}} = (X_d - X_a) \text{ in per-unit system} \tag{41}$$

Similarly, under transient conditions, *i.e.*, when the armature current suddenly changes, the corresponding sudden field-current increment becomes

$$\Delta i_f = i_f - i'_f = \frac{i_f}{e_d}(X_d - X'_d)e_d = k'_d i_d \tag{42}$$

The proportionality factor under transient conditions therefore is:

$$k'_d = \frac{X_d - X'_d}{\text{slope of air-gap line}} = (X_d - X'_d) \text{ in per-unit system} \tag{43}$$

It is thus seen that when the per-unit system of notation is used the field-current increment equals the increment in direct-axis

armature current times the difference between the direct-axis synchronous and transient reactances and is thus readily computable.

The regulator will respond in much the same manner as already discussed for the case where the entire load is suddenly dropped in an effort to restore the terminal voltage to normal. When short circuits are involved, the chances are that, even with high-speed regulators and a responsive excitation system, the voltage will remain below normal as long as the fault remains on. Normal voltage will as a rule first be recovered after clearing, upon which the voltage will undergo oscillatory adjustments until ultimately steady-state conditions are restored. The solution of these problems all require point-by-point methods. Usually the effect of exciter armature constants would be ignored so that the exciter response, once the regulator acts, can be determined independently of the alternator. The details of the solution of these problems may be handled in a number of ways, as previously pointed out. Certain specific methods which have been found practical in transient-stability analyses are discussed below.

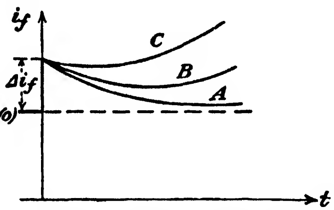


FIG. 278.—Curves illustrating the variation in alternator field current after the sudden occurrence of a fault. *A*, No exciter action. *B*, Exciter action with moderate speed of response. *C*, Exciter action with high speed of response.

Practical Computation of Exciter Response during General Point-by-point Analysis of Transient Stability.—The discussion which is to follow is with particular reference to a system fault which subsequently is cleared by the action of automatic circuit breakers. In other words, it covers exciter response in its relations to the principal transient-stability problem.

When the fault occurs, there is a sudden increment in generator field current, as indicated in Fig. 278. Were there no exciter action, this field current would gradually decrease and ultimately reach its original value (curve *A*, Fig. 278). With exciter action, on the other hand, it will vary in a different manner, depending upon the speed of response of the regulator and the excitation system. With a moderate speed of response, curve *B* might be representative. As will be noted, there is first a drop

in field current and thereafter a rise. Curve C illustrates the effect of an extremely high response rate resulting in an almost immediate field-current rise. These curves are assumed to cover a time interval—starting at the occurrence of the fault—during which the regulator contacts (on account of the large drop in alternator voltage and the presence of a high-speed element giving positive action under these conditions) definitely stay closed permitting full utilization of the excitation system.

With a high-speed regulator the voltage of the exciter may vary with time, as indicated in Fig. 279. Curve A represents a case where during the build-up process the ceiling is not reached, neither is the lowest possible voltage reached during the build-

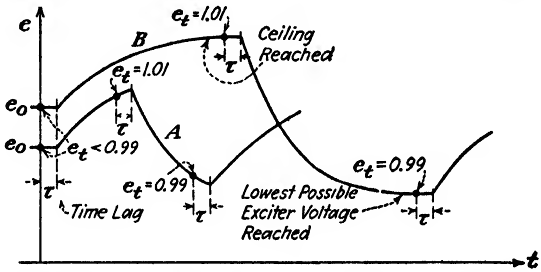


FIG. 279.—Exciter voltage variation with high-speed regulation after the occurrence of a fault and assuming long-delayed clearing. It is assumed that despite the fault normal alternator voltage will be recovered as a result of the increase in field current. Curve A, ceiling and lowest possible exciter voltage not reached during build-up and build-down process. Curve B, ceiling and lowest possible exciter voltage reached during build-up and build-down process.

down process. Owing to the fault, the alternator terminal voltage is much below normal. As will be noted, a time lag has been assumed in the regulator. Hence after the occurrence of the fault, the exciter voltage remains constant at its initial value for a time equal to this time lag. Thereafter it builds up until the alternator voltage has risen to slightly above normal (this being necessary on account of the region of insensitivity), at which point the regulator is actuated so that after the lapse of the aforementioned time lag, build-down of the exciter voltage takes place. The exciter voltage now decreases until the alternator voltage is sufficiently below normal for the regulator once more to be actuated, whereupon, again allowing for the time lag, the exciter voltage once more begins to increase.

The same comments may be applied to curve B, the only difference here being that a constant ceiling voltage (and a

constant minimum voltage) has been assumed to be reached before the regulator is reactuated to initiate the build-down (or build-up) process.

In Fig. 280 are shown reasonably representative and coordinated curves of alternator terminal voltage (positive-sequence) and field current, and also exciter voltage under conditions of a suddenly applied fault and its subsequent (complete) clearing. Initially there is a sudden large drop in terminal voltage and a sudden rise in field current. The exciter voltage remains constant at its initial value for a short time (the regulator time lag) before starting to build up. Both the field current and the terminal

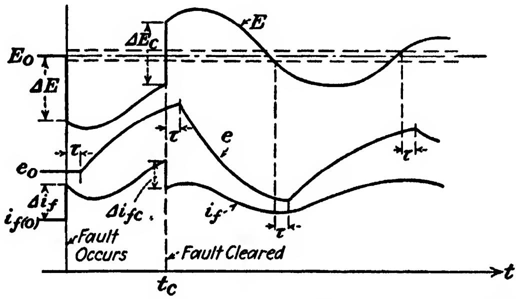


FIG. 280.—Time variation of alternator terminal voltage (positive sequence), field current, and exciter voltage under conditions of a suddenly applied fault and its subsequent (complete) clearing. The excitation system is controlled by a high-speed regulator. It is assumed that the alternator voltage remains subnormal while the fault is on and recovers to well above its normal value upon clearing. This of course may not always be the case.

voltage decrease until a certain amount of exciter-voltage build-up has taken place although usually not in step (as under no-load conditions) due to the reactance drop in the machine which changes with the armature current.

When the fault is cleared at time t_c, it is assumed that the terminal voltage suddenly rises to a value appreciably above normal as a result of the large sudden reduction in lagging reactive armature current. The high-speed element of the regulator, therefore, is immediately actuated forcing the exciter—after expiration of the time lag—to build down its voltage. For reasons already explained, the alternator field current and voltage may still continue to rise somewhat (not necessarily in step) before beginning their downward climb. No further description of the regulation processes and the curves seems

necessary After repeated adjustments, the new steady-state conditions are ultimately reached.

If one assumes that the exciter armature constants are to be ignored and that the use of a high-speed regulator giving the exciter either full build-up or full build-down opportunity, the exciter voltage-time curves may be computed and constructed prior to starting the point-by-point analysis in which it is desired to include the effect of exciter action. In other words, these curves may be determined from exciter characteristics alone. They may be computed for the full range of exciter-voltage variation, *i.e.*, between lowest possible voltage and ceiling, for conditions of build-up and build-down, respectively. Actually during the point-by-point analysis, these extremes may never be reached. In such cases, only the part of the curves which actually apply will, of course, be used.

Two approximate methods of bringing the exciter action into the point-by-point analysis will be covered, *viz.:*

1. Using actual exciter-voltage build-up and build-down characteristics and a single fixed equivalent time constant for the alternator.

2. Assuming constant linear rate of build-up and build-down of exciter voltage (often taken equal to the nominal response rate) and using a single fixed equivalent time constant for the alternator.

The conception of an equivalent alternator time constant and its determination has been previously covered in Chap. XVI. It should be remembered in this connection that the equivalent time constant changes when the fault is cleared owing to the change in external reactance. This change should be allowed for at the proper instant in the point-by-point analysis.

It was also mentioned in Chap. XVI that the equivalent time constant may be approximately corrected for saturation as the analysis progresses. This, however, is practically never done, although it may be desirable to select a time constant which on the average is essentially correct for the range of saturation involved.

In both methods, adjustments are made for each step in the point-by-point calculation in the particular internal machine voltage which is used in the power-output computations. In these computations (see Chap. XIII) one of the voltages in the

quadrature axis, e_d, e_D, e'_d, or the voltage behind direct-axis transient reactance, e_i, are generally used. Exciter action (*i.e.*, the change in flux) will give rise to equal changes in e_d, e_D, e'_d, and hence the increment computed may be applied to any one of them. The change in e_i, the voltage behind transient reactance, will not be rigorously the same. However, this is sometimes assumed and the same increment (*i.e.*, the one which correctly applies to the voltages in the quadrature axis) applied

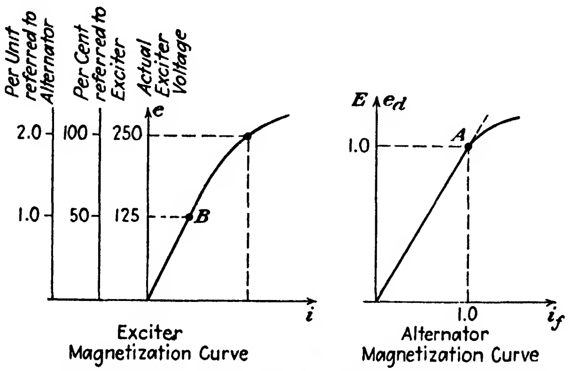


FIG. 281.—Alternator and exciter magnetization curve illustrating how a modified per-unit system of notation may be adopted for the exciter. This refers the exciter quantities to the generator basis for convenience in computation.

to this voltage as well. The following scalar equation interrelate these voltages [see equations (11) and (16), Chap. XIII]:

$$\begin{aligned} e_d &= e_D + i_d(X_d - X_q) = e'_d + i_d(X_d - X'_d) \\ &= e_i \cos(\delta - \delta_i) + i_d(X_d - X'_d) \end{aligned} \tag{44}$$

Assume that in the point-by-point solution, per-unit values of generator quantities are used. When exciter action is to be included, it is convenient to use a modified per-unit system of notation also for exciter quantities. More specifically, these are referred to the generator basis by designating as the unit value of exciter voltage the voltage which will produce unit generator field current, the latter in turn giving unit value of open-circuit voltage. It is obvious that the unit value of exciter voltage so defined will not usually be the rated voltage of the exciter but, as a rule, considerably lower. Figure 281 illustrates these relationships. Point A on the alternator magnetization curve corresponds to normal open-circuit voltage and field current

(the 100 per cent or unit values). Assume that the exciter is rated 250 volts but that only 125 volts are required to produce normal field current in the alternator field winding. The operating point B on the exciter magnetization curve therefore corresponds to point A on the alternator magnetization curve. Referred to the exciter, 250 volts would be the 100 per cent or unit value, and consequently 125 volts (operating point B), would be 50 per cent or 0.5. However, referred to the alternator, 125 volts represents the unit value and the rated voltage of the exciter (*i.e.*, 250 volts) a value of 2.0.

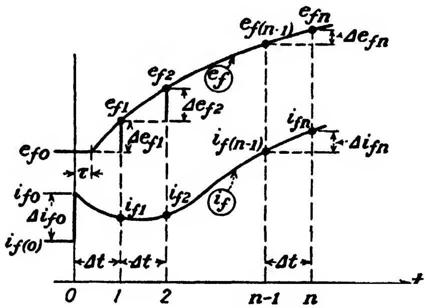


FIG. 282.—Variation in exciter voltage and alternator field current after the sudden occurrence of a short circuit (or other discontinuity). The exciter-voltage build-up curve is considered as a step function in the point-by-point analysis. (Above, the steps are introduced at the end of each interval although, as stated in the text, they may also be introduced at the beginning or at the mid-point of the interval.)

Method I. Using Actual Exciter-voltage Build-up and Build-down Characteristics.—When the precalculated exciter build-up (or build-down) curve is available, it may be utilized in the point-by-point analysis in two ways. It may either be represented as a step function and each step assumed introduced at the beginning, middle, or end of a certain time interval, or the variation during a time interval may be considered linear.

a. Introducing Exciter Build-up Curve as a Step Function.—The exciter build-up curve and the curve of alternator field current are shown in Fig. 282. It is assumed that a short circuit (or other similar discontinuity) has occurred at time equal to zero, resulting in a sudden field-current increment, as previously discussed. If the rise in exciter voltage during the first interval is introduced as a step at the *beginning* of this interval (the

figure indicates step at *end* of interval), the alternator field current during the first interval is given by

$$i_f = \frac{e_{f0} + \Delta e_{f1}}{R_f} - \left(\frac{\Delta e_{f1}}{R_f} - \Delta i_{f0}\right)\epsilon^{-\frac{t}{T}} = \frac{e_{f1}}{R_f} - \left(\frac{e_{f1}}{R_f} - i_{f0}\right)\epsilon^{-\frac{t}{T}} \quad (45)$$

From this, the current at the end of this interval is obtained as

$$i_{f1} = \frac{e_{f1}}{R_f} - \left(\frac{e_{f1}}{R_f} - i_{f0}\right)\epsilon^{-\frac{\Delta t}{T}} \quad (46)$$

Similarly, during the second interval, the current becomes

$$i_f = \frac{e_{f2}}{R_f} - \left(\frac{e_{f2}}{R_f} - i_{f1}\right)\epsilon^{-\frac{t-\Delta t}{T}} \quad (47)$$

and at the end of this interval,

$$i_{f2} = \frac{e_{f2}}{R_f} - \left(\frac{e_{f2}}{R_f} - i_{f1}\right)\epsilon^{-\frac{\Delta t}{T}} \quad (48)$$

Considering the nth interval, it is obvious that the field current is given by

$$i_{fn} = \frac{e_{fn}}{R_f} - \left[\frac{e_{fn}}{R_f} - i_{f(n-1)}\right]\epsilon^{-\frac{\Delta t}{T}} \quad (49)$$

Hence the increment in field current during the nth interval becomes

$$\Delta i_{fn} = i_{fn} - i_{f(n-1)} = \left[\frac{e_{fn}}{R_f} - i_{f(n-1)}\right]\left(1 - \epsilon^{-\frac{\Delta t}{T}}\right) \quad (50)$$

which in the per-unit system defined above may be written

$$\Delta i_{fn} = \Delta e_d = \Delta e'_d = \Delta e_D = [e_{fn} - i_{f(n-1)}]\left(1 - \epsilon^{-\frac{\Delta t}{T}}\right) \quad (51)$$

If, instead of introducing the variation in exciter voltage as a step at the beginning of the interval, the step is introduced as the *end* of the interval (as indicated in Fig. 282), it is readily shown that the field current at the end of the nth interval is given by

$$i_{fn} = \frac{e_{f(n-1)}}{R_f} = \left[\frac{e_{f(n-1)}}{R_f} - i_{f(n-1)}\right]\epsilon^{-\frac{\Delta t}{T}} \quad (52)$$

giving the increment in field current during the nth interval as

$$\Delta i_{fn} = i_{fn} - i_{f(n-1)} = \left[\frac{e_{f(n-1)}}{R_f} - i_{f(n-1)}\right]\left(1 - \epsilon^{-\frac{\Delta t}{T}}\right) \quad (53)$$

which in turn in the per-unit system may be written

$$\Delta i_{fn} = \Delta e_d = \Delta e'_d = \Delta e_D = (e_f - i_f)_{n-1}\left(1 - \epsilon^{-\frac{\Delta t}{T}}\right) \quad (54)$$

By replacing the exponential by a power series, and retaining the first two terms only, this equation reduces to [equation (51) can be similarly reduced]

$$\Delta i_{fn} = \Delta e_d = \Delta e'_d = \Delta e_D = (e_f - i_f)_{n-1}\frac{\Delta t}{T} \quad (55)$$

From a computational standpoint, equation (55) is slightly more convenient than equation (54) and usually gives more than sufficient accuracy with the ordinary length of time interval and range of time constants. The comparison in the following tabulation brings out this fact.[1]

Δt sec.	T sec.	$\frac{\Delta t}{T}$	$\epsilon^{-\frac{\Delta t}{T}}$	$1 - \epsilon^{-\frac{\Delta t}{T}}$	Difference per cent
0.05	1.0	0.05	0.951	0 049	2.0
0.10	1.0	0.10	0.905	0 095	5.0
0.05	5.0	0.01	0 990	0.01	0
0.10	5.0	0.02	0.980	0.02	0

b. Considering Linear Variation within Time Interval.—It is reasonable to expect improved accuracy when the actual exciter-voltage variation is considered linear within the time interval. In order to utilize this scheme in a point-by-point solution, an expression for the response of an inductive circuit

[1] A hydroelectric generator having, for example, 100 per cent synchronous and 30 per cent transient reactance would generally have a direct-axis transient, open-circuit time constant of about 5 sec. The corresponding short-circuit time constant for a symmetrical three-phase short circuit at the terminals would then be about 1.5 sec. Dissymmetrical short circuits and, in general, short circuits not directly at the machine would result in higher values. Hence even for the worst case assuming the standard time interval of 0.05 sec., the error introduced by the simplified representation of the exponential would be less than 2 per cent.

to a linearly varying voltage must be available. Consider this voltage to be of the form

$$e_f = e_{f0} + at \tag{56}$$

The superposition theorem[1] may conveniently be used and the field current expressed as follows:

$$i_f = i_{f0} + A(0)f(t) + \int_0^t f(\lambda)A'(t-\lambda)d\lambda \tag{57}$$

In this it is assumed that the circuit initially carries a steady-state current corresponding to the constant part of the voltage in equation (56). The indicial admittance, its derivative, and the voltage function to be inserted in the above equation are

$$\left.\begin{aligned} A(t) &= \frac{1}{R}\left(1 - \epsilon^{-\frac{t}{T}}\right) & A(0) &= 0 \\ A'(t) &= \frac{1}{RT}\epsilon^{-\frac{t}{T}} & A'(t-\lambda) &= \frac{1}{RT}\epsilon^{-\frac{t-\lambda}{T}} \\ f(t) &= at & f(\lambda) &= a\lambda \end{aligned}\right\} \tag{58}$$

Hence

$$\begin{aligned} i_f &= i_{f0} + \frac{a}{RT}\int^t \lambda\epsilon^{-\frac{t-\lambda}{T}}d\lambda \\ &= i_{f0} + \frac{a}{RT}\epsilon^{-\frac{t}{T}}\left[T^2\epsilon^{\frac{t}{T}}\left(\frac{\lambda}{T} - 1\right)\right]_0^t \\ &= i_{f0} + \frac{a}{R}\left[t - T\left(1 - \epsilon^{-\frac{t}{T}}\right)\right] \end{aligned} \tag{59}$$

If initially there had been a sudden readjustment in current (so that the initial current $i_{f0} = i_{f(0)} + \Delta i_{f0}$ would not be a steady-state current), it is evident that the above solution would be modified to

$$i_f = i_{f(0)} + \Delta i_{f0}\epsilon^{-\frac{t}{T}} + \frac{a}{R}\left[t - T\left(1 - \epsilon^{-\frac{t}{T}}\right)\right] \tag{60}$$

These general solutions may now be made use of in the point-by-point process of analysis. Referring to Fig. 282 and considering the presence of a time lag τ, the field current during the first interval is given by

[1] See reference in footnote on p. 549.

$$i_f = i_{f(0)} + \Delta i_{f0}\epsilon^{-\frac{t}{T}} + \frac{a_1}{R_f}\left[(t-\tau) - T\left(1 - \epsilon^{-\frac{t-\tau}{T}}\right)\right] \tag{61}$$

where

$$i_{f(0)} = \frac{e_{f0}}{R_f} \qquad \text{and} \qquad a_1 = \frac{e_{f1} - e_{f0}}{\Delta t - \tau} = \frac{\Delta e_{f1}}{\Delta t - \tau} \tag{62}$$

At the end of this interval, the current becomes

$$\begin{aligned} i_{f1} &= \frac{e_{f0}}{R_f} + \Delta i_{f0}\epsilon^{-\frac{\Delta t}{T}} + \frac{\Delta e_{f1}}{R_f}\left[1 - \frac{T}{\Delta t - \tau}\left(1 - \epsilon^{-\frac{\Delta t-\tau}{T}}\right)\right] \\ &= \frac{e_{f0}}{R_f} - \left(\frac{e_{f0}}{R_f} - i_{f0}\right)\epsilon^{-\frac{\Delta t}{T}} + \frac{e_{f1} - e_{f0}}{R_f}\left[1 - \frac{T}{\Delta t - \tau}\left(1 - \epsilon^{-\frac{\Delta t-\tau}{T}}\right)\right] \end{aligned} \tag{63}$$

Similarly during the second interval, the current is given by

$$\begin{aligned} i_f &= \frac{e_{f1}}{R_f} + \left(i_{f1} - \frac{e_{f1}}{R_f}\right)\epsilon^{-\frac{t-\Delta t}{T}} + \frac{a_2}{R_f}\left[(t - \Delta t) - T\left(1 - \epsilon^{-\frac{t-\Delta t}{T}}\right)\right] \\ &= \frac{e_{f1}}{R_f} - \left(\frac{e_{f1}}{R_f} - i_{f1}\right)\epsilon^{-\frac{t-\Delta t}{T}} + \frac{e_{f2} - e_{f1}}{R_f\Delta t}\left[(t - \Delta t) - \right. \\ &\qquad \left. T\left(1 - \epsilon^{-\frac{t-\Delta t}{T}}\right)\right] \end{aligned} \tag{64}$$

and the value at the end of this interval is

$$i_{f2} = \frac{e_{f1}}{R_f} - \left(\frac{e_{f1}}{R_f} - i_{f1}\right)\epsilon^{-\frac{\Delta t}{T}} + \frac{e_{f2} - e_{f1}}{R_f}\left[1 - \frac{T}{\Delta t}\left(1 - \epsilon^{-\frac{\Delta t}{T}}\right)\right] \tag{65}$$

Applied to the nth interval, this becomes

$$\begin{aligned} i_{fn} &= \frac{e_{f(n-1)}}{R_f} - \left[\frac{e_{f(n-1)}}{R_f} - i_{f(n-1)}\right]\epsilon^{-\frac{\Delta t}{T}} + \frac{e_{fn} - e_{f(n-1)}}{R_f}\left[1 - \right. \\ &\qquad \left. \frac{T}{\Delta t}\left(1 - \epsilon^{-\frac{\Delta t}{T}}\right)\right] \end{aligned} \tag{66}$$

The field-current increment during the nth interval is consequently

$$\begin{aligned} \Delta i_{fn} &= \left[\frac{e_{f(n-1)}}{R_f} - i_{f(n-1)}\right]\left(1 - \epsilon^{-\frac{\Delta t}{T}}\right) + \frac{e_{fn} - e_{f(n-1)}}{R_f} \\ &\qquad \left[1 - \frac{T}{\Delta t}\left(1 - \epsilon^{-\frac{\Delta t}{T}}\right)\right] \end{aligned} \tag{67}$$

When the exponential is approximated by the first two terms of the series expansion, this reduces to

$$\Delta i_{fn} = \left[\frac{e_{f(n-1)}}{R_f} - i_{f(n-1)}\right]\frac{\Delta t}{T} \tag{68}$$

which in per-unit notation may be written

$$\Delta i_{fn} = \Delta e_d = \Delta e'_d = (e_f - i_f)_{n-1} \cdot \frac{\Delta t}{T} \tag{69}$$

In the approximate solution the formulation for the nth interval is universally applicable to any interval. When the complete solution is used, on the other hand, the nth-interval formulation is not at once applicable to an interval in which a time lag has to be considered, as, for instance, the first. In such cases the two Δt's within the last bracket of the expression must be replaced by $\Delta t - \tau$.

It is interesting to note that equations (55) and (69) are identical. Thus consideration of the exciter build-up (or build-down) curve as a step function with the steps introduced at the end of a time interval and as a linear function within each interval leads to the same "approximate" solution (substantially "exact" for all practical purposes as previously pointed out).

Method II. Assuming Constant Linear Rate of Build-up (and Build-down) of Exciter Voltage.—When conditions are such that excursion over the complete range of the exciter-voltage time curve does not take place, it may give sufficient accuracy to consider the exciter voltage as varying linearly with time.[1] This assumption is actually sometimes utilized, the slope of the straight line frequently being made equal to the nominal response rate. The latter, as will be recalled, is defined with respect to

[1] Reference may be made to Fig. 265, Chap. XVI. It will be noted that the exciter-voltage build-up curve does not deviate appreciably from a straight line between 175 and 275 volts, corresponding to an average build-up rate of 128 volts per sec. (and indeed not much between 150 and 300 volts, giving an average build-up rate of 124 volts per sec.). Nominal slip-ring voltage which for this exciter may be expected to fall in the range from 180 volts to 200 volts, depending upon application, is, as will be noted, in this particular case located somewhat below the mid-point of the straight portion. It is also of interest to note that the nominal response rate for this exciter is 133 volts per sec. at a nominal slip-ring voltage of 188 volts (computed in Example 8, Chap. XV).

build-up and hence would give a nearly correct relationship under build-up conditions. If nominal response is used also during build-down, the slope of the straight-line relation will usually not be steep enough, since as a rule the rate of change during build-down exceeds that during build-up. It should be noted that this method assumes linear variation during the entire build-up and build-down process and is therefore entirely different from the one previously discussed, which merely simulated the actual exciter build-up curve by a straight line within each short interval of time.

It is obvious that exciter action in this case may be introduced into the point-by-point analysis by the same methods as were described under Method I. A straight line would merely be used instead of the actual exciter build-up and build-down curves. The exciter-voltage variation can be introduced either as steps in each interval or by the previously described method involving linear variation during the interval. The only difference in the latter case would be that, with a definitely continuous linear rate of build-up, there would be no change in slope from interval to interval. Change in slope would occur only when the process of build-up changes to one of build-down, and vice versa.

When the variation is assumed definitely linear, however, it is possible to establish a formal solution for alternator field current for each of the two processes, build-up and build-down, respectively. Values of field current from these expressions would, of course, have to be calculated at the end of each time interval in order to reflect exciter response in the general point-by-point analysis. The validity of the expression, however, would not be limited to any particular short interval of time. This method has essentially academic interest only, since in the practical case it would no doubt be more convenient, even when definitely linear variations are assumed, to use one of the methods already described.

From Fig. 283, the voltage during the build-up process, considering a time lag τ, is given by

$$e_f = e_{f0} + a(t - \tau) \tag{70}$$

By making use of equation (60), the expression for field current becomes

$$i_f = i_{f(0)} + \Delta i_{f0}\epsilon^{-\frac{t}{T}} + \frac{a}{R_f}\left[(t - \tau) - T\left(1 - \epsilon^{-\frac{t-\tau}{T}}\right)\right] \quad (71)$$

The increment in field current, exclusive of the sudden initial change, is therefore

$$\Delta i_f = i_f - i_{f0} = -\Delta i_{f0}\left(1 - \epsilon^{-\frac{t}{T}}\right) + \frac{a}{R_f}\left[(t - \tau) - T\left(1 - \epsilon^{-\frac{t-\tau}{T}}\right)\right] \quad (72)$$

which, in the per-unit system, may be written

$$\Delta i_f = \Delta e_d = \Delta e_d' = \Delta e_D = -\Delta i_{f0}\left(1 - \epsilon^{-\frac{t}{T}}\right) + a\left[(t - \tau) - T\left(1 - \epsilon^{-\frac{t-\tau}{T}}\right)\right] \quad (73)$$

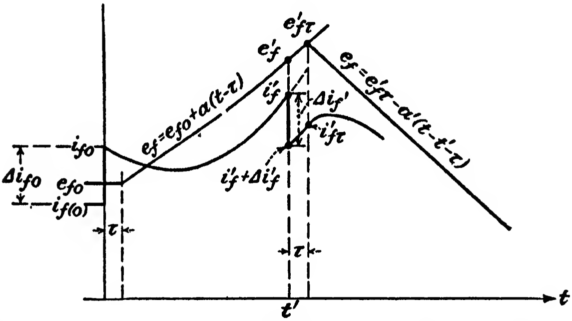


FIG. 283.—Exciter-voltage and alternator field-current variation after the sudden occurrence of a short circuit and its subsequent clearing. The exciter build-up and build-down curves are approximated by linear relationships.

Assuming first that the time constant remains unchanged the above expressions may be used until regulator action takes place, changing over from build-up to build-down. In Fig. 283 it is assumed that this occurs at time t' simultaneously with a discontinuity in the field current (as a result of the short circuit being cleared, for instance) and a time lag has also been considered. The exciter voltage during build-down may then be expressed as

$$e_f = e'_{f\tau} - a'(t - t' - \tau) \quad (74)$$

The field current from the introduction of the discontinuity to the actual change-over from build-up to build-down, *i.e.*, for

$t' < t < (t' + \tau)$ is given by

$$i_f = i_{f(0)} + \Delta i_{f0}\epsilon^{-\frac{t}{T}} + \Delta i'_f\epsilon^{-\frac{t-t'}{T}} + \frac{a}{R_f}\left[(t - \tau) - T\left(1 - \epsilon^{-\frac{t-\tau}{T}}\right)\right] \quad (75)$$

After build-down has begun, *i.e.*, for $t > t' + \tau$ the expression for field current becomes

$$i_f = i_{f(0)} + \Delta i_{f0}\epsilon^{-\frac{t}{T}} + \Delta i'_f\epsilon^{-\frac{t-t'}{T}} + \frac{a}{R_f}\left[t' - T\left(1 - \epsilon^{-\frac{t-\tau}{T}}\right)\right]\epsilon^{-\frac{t-t'-\tau}{T}} - \frac{a'}{R_f}\left[(t - t' - \tau) - T\left(1 - \epsilon^{-\frac{t-t'-\tau}{T}}\right)\right] \quad (76)$$

Should the constant response rate numerically be the same for build-up and build-down, *i.e.*, $a = a'$, the foregoing equation reduces to

$$i_f = i_{f(0)} + \Delta i_{f0}\epsilon^{-\frac{t}{T}} + \Delta i'_f\epsilon^{-\frac{t-t'}{T}} - \frac{a}{R_f}\left[(t - \tau) - t'\left(1 + \epsilon^{-\frac{t-t'-\tau}{T}}\right) - T\left(1 + \epsilon^{-\frac{t-\tau}{T}} - 2\epsilon^{-\frac{t-t'-\tau}{T}}\right)\right] \quad (77)$$

It may possibly be more convenient to use equation (75) in the following modified form

$$\begin{aligned} i_f &= (i'_f + \Delta i'_f)\epsilon^{-\frac{t-t'}{T}} + \frac{e'_f}{R_f}\left(1 - \epsilon^{-\frac{t-t'}{T}}\right) + \frac{a}{R_f}\left[(t - t') - T\left(1 - \epsilon^{-\frac{t-t'}{T}}\right)\right] \\ &= (i'_f + \Delta i'_f)\epsilon^{-\frac{t-t'}{T}} + \frac{e'_f - aT}{R_f}\left(1 - \epsilon^{-\frac{t-t'}{T}}\right) + \frac{a}{R_f}(t - t') \end{aligned} \quad (78)$$

where

$$i'_f = i_{f(0)} + \Delta i_{f0}\epsilon^{-\frac{t'}{T}} + \frac{a}{R_f}\left[(t' - \tau) - T\left(1 - \epsilon^{-\frac{t'-\tau}{T}}\right)\right] \quad (79)$$

$$e'_f = e_{f0} + a(t' - \tau) \qquad \text{(or read from plot)} \quad (80)$$

In a similar manner equation (76) may be modified to

$$i_f = i'_{f\tau}\epsilon^{-\frac{t-t'-\tau}{T}} + \frac{e'_{f\tau}}{R_f}\left(1 - \epsilon^{-\frac{t-t'-\tau}{T}}\right) - \frac{a'}{R_f}\left[(t - t' - \tau) - T\left(1 - \epsilon^{-\frac{t-t'-\tau}{T}}\right)\right]$$

$$= i'_{f\tau}\epsilon^{-\frac{t-t'-\tau}{T}} + \frac{e'_{f\tau} - a'T}{R_f}\left(1 - \epsilon^{-\frac{t-t'-\tau}{T}}\right) - \frac{a'}{R_f}(t - t' - \tau) \tag{81}$$

where

$$i'_{f\tau} = i_{f(0)} + \Delta i_{f0}\epsilon^{-\frac{t'+\tau}{T}} + \Delta i'_f\epsilon^{-\frac{\tau}{T}} + \frac{a}{R_f}\left[t' - T\left(1 - \epsilon^{-\frac{t'}{T}}\right)\right]$$

$$= (i'_f + \Delta i'_f)\epsilon^{-\frac{\tau}{T}} + \frac{e'_f}{R_f}\left(1 - \epsilon^{-\frac{\tau}{T}}\right) + \frac{a}{R_f}\left[\tau - T\left(1 - \epsilon^{-\frac{\tau}{T}}\right)\right]$$

$$= (i'_f + \Delta i'_f)\epsilon^{-\frac{\tau}{T}} + \frac{e'_f - aT}{R_f}\left(1 - \epsilon^{-\frac{\tau}{T}}\right) + \frac{a}{R_f}\tau \tag{82}$$

$$e'_{f\tau} = e_{f0} + at' \qquad \text{(or read from plot)} \tag{83}$$

If the time constant changes from T to T' at the discontinuity preceding the initiation of the build-down process, equations (75), (76), and (77), and the first part of (82) *should not be used* (except after suitable modification). In this case equations (79), (80), and (83) apply as written and equations (78), (81), and (82) (*last two parts only*) upon replacing T by the new time constant T'.

Should it be desired to readjust the time scale and consider time equal to zero whenever a change-over from build-up to build-down, or vice versa, is initiated this can be taken care of by making $t' = 0$ in equations (74) to (78), inclusive, and (81). It does not, of course, affect equations (79), (80), (82) and (83).

EXAMPLE 4

Statement of Problem

The system of Fig. 284(a) represents a hydroelectric generating station supplying power over a double-circuit transmission line to a receiving-end system of such size that it may be simulated by an infinite bus. It is desired to investigate the effect of different exciter-response rates on the field current and flux linkages in the hydroelectric generators during a line-to-line fault on one of the transmission circuits just outside the sending-end, high-tension bus.

A linear build-up of exciter voltage will be assumed in order to avoid point-by-point calculations. Unit rate of response will be defined as a voltage build-up per second equal to the slip-ring voltage of the alternator required to produce 100 per cent field current. Thus, for a 250-volt exciter

on a generator requiring 125 volts to supply 100 per cent field current, unit rate of response is 125 volts per sec.[1] The following per unit response rates [equal numerically to a in equation (70) *et seq.* when per unit values are used] are to be employed in this example: a = 0, 1, 4, 8, 12, 16, 20, 30, 40, 50.

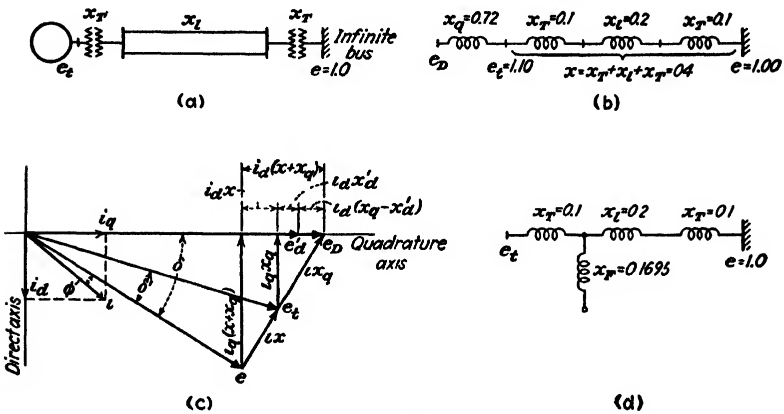


FIG. 284.—(*a*) Hydroelectric generating station supplying power to large receiving-end system (infinite bus) over a two-circuit transmission line. (*b*) Equivalent circuit of the system in (*a*) for normal conditions of operation. (*c*) Vector diagram for the system in (*a*) for normal conditions of operation. (*d*) Equivalent circuit for the system in (*a*) beyond the generator terminals with one of the transmission lines faulted (Example 4).

In per unit on the generator base the generator and circuit constants are:

$X_d = 0.99$	$X_q = 0.72$	$X_T = 0.1$ at each end
$X'_d = 0.36$	$X'_q = 0.72$	$X_l = 0.2$ (two circuits)
$X''_d = 0.26$	$X''_q = 0.32$	$T'_{d0} = 5.0$ sec.
$X^- = 0.29$		$\tau = 0.05$ sec.

The infinite bus voltage is $e = 1.00$. Before the fault occurs the generator terminal voltage is $e_t = 1.10$, and the generator power output is $P = 1.00$.

For this system with the line-to-line fault on one transmission circuit near the sending-end high-tension bus, compute and plot curves versus time of

a. Change in field current.

b. Voltage e'_d corresponding to the flux linkages in the direct axis.

c. Time-average value of flux linkages (e'_d) versus time.

These curves should be plotted for values of time up to 0.6 sec. after the fault occurs. Clearing of the fault is not to be considered.

[1] If nominal slip-ring voltage is 188 volts, unit response rate as defined above would correspond to a response rate with respect to nominal slip-ring voltage of $^{125}/_{188} = 0.665$.

Solution

Computation of Prefault Values.—The prefault equivalent circuit is given in Fig. 284(*b*) and the corresponding vector diagram in (*c*). All currents and voltages in the computations below are taken with the infinite-bus voltage e as a reference.

Since

$$P = \frac{ee_t}{X} \sin \delta' \tag{a}$$

$$1.00 = \frac{1.00 \times 1.10}{0.4} \sin \delta'$$

or

$$\delta' = 21.3°$$

From the vector diagram

$$e_t = e + jiX \text{ (vectorially)} \tag{b}$$

or

$$\begin{aligned} i &= \frac{e_t/\underline{\delta'} - e/\underline{0°}}{X/\underline{90°}} \\ &= \frac{1.10/\underline{21.3°} - 1.00/\underline{0°}}{0.4/\underline{90°}} \\ &= 1.00 - j0.0625 = 1.00/\underline{-3.6°} \end{aligned} \tag{c}$$

Also

$$\begin{aligned} e_D &= e + ji(X + X_q) \text{ (vectorially)} \\ &= 1.00/\underline{0°} + j(1.00 - j0.0625)(0.4 + 0.72) \\ &= 1.07 + j1.12 = 1.55/\underline{46.3°} \end{aligned} \tag{d}$$

The direct-axis component of current is, therefore,

$$\begin{aligned} i_d &= i \sin (\delta + \phi) \\ &= 1.00 \sin (46.3° + 3.6°) \\ &= 0.765 \end{aligned} \tag{e}$$

Finally,

$$\begin{aligned} e'_d &= e_D - i_d(X_q - X'_d) \text{ (scalar equation)} \\ &= 1.55 - 0.765\ (0.72 - 0.36) \\ &= 1.27 \end{aligned} \tag{f}$$

Computation of Values during Fault.—The equivalent circuit external to the machine during the faulted period is given in Fig. 284(*d*). On the basis of equation (48), Chap. XI, the reactance of the fault shunt is

$$\begin{aligned} X_F &= \frac{(X^- + X_T)(X_l + X_T)}{X^- + X_l + 2X_T} \\ &= \frac{(0.29 + 0.1)(0.2 + 0.1)}{0.29 + 0.2 + 0.2} = 0.1695 \end{aligned} \tag{g}$$

The generator time constant must be adjusted for external reactance in accordance with equation (54), Chap. XVI. In this equation X_e, the driving-point reactance viewed from the generator terminals, is

$$X_e = X_T + \frac{X_F(X_l + X_T)}{X_F + X_l + X_T} \tag{h}$$

$$= 0.1 + \frac{0.1695(0.2 + 0.1)}{0.1695 + 0.2 + 0.1} = 0.208$$

The equivalent time constant is, therefore,

$$T = T'_{d0}\frac{X'_d + X_e}{X_d + X_e} \tag{i}$$

$$= 5.0\frac{0.36 + 0.208}{0.99 + 0.208} = 2.37 \text{ sec.}$$

For the first instant after the fault occurs, e'_d and δ retain their prefault values and hence are known. The value of i'_d just after fault may, therefore, be computed from equation (40), Chap. XII. In this case,

$$X_m = X_F = 0.1695,$$
$$X_1 = X_T = 0.1,$$
$$X_2 = X_e + X_T = 0.3,$$

and, consequently,

$$i_d = \frac{e'_d - \dfrac{X_m}{X_2 + X_m}e\cos\delta}{X'_d + X_1 + \dfrac{X_2X_m}{X_2 + X_m}} \text{ (scalar equation)} \tag{j}$$

$$= \frac{1.27 - \dfrac{0.1695}{0.3 + 0.1695} \times 1.00\cos 46.3°}{0.36 + 0.1 + \dfrac{0.3 + 0.1695}{0.3 + 0.1695}}$$

$$= 1.796 \text{ immediately after fault}$$

The change in i_d which takes place suddenly upon occurrence of the fault is

$$\Delta i_d = i_{d(\text{after fault})} - i_{d(\text{before fault})} \tag{k}$$

$$= 1.796 - 0.765 = 1.031$$

The sudden change in field current which must take place in order to maintain constant flux linkages at this first instant is, therefore, approximately [see equation (42)]

$$\Delta i_{f0} = (X_d - X'_d)\Delta i_d \tag{l}$$

$$= (0.99 - 0.36)1.031 = 0.650$$

Computation of Field-current and Flux-linkage Transient.—Values of Δi_f and $\Delta e'_d$ may now be computed by means of equation (73). The principal

parts of these computations are summarized in Table 71. It should be noted that at any time t

$$e'_d = e'_{d0} + \Delta e'_d = 1.27 + \Delta e'_d \tag{m}$$

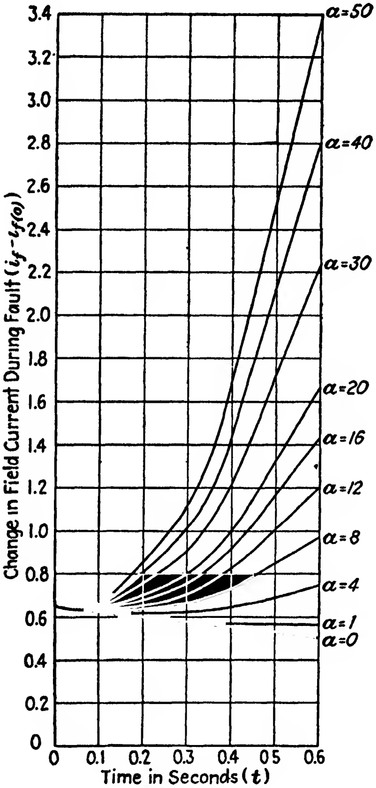


FIG. 285.

FIG. 285.—Change in field current in the generators of the system in Fig. 284 during a line-to-line fault just outside the sending-end high-tension bus with different values of per-unit exciter-response rate (Example 4).

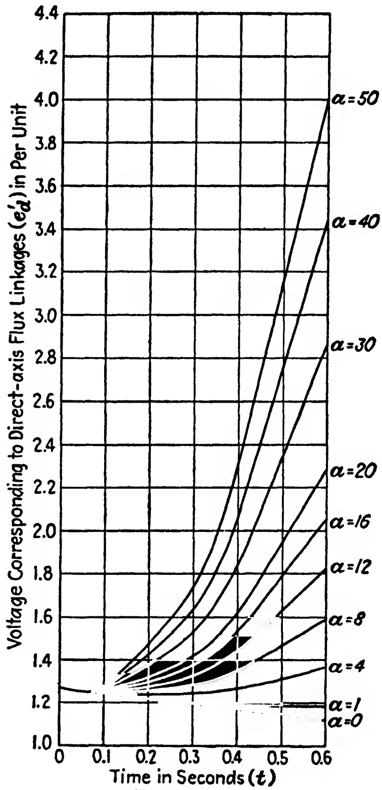


FIG. 286.

FIG. 286.—Change in direct-axis flux linkages in the generators of the system in Fig. 284 during a line-to-line fault just outside the sending-end high-tension bus with different values of per-unit exciter-response rate (Example 4).

and total change in field current

$$i_f - i_{f(0)} = \Delta i_{f0} + \Delta i_f = 0.650 + \Delta i_f \tag{n}$$

where the values of e'_d and i_f are obtained from Table 71 for the appropriate values of t and a.

a. Curves of change in field current ($i_f - i_{f(0)}$) in per unit versus time for the various exciter-response rates considered are given in Fig. 285.

b. Curves of e'_d in per unit versus time for the various exciter-response rates are given in Fig. 286.

c. The time-average values of flux linkages versus time are plotted in Fig. 287. The ordinates of this curve have been normalized by expressing the values as fractions of the prefault value. A point on one of these curves for the time t is computed by finding the average ordinate of the corresponding curve in Fig. 286 up to the time t (equals area in per unit seconds under this curve between 0 and t divided by t) and dividing this by the prefault value of $e'_d(= 1.27)$. The points from which these curves are plotted are summarized in Table 72. Figure 287 shows, for example, that for an exciter-response rate of $a = 16$ and a fault duration of 0.2 sec., the average value of e'_d during the fault is 0.999 of the initial value.

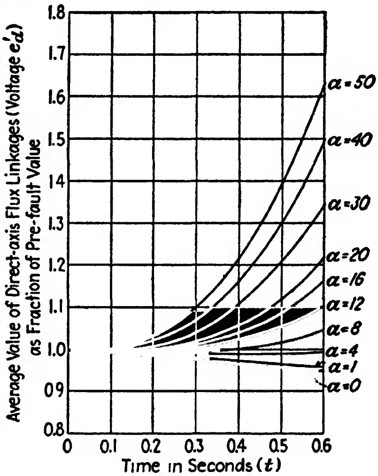


FIG. 287.—Average value of direct-axis flux linkages in the generators of the system in Fig. 284 during a line-to-line fault just outside the sending-end high-tension bus with different values of per-unit exciter-response rate (Example 4).

Although computed for one specific system and one type of fault, Figs. 285, 286, and 287 yield considerable general information regarding the behavior of field current and flux linkages during a fault for a wide range of response rates. The curves for $a = 0$ correspond to an unregulated, handcontrolled generator; those for $a = 50$ correspond to a generator with ultra high-speed excitation (6,250 volts per sec. if unit slip-ring voltage is 125 volts). It will be noted that for a fault duration of 0.2 sec., a common value, it requires a response rate of $a = 16$ (2,000 volts per sec. if unit slip-ring voltage is 125 volts) to maintain an average value of e'_d during fault equal to the prefault value.

The figures quoted are based upon a line-to-line fault. If instead a double line-to-ground fault had been considered, the equivalent time constant would have been somewhat smaller and the opportunity for field-current build-up as a result of exciter action correspondingly greater.

It has been previously pointed out that in stability calculations the assumption of *constant* flux linkages is frequently made. The accuracy of this assumption evidently depends upon the exciter-response rate, the electrical constants of the alternator as well as on the type and location of the fault. It also very definitely depends upon the length of time considered and whether or not fault clearing takes place within this time. In most large systems the half period of machine oscillations, which also represents

TABLE 70.—CALCULATION OF FIELD-CURRENT AND FLUX-LINKAGE VARIATIONS AS A RESULT OF EXCITER-VOLTAGE BUILD-UP
(Example 4)

t sec.	$\Delta i_{f0}\left(1-\epsilon^{-\frac{t}{T}}\right)$ per unit	$\left[(t-\tau)-T\left(1-\epsilon^{-\frac{t-\tau}{T}}\right)\right]$ per unit	$\Delta i_f = \Delta e'_d$ per unit									
			$a = 0$	$a = 1$	$a = 4$	$a = 8$	$a = 12$	$a = 16$	$a = 20$	$a = 30$	$a = 40$	$a = 50$
0	0	0	0	0	0	0	0	0	0	0	0	0
0.03	0.00910	0	−0.0091	−0.0091	−0.0091	−0.0091	−0.0091	−0.0091	−0.0091	−0.0091	−0.0091	−0.0091
0.05	0.01366	0	−0.0137	−0.0137	−0.0137	−0.0137	−0.0137	−0.0137	−0.0137	−0.0137	−0.0137	−0.0137
0.10	0.0267	0.0002	−0.0267	−0.0265	−0.0259	−0.0251	−0.0243	−0.0235	−0.0227	−0.0207	−0.0187	−0.0167
0.15	0.0396	0.0028	−0.0396	−0.0368	−0.0284	−0.0172	−0.0060	0.0052	0.0164	0.0444	0.0724	0.1004
0.20	0.0526	0.0055	−0.0526	−0.0471	−0.0306	−0.0086	0.0134	0.0354	0.0574	0.1124	0.1674	0.222
0.30	0.0774	0.0110	−0.0774	−0.0664	−0.0334	0.0106	0.0546	0.0986	0.1426	0.253	0.363	0.473
0.40	0.1015	0.0230	−0.1015	−0.0785	−0.0095	0.0825	0.1745	0.267	0.358	0.588	0.818	1.048
0.50	0.1237	0.0400	−0.1237	−0.0837	0.0363	0.1963	0.356	0.516	0.676	1.076	1.476	1.876
0.60	0.1458	0.0580	−0.1458	−0.0878	0.0862	0.318	0.550	0.782	1.014	1.594	2.17	2.75

the critical time, is of the order of ½ sec. or slightly less. Of particular interest to stability, therefore, is the average value of flux linkages during this interval. If the fault remains on so as to maintain continuously a terminal voltage considerably below normal, exciter build-up will take place until the ceiling voltage is reached. If, on the other hand, the fault is cleared (in the usual case at about 0.2 sec.), the situation is governed by whether the terminal voltage rise occurring at the instant of clearing is sufficient to bring this voltage back to normal or perchance to a value above normal or whether this voltage still will remain subnormal. It is believed that in most systems the latter will be the case, when the fault is not too distant from the generator. With this situation, exciter build-up and improvements in flux linkages will continue, at least for a while, also after clearing, thus contributing to a higher average value over the critical period.

If, on the other hand, the voltage upon clearing reaches normal or overshoots normal, the regulator contacts will open and the exciter (after expiration of the time lag) begin to build down. This naturally will cancel a part of the build-up achieved during the fault-on period and leave a smaller average value of flux linkages over the critical period.

It should be recalled that the equivalent time constant increases when the fault is cleared. The result of this is to make the effect of exciter action less pronounced whether the former be one of build-up or one of build-down.

TABLE 71.—AVERAGE VALUE OF FLUX LINKAGES EXPRESSED AS FRACTION OF INITIAL VALUE
(Example 4)

Per unit response rate	t = 0.2 sec.	t = 0.4 sec.	t = 0.6 sec.
a = 0	0.986	0 965	0.949
a = 1	0 986	0 977	0.957
a = 4	0.992	0.986	0.998
a = 8	0.992	1.000	1.053
a = 12	0 996	1 022	1.107
a = 16	0 999	1.042	1.168
a = 20	1.002	1.060	1.222
a = 30	1 012	1.112	1.352
a = 40	1.020	1.162	1.500
a = 50	1.032	1.218	1 639

CHAPTER XVIII

INERTIA AND GROUNDING

The Inertia Constant.—In the general differential equation for the oscillation of a rotating machine, *viz.*,

$$M\frac{d^2\delta}{dt^2} = P_i - P \tag{1}$$

the *inertia constant M* enters. The inertia constant of a machine may be defined as the shaft power required to produce unit angular acceleration (retardation) of the rotor with the latter running at rated speed. The inertia constant must, of course, be adjusted to correspond to the units used in the equation. When the power is expressed in kilowatts and the angle in electrical degrees the inertia constant should obviously be given in kilowatts per electrical degree per second squared (kw. per elec. deg. per sec.^{2}).

It is usually convenient to have available formulas for the inertia constant in terms of the moment of inertia or the stored energy of the rotating parts. Let

M = inertia constant in kilowatts per electrical degree per second squared
P = rating of the machine in kilovolt-amperes
WR^2 = moment of inertia in pound-feet squared
H = stored energy in kilowatt-seconds per kilovolt-ampere
n = speed in r.p.m.
f = frequency in cycles per second
p = number of poles

If the mechanical moment of inertia (divided by the acceleration of gravity) is multiplied by the proper factors converting from mechanical power units to kilowatts and from mechanical radians to electrical degrees, the inertia constant is obtained as

$$M = \frac{WR^2}{g}\left(\frac{2\pi n}{60} \cdot \frac{0.746}{550}\right)\left(\frac{\pi}{180} \cdot \frac{2}{p}\right)$$
$$= \frac{4\pi^2 \times 0.746}{32.2 \times 60 \times 550 \times 180} \cdot \frac{WR^2 n}{p}$$
$$= \frac{15.38 \times 10^{-8} WR^2 n}{p} \qquad (2)$$

By substituting for the number of poles in terms of speed and frequency ($p = 120f/n$), the inertia constant is also given by

$$M = \frac{1.28 \times 10^{-9} WR^2 n^2}{f} \qquad (3)$$

The stored energy in kilowatt-seconds per kilovolt-ampere in terms of the moment of inertia and the speed becomes

$$H = \frac{1}{P} \cdot \frac{WR^2}{2g}\left(\frac{2\pi n}{60}\right)^2\left(\frac{0.746}{550}\right)$$
$$= \frac{4\pi^2 \times 0.746}{2 \times 32.2 \times 3{,}600 \times 550} \cdot \frac{WR^2 n^2}{P}$$
$$= \frac{0.231 \times 10^{-6} WR^2 n^2}{P} \qquad (4)$$

In terms of the inertia constant, the stored energy may be written

$$PH = \frac{M}{2}\left(\frac{n \times 360}{60}\right)\left(\frac{p}{2}\right) = \frac{M}{2}\left(\frac{360n}{60}\right)\left(\frac{120f}{2n}\right) = M\ 180f \qquad (5)$$

which solved for the inertia constant gives

$$M = \frac{HP}{180f} \qquad (6)$$

The inertia constant may thus be calculated from either one of equations (2), (3), and (6) according to the particular data at hand. Evidently the inertia constant to be used in the swing equation must include the mass of the prime mover or shaft load as well as that of the electrical machine. While the moments of inertia vary widely, depending upon the size and speed of the machine, the stored energy per kilovolt-ampere has a *characteristic and fairly constant value* for the various classes of machines. This fact can frequently be made use of to advantage when specific data of the machines in question are lacking. In Table 72 will be found a collection of average values of stored energy.

EXAMPLE 1

Statement of Problem

A 32,000-kva., 48-pole, 150-r.p.m., 60-cycle hydroelectric generator has a moment of inertia WR^2 of 19×10^6 lb.-ft.^{2}, including the turbine. Calculate the inertia constant of the unit.

Solution

If formula (2) is used, the inertia constant becomes

$$M = \frac{15.38 \times 10^{-8} \times 19 \times 10^6 \times 150}{48} = 9.13 \text{ kw. per elec. deg. per sec.}^2$$

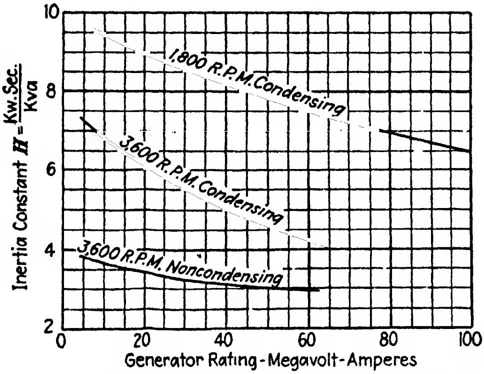


Fig. 288.—Inertia constants of large turbogenerators, turbine included.

The Effect of Inertia.—It is evident that the inertia influences the oscillations of a rotating machine when its equilibrium has been upset. Not only is the value of the inertia a factor to be

Table 72.—Average Values of Stored Energy in Rotating Machines[1]

Machine Type	Stored Energy at Rated Speed, kw.-sec. per kva.
Turbogenerators:	
Steam turbines	Select values from Fig. 288.
Hydrogenerators:	
Hydraulic turbines	Select values from Fig. 289.
Synchronous motors	2.00
Synchronous condensers:[2]	
Large	1.25
Small	1.00
Rotary converters	2.00
Induction motors	0.50

[1] Principally obtained from "First Report on Power System Stability," by the A.I.E.E. Subcommittee on Interconnection and Stability Factors, *Elec. Eng.*, February, 1937.

[2] Hydrogen cooled, 25 per cent less.

considered, but, in a system where several machines may be involved, the distribution of inertia plays an important role. Before discussing the broader aspects of the effect of inertia, it will be well to confine the attention to a few elementary cases.

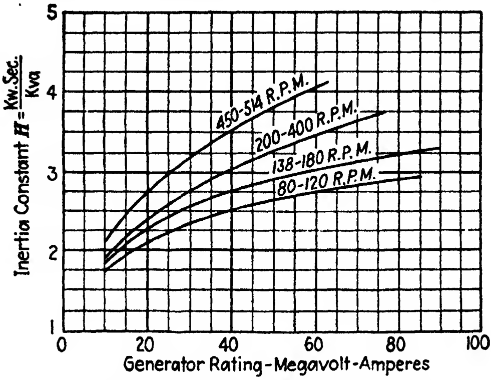


FIG. 289.—Inertia constants of large vertical-type water-wheel generators, including allowance of 15 per cent for water wheels.

Effect of Inertia during Small Oscillations.—The influence of any factor entering a differential equation can be ascertained when a formal solution of the equation is at hand. As already stated, it is generally not practicable to establish such solutions of the machine equation, even under simplified assumptions in regard to input, flux linkages, etc. This is due to the sinusoidal variation of the output. If consideration is limited to oscillations of small amplitude, however, it may be permissible, as a reasonably close approximation, to assume that the *output varies linearly with the displacement angle.* This is equivalent to considering the small portion of the power-angle curve between the limits of the oscillations as a straight line. In such cases formal solutions of the differential equation are possible when the input and the flux linkages are assumed constant.

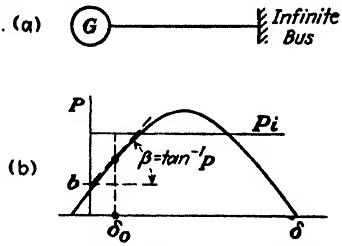


FIG. 290.—(*a*) Generator supplying a large system (infinite bus) over a transmission circuit. (*b*) Power-angle curve for system in (*a*).

Synchronous Machine Connected to Infinite Bus.—The system is shown in Fig. 290(*a*) and the power-angle curve in (*b*). The

acceleration of the synchronous machine is given by

$$\frac{d^2\delta}{dt^2} = \frac{\Delta P}{M} = \frac{P_i - P}{M} \tag{7}$$

The approximate linear output relation may be written

$$P = p\delta + b \tag{8}$$

It is logical to consider the slope of this straight line equal to the slope of the actual power-angle curve at the initial operating point. Hence, when angles are expressed in degrees

$$p = \frac{\pi}{180} \cdot \frac{E_1E_2}{Z_{12}} \cos (\delta_0 - \alpha_{12}) \tag{9}$$

The quantity b in equation (8) depends on the position of the straight line with respect to the sinusoid. If it is placed tangent to the latter at the initial operating point, b is obviously given by

$$b = P_0 - p\delta_0 \tag{10}$$

where

$$P_0 = \frac{E_1^2}{Z_{11}} \sin \alpha_{11} + \frac{E_1E_2}{Z_{12}} \sin (\delta_0 - \alpha_{12}) \tag{11}$$

If, on the other hand, the straight line is located as a chord to the power-angle curve, which may be still more accurate, the determination of b must be based on one of the points of intersection between the two. Equations (10) and (11) may evidently still be applied if P_k be substituted for P_0 and δ_k for δ_0, the former being the power and the latter the angle at one of the points of intersection.

Substituting equation (8) in equation (7), the modified differential equation becomes

$$\frac{d^2\delta}{dt^2} = \frac{P_i - p\delta - b}{M} = \frac{P_i - b}{M} - \frac{p}{M}\delta \tag{12}$$

This may be changed to

$$d\left[\left(\frac{d\delta}{dt}\right)^2\right] = 2\left[\frac{P_i - b}{M} - \frac{p}{M}\delta\right]d\delta \tag{13}$$

from which by integration

$$\frac{d\delta}{dt} = \sqrt{2\frac{P_i - b}{M}\delta - \frac{p}{M}\delta^2 + \frac{C_1}{M}} \tag{14}$$

Separating variables and integrating again results in

$$\sin^{-1}\frac{p\delta - P_i - b}{\sqrt{(P_i - b)^2 + pC_1}} = \sqrt{\frac{p}{M}}t + C_2 \tag{15}$$

which solved for the angle gives

$$\delta = \frac{P_i - b}{p} + \sqrt{\left(\frac{P_i - b}{p}\right)^2 + \frac{C_1}{p}}\sin\left(\sqrt{\frac{p}{M}}t + C_2\right) \tag{16}$$

The constants of integration C_1 and C_2 are determined from initial conditions. When

$$t = 0: \quad \frac{d\delta}{dt} = 0 \qquad \text{and} \qquad \delta = \delta_0$$

By substituting in equation (14), C_1 becomes

$$C_1 = p\delta_0^2 - 2(P_i - b)\delta_0 \tag{17}$$

and hence the angle when this value is inserted in equation (16) is

$$\delta = \frac{P_i - b}{p} + \left(\frac{P_i - b}{p} - \delta_0\right)\sin\left(\sqrt{\frac{p}{M}}t + C_2\right) \tag{18}$$

By applying the initial conditions to this equation, the constant C_2 is found to be

$$C_2 = \sin^{-1}\frac{\delta_0 - \dfrac{P_i - b}{p}}{\dfrac{P_i - b}{p} - \delta_0} = \sin^{-1}(-1) = -90 \text{ deg.} \tag{19}$$

The final solution for the angle is consequently

$$\delta = \frac{P_i - b}{p} - \left(\frac{P_i - b}{p} - \delta_0\right)\cos\sqrt{\frac{p}{M}}t \tag{20}$$

Inspection of this equation shows in the first place that when a linear power-angle relationship is assumed the oscillations are sinusoidal with respect to time. This is very nearly the type of oscillations which actually would be obtained when the amplitudes are small. Furthermore the equation indicates that the amplitudes are directly proportional to the initial power differential but entirely independent of the inertia of the machine.

The period of the oscillations, however, is affected by the inertia, being proportional to the square root of the inertia constant. The period is given by

$$T = 2\pi\sqrt{\frac{M}{p}} \tag{21}$$

Synchronous Generator and Motor Connected by Reactance Tie.—This system [Fig. 291(*a*)] is reducible to the synchronous-machine-infinite-bus system in (*b*). Since the circuit contains reactance only, the power-angle curve is an undisplaced sinusoid as shown in (*c*)

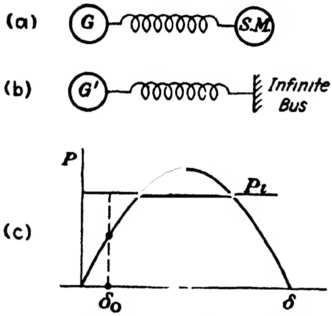


FIG 291 —(*a*) Simple two-machine system consisting of a generator supplying power to a synchronous motor over a reactance tie (*b*) Equivalent system replacing (*a*) (*c*) Power-angle curve for equivalent system in (*b*)

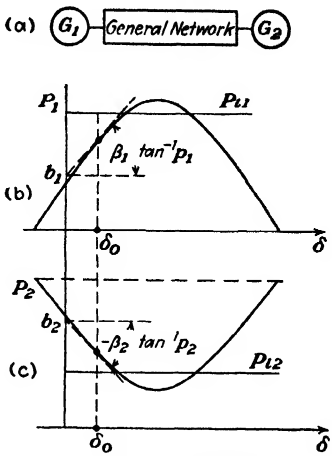


FIG 292 (*a*) General two-machine system (*b*) Power-angle curve for generator G_1 (*c*) Power-angle curve for generator G_2

This case, therefore, is identical with the preceding one and the same equations hold The resultant inertia constant of the two machines as given by

$$M = \frac{M_1 M_2}{M_1 + M_2} \tag{22}$$

must, of course, be used.

Two Synchronous Machines Connected to a General Network.—The system is shown in Fig. 292(*a*) and the power-angle curves of the two machines (considered as generators feeding into the

network) in (*b*) and (*c*), respectively. The equations for the actual output of the two machines are

$$P_1 = \frac{E_1^2}{Z_{11}} \sin \alpha_{11} + \frac{E_1E_2}{Z_{12}} \sin (\delta - \alpha_{12}) \tag{23}$$

and

$$P_2 = \frac{E_2^2}{Z_{22}} \sin \alpha_{22} - \frac{E_1E_2}{Z_{12}} \sin (\delta + \alpha_{12}) \tag{24}$$

while the approximate linear relations to be used in the analysis may be written

$$P_1 = p_1\delta + b_1 \tag{25}$$

and

$$P_2 = p_2\delta + b_2 \tag{26}$$

Adjusting the slopes of the straight lines to correspond to the slopes of the power-angle curves at the initial operating points, the proportionality factors become

$$p_1 = \frac{\pi}{180} \cdot \frac{E_1E_2}{Z_{12}} \cos (\delta_0 - \alpha_{12}) \tag{27}$$

and

$$p_2 = -\frac{\pi}{180} \cdot \frac{E_1E_2}{Z_{12}} \cos (\delta_0 + \alpha_{12}) \tag{28}$$

If the straight lines are placed tangent to the power-angle curves at the initial operating points, the intercepts b_1 and b_2 are given by

$$b_1 = P_{10} - p_1\delta_0 \tag{29}$$

and

$$b_2 = P_{20} - p_2\delta_0 \tag{30}$$

where

$$P_{10} = \frac{E_1^2}{Z_{11}} \sin \alpha_{11} + \frac{E_1E_2}{Z_{12}} \sin (\delta_0 - \alpha_{12}) \tag{31}$$

and

$$P_{20} = \frac{E_2^2}{Z_{22}} \sin \alpha_{22} - \frac{E_1E_2}{Z_{12}} \sin (\delta_0 + \alpha_{12}) \tag{32}$$

If the straight lines are located as chords, equations (29) to (32), inclusive, may still be used with modifications similar to those

pointed out in connection with the previously discussed infinite-bus system.

The relative acceleration of the two machines is given by

$$\frac{d^2\delta}{dt^2} = \frac{\Delta P_1}{M_1} - \frac{\Delta P_2}{M_2} = \frac{P_{i1} - P_1}{M_1} - \frac{P_{i2} - P_2}{M_2} \tag{33}$$

which upon introduction of the approximate output relations [equations (25) and (26)] may be written

$$\begin{aligned} \frac{d^2\delta}{dt^2} &= \frac{P_{i1} - b_1}{M_1} - \frac{P_{i2} - b_2}{M_2} - \left(\frac{p_1}{M_1} - \frac{p_2}{M_2}\right)\delta \\ &= \frac{M_2(P_{i1} - b_1) - M_1(P_{i2} - b_2)}{M_1M_2} - \frac{M_2p_1 - M_1p_2}{M_1M_2}\delta \end{aligned} \tag{34}$$

From the analogy between this and equation (13), the final solution for the present case may, by reference to equation (21), at once be written down by inspection.[1] It becomes

$$\delta = \frac{M_2(P_{i1} - b_1) - M_1(P_{i2} - b_2)}{M_2p_1 - M_1p_2} - \left(\frac{M_2(P_{i1} - b_1) - M_1(P_{i2} - b_2)}{M_2p_1 - M_1p_2} - \delta_0\right) \cos\sqrt{\frac{M_2p_1 - M_1p_2}{M_1M_2}}t \tag{35}$$

[1] The equivalent representation of the general two-machine system may also be used (and perchance to advantage) for establishing the angle-time relations for small oscillations. In terms of equivalent output, input, and inertia [see equations (13), (14), and (15) in Chap. XIV and also Fig. 290(*b*)] the expression for relative acceleration is

$$\frac{d^2\delta}{dt^2} = \frac{P'_i - P'}{M_0} = \frac{\Delta P'}{M_0} \tag{a}$$

where

$$P'_i = \frac{M_2P_{i1} - M_1P_{i2}}{M_1 + M_2} \tag{b}$$

$$P' = \frac{M_2P_1 - M_1P_2}{M_1 + M_2} \tag{c}$$

The approximate linear equivalent-output relation becomes:

$$P' = p'\delta + b' \tag{d}$$

in which

This equation shows that also in the general two-machine case the oscillations are sinusoidal when linear' output relations are assumed. It will further be seen that *the amplitude as well as the period is influenced by the inertia of the machines.* It appears as if it were possible in this case, by the proper distribution of inertia, to make the amplitude zero. Zero amplitude requires that

$$M_2(P_{i1} - b_1) - M_1(P_{i2} - b_2) - M_2p_1\delta_0 + M_1p_2\delta_0 = 0 \quad (36)$$

which solved for the ratio of the inertia constants gives

$$\frac{M_1}{M_2} = \frac{P_{i1} - (p_1\delta_0 + b_1)}{P_{i2} - (p_2\delta_0 + b_2)} = \frac{\Delta P_{10}}{\Delta P_{20}} \quad (37)$$

The ratio of the initial power differentials is, *in general,* negative. Under this condition, it is obvious that zero amplitude cannot be obtained with finite inertia in the machines. In order fully to suppress the oscillations, infinite inertia would be necessary in both machines. Cases are conceivable, however, where the initial power differentials will have the same sign giving a positive ratio.

$$p' = \left(\frac{dP'}{d\delta}\right)_{\delta_0} \quad (e)$$

and

$$b' = P'_0 - p'\delta_0 \quad (f)$$

The constants p' and b' line up with p_1, p_2, b_1, and b_2 as follows:

$$p' = \frac{1}{M_1 + M_2}\left[M_2\left(\frac{dP_1}{d\delta}\right)_{\delta_0} - M_1\left(\frac{dP_2}{d\delta}\right)_{\delta_0}\right] = \frac{M_2p_1 - M_1p_2}{M_1 + M_2} \quad (g)$$

and

$$b' = \frac{M_2P_{01} - M_1P_{02}}{M_1 + M_2} - \frac{M_2p_1 - M_1p_2}{M_1 + M_2}\delta_0$$
$$= \frac{1}{M_1 + M_2}[M_2(P_{01} - p_1\delta_0) - M_1(P_{02} - p_2\delta_0)] = \frac{M_2b_1 - M_1b_2}{M_1 + M_2} \quad (h)$$

By analogy to equation (20), the solution of equation (a) becomes:

$$\delta = \frac{P'_i - b'}{p'} - \left(\frac{P'_i - b'}{p'} - \delta_0\right)\cos\sqrt{\frac{p'}{M_0}}t \quad (i)$$

This simple form may be useful for many purposes and reduces to equation (35) by substitution for p' and b' from equations (g) and (h). According to equation (i) no oscillations will occur when $P'_i = P'_0$, a relation which may be expanded to give equations (36) and (37).

This would result in zero oscillations of the machines with respect to each other (the initial angle between them being maintained), both drifting away together from synchronous speed. These statements are predicated on a single discontinuity; for instance, a fault which is permitted to hang on. If clearing takes place or, in general, several discontinuities enter into the picture, they need modification. As a matter of fact, in such cases oscillations are bound to occur.

With a given finite total inertia the distribution which will give rise to a minimum of amplitude may be investigated. The premise involving fixed total inertia is usually not a practical one and the subsequent discussion based on it, therefore, is of essentially academic interest only. Let the aggregate inertia in the system be

$$M = M_1 + M_2 \tag{38}$$

The maximum angle reached during the oscillations is

$$\delta_m = 2\frac{M_2(P_{i1} - b_1) - M_1(P_{i2} - b_2)}{M_2p_1 - M_1p_2} - \delta_0 \tag{39}$$

which when substituting for M_2 from equation (38) may be written

$$\delta_m = 2\frac{M(P_{i1} - b_1) - M_1(P_{i1} - b_1 + P_{i2} - b_2)}{Mp_1 - M_1(p_1 + p_2)} - \delta_0 \tag{40}$$

Differentiation with respect to M_1 gives

$$\frac{d\delta_m}{dM_1} = -2M\frac{p_1(P_{i2} - b_2) - p_2(P_{i1} - b_1)}{[Mp_1 - M_1(p_1 + p_2)]^2} \tag{41}$$

By a similar procedure, the derivative with respect to M_2 may be formed. This appears to be equal and opposite to the derivative with respect to M_1. Consequently,

$$\frac{d\delta_m}{dM_1} = -\frac{d\delta_m}{dM_2} = -2M\frac{p_i(P_{i2} - b_2) - p_2(P_{i1} - b_1)}{(M_2p_1 - M_1p_2)^2} \tag{42}$$

Obviously the ability of these derivatives to equal zero has no connection whatsoever with the inertia and its distribution. Hence the maximum angle [equation (40)] has no functional maximum or minimum values occurring at some specific inertia distribution. For ordinary systems the coefficients p_1 and p_2

have opposite sign. Furthermore, $P_{i1} - b_1$ and $P_{i2} - b_2$ usually have opposite signs. Hence the two terms in the numerator of equation (42) are inherently of the same sign, either positive or negative, and the denominator is positive. The sign of the derivatives, therefore, is independent of the distribution of inertia, the latter affecting their magnitudes only.

On the basis of these relationships, it appears that, from the standpoint of keeping the oscillations small, it would apparently be advantageous to concentrate all the inertia at one end of the system according to whether the derivative [equation (41) or (42)] with respect to the inertia constant at that end is negative or positive considering, however, at the same time the sign of $p_1 + p_2$. Hence, if $d\delta_m/dM_1$ is negative and $p_1 + p_2$ positive, M_1 should be taken equal to M, and M_2 equal to zero; and if both $d\delta_m/dM_1$ and $p_1 + p_2$ are positive, M_1 should be taken equal to zero, and M_2 equal to M. With zero inertia at one end, the frequency of the oscillations would be infinite and the period zero. Of course, such a distribution of inertia has theoretical interest only and can be only remotely approached in a practical system.

The period of oscillation is given by

$$T = 2\pi\sqrt{\frac{M_1M_2}{M_2p_1 - M_1p_2}} \tag{43}$$

Zero inertia at *one end* (or *both ends*) will make this zero, and infinite inertia at *both ends*, infinite. With finite total inertia, a certain distribution will result in maximum period. Substituting for M_2, by means of equation (38), the quantity under the radical in equation (43), designated by μ, may be written

$$\mu = \frac{M_1(M - M_1)}{Mp_1 - M_1(p_1 + p_2)} \tag{44}$$

The derivative of μ with respect to M_1 becomes upon contraction

$$\frac{d\mu}{dM_1} = \frac{M_1^2(p_1 + p_2) - 2M_1Mp_1 + M^2p_1}{[Mp_1 - M_1(p_1 + p_2)]^2} \tag{45}$$

For maximum period the derivative must be zero. Hence

$$M_1^2 - \frac{2p_1M}{p_1 + p_2}M_1 + \frac{p_1M^2}{p_1 + p_2} = 0 \tag{46}$$

which solved for M_1 gives

$$M_1 = \frac{p_1 \pm \sqrt{-p_1p_2}}{p_1 + p_2} M \tag{47}$$

The inertia constant M_2 becomes

$$M_2 = \frac{p_2 \mp \sqrt{-p_1p_2}}{p_1 + p_2} M \tag{48}$$

For the special case of $p_1 = -p_2$, the above equations yield as solutions

$$M_1 = -M_2 = \infty \qquad \text{and} \qquad M_1 = M_2 = \frac{M}{2}$$

The infinite values are trivial, but the others are recognized as correct for a two-machine system containing reactance only.

By inserting for the p's in equations (47) and (48) by means of equations (27) and (28), the inertia constants for maximum period in terms of the aggregate inertia in the system may be written

$$M_1 = \frac{\cos(\delta_0 - \alpha_{12}) \pm \sqrt{\cos(\delta_0 - \alpha_{12}) \cos(\delta_0 + \alpha_{12})}}{2 \sin \delta_0 \sin \alpha_{12}} M \tag{49}$$

and

$$M_2 = \frac{-\cos(\delta_0 + \alpha_{12}) \mp \sqrt{\cos(\delta_0 - \alpha_{12}) \cos(\delta_0 + \alpha_{12})}}{2 \sin \delta_0 \sin \alpha_{12}} M \tag{50}$$

while the ratio of the inertia constants becomes

$$\frac{M_1}{M_2} = \frac{p_1 \pm \sqrt{-p_1p_2}}{p_2 \mp \sqrt{-p_1p_2}} = \pm\sqrt{-\frac{p_1}{p_2}} = \pm\sqrt{\frac{\cos(\delta_0 - \alpha_{12})}{\cos(\delta_0 + \alpha_{12})}} \tag{51}$$

EXAMPLE 2

Statement of Problem

Two generating stations G_1 and G_2 supply power to a general network as indicated in Fig. 292(a). The load on the system is such that the displacement angle between machines is $\delta_0 = 24.5$ deg. A line-to-line fault occurs in the neighborhood of generating station G_1. With this fault on the system, the general circuit constant B_0 equals $-0.238 + j2.025 = 2.04\underline{/96.7°}$ per unit.

Using the analysis for small oscillations, determine the distribution of inertia which would make the period a maximum.'

Solution

The desired ratio of the inertia constants is given by equation (51). Since the general circuit constant B_0 also represents the transfer impedance, the angle α_{12} entering into the above equation is -6.7 deg. Hence, inserting numerical values,

$$\frac{M_1}{M_2} = \sqrt{\frac{\cos(\delta_0 - \alpha_{12})}{\cos(\delta + \alpha_{12})}} = \sqrt{\frac{\cos(24.5° + 6.7°)}{\cos(24.5° - 6.7°)}} = \sqrt{\frac{\cos 31.2°}{\cos 17.8°}}$$

$$= \sqrt{\frac{0.855}{0.952}} = \sqrt{0.899} = 0.949$$

NOTE: For the system described and the condition of fault assumed, it is probable that the oscillations under sustained fault conditions would become fairly large. Hence the above result may actually be far from exact.

Effect of Inertia during Large Oscillations in Two-machine Systems.—In most cases the shock impressed on a system, due to a fault or due to the application of additional load, will be sufficiently severe to cause rather large oscillations. In such instances the *outputs do not vary linearly with the angle.* In order to determine the effect of inertia distribution as regards magnitude and period of the oscillations, it is, in general, necessary to carry through a number of calculations. Usually such calculations would have to cover several values of load as well as several types of fault in order to yield a sufficiently comprehensive picture on which to base conclusions. Application of results from an approximate analysis based on small oscillations may be used as a rough approximation. It is fair to assume that the general character of the effect would be much the same. However, results from such an approximate analysis should be considered *as indicative only and not as representing precise* results. They will be more in the nature of a qualitative rather than a quantitative indication.

EXAMPLE 3

Statement of Problem

A steam-turbine-driven synchronous generator supplies power to a synchronous motor over a reactance tie as shown in Fig. 291(*a*). On a suitable kilovolt-ampere basis the transient reactances of the generator and motor are 20 per cent each, and the reactance of the tie is 40 per cent. Base-load power (equal to the rating of the generator) is transmitted at normal (100 per

cent) bus voltages. The inertia constant of the turbogenerator unit M_1 equals 9×10^{-4} per unit per elec. deg. per sec.[2] A 10 per cent load increment is suddenly applied to the shaft of the motor.

a. Determine the maximum amplitude of the oscillation and the time at which it occurs, using the simplified analysis for small oscillations. Express the time in terms of the inertia constants of the machines.

b. Calculate and plot the actual swing curve and determine from this the maximum amplitude and the time at which it occurs, as in (*a*).

c. Compare and discuss the results obtained by the two methods.

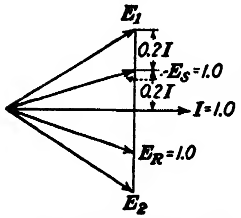


FIG. 293.—Vector diagram for the two-machine system in Fig. 291(*a*) (Example 3).

Solution

The normal inertia constant of a motor of rating equal to that of the generator would be approximately 2×10^{-4} per unit per elec. deg. per sec.[2]

The internal voltage E_1 of the generator and E_2 of the motor are (see vector diagram, Fig. 293)

$$E_1 = E_2 = \sqrt{1.0^2 - 0.2^2 + 0.4^2} = \sqrt{1.12} = 1.06$$

The transfer reactance is

$$X_{12} = 0.2 + 0.4 + 0.2 = 0.8$$

and the initial operating angle

$$\delta_0 = \sin^{-1}\frac{P_2X_{12}}{E_1E_2} = \sin^{-1}\frac{0.8}{1.12} = \sin^{-1} 0.713 = 45.5 \text{ deg.}$$

$$p = \frac{\pi}{180}\,\frac{E_1E_2}{X_{12}}\cos\delta_0 = \frac{\pi \times 1.12 \times 0.701}{180 \times 0.8} = 0.0171$$

$$b = P_0 - p\delta_0 = 1.0 - 0.0171 \times 45.5 = 1.0 - 0.778 = 0.222$$

a. The amplitude (maximum angle) is *independent* of the inertia [see equation (20)] and is

$$\delta_m = 2\frac{P_i - b}{p} - \delta_0 = \frac{2(1.1 - 0.222)}{0.0171} - 45.5$$

$$102.5 - 45.5 = 57.0 \text{ deg.}$$

A half-period τ_m is required to reach maximum angle. Hence from equation (21)

$$\tau_m = \pi\sqrt{\frac{M}{p}} = \frac{\pi\sqrt{M}}{\sqrt{0.0171}} = 24.0\sqrt{M}$$

b. The method described in Chap. XIII, involving precalculated curves will be used.

$$P_0 = 1.0 \qquad P_i = 1.1 \qquad P_m = \frac{1.12}{0.8} = 1.40$$

$$\frac{P_0}{P_m} = \frac{1.0}{1.4} = 0.713 \qquad P = \frac{P_i}{P_m} = \frac{1.1}{1.4} = 0.786$$

Using these values, the following (approximate) swing-curve data are obtained from Figs. 185 and 186 in Chap. XIII (interpolating by inspection only):

τ'	δ	τ'	δ
0	45.5	5	59.2
1	47.4	6	55.1
2	51.4	7	49.6
3	57.5	8	45.0
4	59.0		

The curve is plotted in Fig. 294 and gives a maximum angle of 59.5 deg. occurring at $\tau' = 4.3$, approximately. The relationship between actual time t and modified τ' in the precalculated swing curves as used is given by

$$t = \tau' \sqrt{\frac{180M}{\pi}} = 7.57\tau'\sqrt{M}$$

so that

$$\tau_m = 7.57 \times 4.3\sqrt{M} = 32.2\sqrt{M}$$

c. The maximum angles as found by the approximate and correct methods are 57.0 and 59.5 deg., respectively. This represents a good check between the two. There is a large discrepancy between the time relations, the approximate method for the same aggregate inertia giving a time required to reach maximum angle which is 25 per cent too short.

Effect of Inertia in Compound Systems.—It is impracticable to state anything very specific in regard to the effect of inertia in a large compound power system. The following general statements, however, may be made: (1) the magnitude and (2) the distribution of inertia influence the stability situation.

In regard to the magnitude of the inertia, a machine of normal design has associated with it a more or less fixed amount of inertia. This fact has previously been called attention to and is reflected in the figures collected in Table 72. However, by modification in design, it may also be possible to change the amount of inertia in a certain machine and make it larger or

smaller than what usually is standard. Deviation from standard design, however, usually will add to the cost of a machine. If, hence, an increase in inertia should appear desirable from a stability standpoint, its economic justification must be determined by balancing the benefits obtained against the increased cost. It will probably be found in most instances that it does not pay to specify a design which deviates very appreciably from the normal one.

If it were possible to lay out a power system right from the beginning to supply a specified load in a perfectly defined load area, it might be possible to so locate the stations and select their characteristics, including the inertia, that the best possible

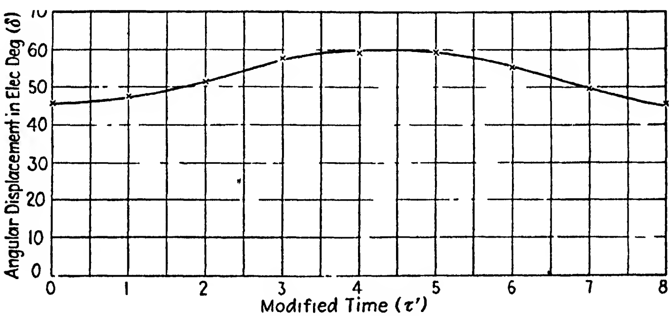


FIG. 294.—Swing curve for the system in Fig. 291(a) determined from precalculated data (Example 3).

stability situation would result. In other words, the system could then be designed for optimum stability. Unfortunately, however, the engineer is never confronted with such an ideal problem. The power systems of today have all grown from humble beginnings and have successively been extended by additions of generating stations, substations, and circuits as the load area and the load demand grew. Hence an ideal design and layout can never be attained from the stability standpoint. This does not mean, however, that attention should not be given to the effect of inertia when an addition is made.

An important problem from the stability standpoint is that of a station located at some distance supplying power over a transmission line to an already existing system containing generating stations, transmission and distribution networks, and loads. This has previously been emphasized, particularly in Chap. VIII.

From the stability standpoint, this problem is somewhat more clear-cut in that frequently it is sufficient to examine the stability of the distant station with respect to the receiving-end system considered as a unit. As far as inertia goes, the considerations limit themselves to the determination of the amount of inertia which it might be beneficial to have in the distant station. A decision must be based on a technical investigation as well as on economics. The aggregate inertia in the receiving-end system is usually much larger than that in the distant station. Under these circumstances, any increase in the inertia of the remote generators should definitely tend to improve the stability situation. The effect of increased inertia is primarily reflected in a longer period of oscillation. The results of this are substantially that (1) with a given fault clearing time and power transmitted, the amplitude of the oscillations will be reduced, (2) with a given fault clearing time, the power transmitted may be increased without loss of stability, *i.e.*, the transient-stability limit increases, and (3) with a given value of power transmitted, the clearing time may be lengthened. These effects, however, are all relatively moderate.[1] As a matter of fact, as already pointed out, the decision usually rests on whether or not it is economically justifiable to change the inertia appreciably from what is normal in machines of the rating to be installed.

Effect of Grounding the System's Neutral Points.—Until recently it has been the more or less standardized practice in this country to ground solidly the high-tension neutral points of all transformers. From this fact the inference should not immediately be drawn, however, that solid grounding always will be the best. As a matter of fact, as the systems grow in size and the effect of short circuits becomes more severe, it will probably be beneficial in most instances to introduce a certain amount of impedance in the ground connections. It may be mentioned in this connection, for instance, that on many systems in Europe it has been the practice for a number of years to omit grounding entirely. This indeed is beneficial from a stability standpoint, but it does increase the severity of the overvoltages during arcing

[1] Reference may be made to the previously mentioned report of the A.I.E.E. Subcommittee on Interconnection and Stability Factors, *loc. cit.*, in which a few curves indicating the effect of generator inertia upon system stability will be found.

grounds.[1] Overvoltages due to such grounds are materially reduced by solid grounding.

The zero-sequence impedances of a system depend largely upon the scheme employed for grounding the system's neutral points. Introducing impedance in the grounds will increase the zero-sequence impedances of the system and will, therefore, affect the performance of the system under single-phase line-to-ground, as well as under double-phase line-to-ground, fault conditions. If it were practicable to omit grounding entirely (as has been done on many systems in Europe for quite a number of years), the effect of single-phase line-to-ground faults would be nil, and double-phase-to-ground faults would become the equivalent of line-to-line faults. Operation of systems without grounded neutrals, however, requires that the transformers be insulated for full line voltage to ground. Hence the cost of the transformers for such operation is increased.

No general rule can be given for the amount of ground impedance which can be used to advantage. It is believed, however, that, when more experience has been gained and more information accumulated,[2] certain approximate working rules may be established which can be applied in less important cases which do not warrant complete analysis. Each system offers an individual problem, and before deciding upon a grounding scheme, an analysis of the characteristics of the particular system under various conditions of line-to-ground faults is necessary. A question which also arises is whether resistance or reactance should be used for grounding.

The purpose of the grounding may be (1) to reduce the shock impressed on the system by limiting the flow of short-circuit current, or (2) to introduce a braking effect which will reduce the amplitude of oscillations when a fault occurs.

If reactors are used in the neutrals, the first-mentioned effect is obtained. By the introduction of the reactance, the fault

[1] Lewis, W. W., "Transmission Line Engineering," pp. 222*ff.*, McGraw-Hill Book Company, Inc., New York, 1928; Peters, J. F. and J. Slepian, "Voltages Induced by Arcing Grounds," Vol. 42, pp. 478–489, *Trans. A.I.E.E.*, 1923.

[2] A limited number of results are given in R. D. Booth and O. G. C. Dahl, "Power System Stability, a Non-mathematical Review," *Gen. Elec. Rev.*, p. 677, December, 1930; and the report of the A.I.E.E. Subcommittee on Interconnection and Stability Factors, *loc. cit.*

current is reduced, the amounts of power which may flow past the fault are increased, and the shock impressed on the system decreased. If a resistor is used, either of the effects may be obtained, depending upon the value of the resistance. If the resistance is comparatively high, it will have a current-limiting effect as described for the reactor. If the resistance is less, it will not limit the current to an appreciable extent, but will absorb power while the fault currents flow. The fact that the resistance absorbs power increases the outputs of the generators and hence reduces the power differentials which create the oscillations. (This statement is, of course, based on the assumption that the fault causes the critical machine to speed up. Thus, in the neighborhood of synchronous condensers and actual or "equivalent" synchronous motors which slow down under fault conditions, resistance grounding would generally not be desirable.)

Which scheme is the better—the one utilizing reactors or the one utilizing resistors—depends upon the particular system layout and upon its inherent characteristics. Again no general rule can be given. The proper use of resistors and reactors for grounding can be determined only on the basis of facts obtained through detailed analysis of the performance of the particular system. It is entirely possible that in certain cases it may be advantageous to ground some high-tension neutral points through resistors and others through reactors. Furthermore, cases are conceivable where it would be desirable to make provisions for changing over from one type of grounding to another, depending upon operating conditions.[1]

It may be said, in general, that the selection of the proper grounding system is pretty much of a compromise. The effect of grounding is not by any means independent of the position of the fault. One grounding scheme may be very effective with a fault in a certain position on the transmission line and much less effective when the fault occurs at some other point on the same line. Furthermore, a grounding scheme which materially reduces the swings of the machines at, for instance, the sending end of the system may not so affect the machines at the other end of the system. All these things must be taken

[1] As an example may be quoted a hydroelectric station where the water situation permits generation during a part of the year only, while during the remainder the generators operate as synchronous condensers.

into account and properly weighed before a final decision is made as to what type of ground connections should be used. In selecting a grounding system, it may be worth while to keep three items definitely in mind: (1) the phase displacements between the generators at the sending and receiving ends of the system: (2) the system drift or speed change of the system as a whole; and (3) the insulation stresses to which the transformers

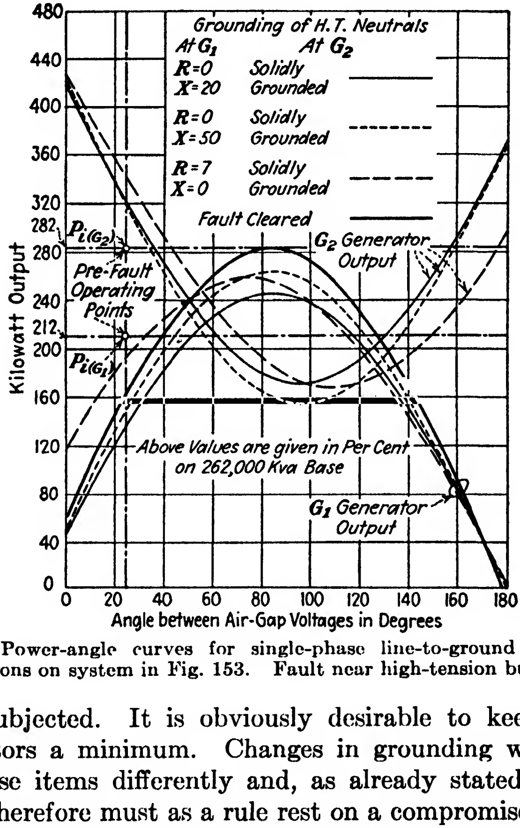


FIG. 295.—Power-angle curves for single-phase line-to-ground short-circuit conditions on system in Fig. 153. Fault near high-tension bus at G_1.

will be subjected. It is obviously desirable to keep each of these factors a minimum. Changes in grounding will usually affect these items differently and, as already stated, the final decision therefore must as a rule rest on a compromise.

Power Output during Fault Conditions as Affected by Grounding.—In order to get a picture of the influence of grounding on the outputs of the synchronous machines during fault conditions, consider the two-generator system shown in Fig. 153, Chap. XI. This represents a distantly located generating station supplying power over a transmission line to a receiving-end system which

for the purpose of analysis has been represented by an equivalent generator and an equivalent load directly at the receiving-end bus. The effect on the power output is best visualized by power-angle curves. Figures 295 and 296 show a few such curves for the two generating stations in the considered system under line-to-ground short-circuit conditions.[1] The system is described in further detail in Example 1, Chap. XI.

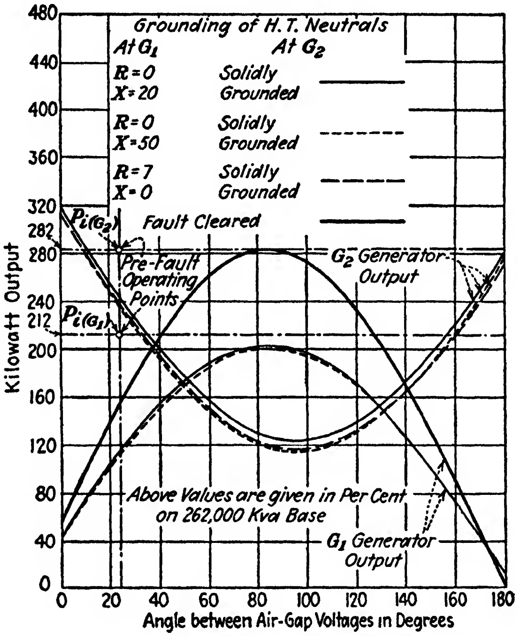


FIG. 296.—Power-angle curves for single-phase line-to-ground short-circuit conditions on system in Fig. 153. Fault near high-tension bus at G_2.

In Fig. 295 the short circuit is near the sending-end bus and in Fig. 296 near the receiving-end bus. For all these curves the high-tension neutral points of the receiving-end transformers are solidly grounded, while the sending-end neutral points are grounded through 20 per cent reactance, 50 per cent reactance, and 7 per cent resistance, respectively, the percentages being given on the basis of the rating of the hydroelectric station. These power-angle curves serve to show the effect of the methods of grounding indicated, and are useful (although several addi-

[1] See paper by BOOTH and DAHL, *loc. cit.*

tional curves would be required) in selecting an appropriate scheme of grounding of the high-tension neutral points in the particular layout considered.

It will be noted from examination of Fig. 296 that approximately the same effect on generator performance and loading can be obtained for fault conditions near the receiving end of the line by the use of a 7 per cent resistor as with a 50 per cent reactor. It is to be expected that the voltage stresses on the transformers will be less with the 7 per cent resistor than with the larger reactor. It will be noted in Fig. 295 that the use of the resistor will materially increase the output of the sending-end generating station at the time of fault. It will also be noted, however, that it similarly increases the load imposed on the generators at the receiving-end of this system. The increment of load placed on the receiving end of the system will, of course, neutralize to some extent the beneficial effects of the increment of load obtained on the hydroelectric station by slowing down the receiving-end generators; possibly not in the ratio of kilowatts of load imposed, owing to the fact that inertia of the receiving-end system will, in all probability, be materially greater than that in the sending end, because of its probable greater generator capacity and because of the inertia of rotating machines in the load. There is, however, a definite tendency to produce a system drift which may cause hunting between the two ends of the system as the governors respond to this drift. The superposition of this additional requirement for synchronizing power at the time when the ability of the system is restricted, first by the placing of a fault on one of the circuits and then later by the loss of the faulted circuit, is, of course, undesirable. The use of a resistance ground must be carefully analyzed to insure that apparent benefits are real.

Additional Voltage Stresses on the Transformers Caused by Grounding.—It has already been mentioned that grounding of the transformer neutral points will impose additional voltage stresses on the transformers. Instead of being at ground potential, the grounded side of the windings will, when ground impedance is used, during fault conditions be subjected to a voltage to ground. This voltage can be estimated by calculating the zero-sequence current which will flow. In connection with a stability study this may be done as the point-by-point solution

progresses, and the *maximum value* encountered used for the purpose of determining the voltage stresses. Relationships for the zero-sequence current desired can always be worked out in terms of the machine voltages and the circuit constants. For the particular two-generator system in Fig. 153, Chap. XI, for instance (discussed in some detail in Chap. XI and using the notation therein adopted), the following expressions may readily be established, attention being confined to the grounding impedance at the sending-end transformer bank.

Line-to-ground Fault on Phase a:

$$I_1^0 = \frac{1}{Z_1^0 + Z_1^- \dfrac{(1 + Z_1^0)/Z^0}{(1 + Z_1^-)/Z^-}} \left[\left(1 - \frac{Z_1^+ D}{B}\right) E_{1a}^+ + \frac{Z_1^+}{B} E_{2a}^+ \right] \tag{52}$$

Double Line-to-ground Fault on Phases b and c:

$$I_1^0 = -\frac{1}{Z_1^0} \left[\left(1 - \frac{Z_1^+ D}{B}\right) E_{1a}^+ + \frac{Z_1^+}{B} E_{2a}^+ \right] \tag{53}$$

The zero-sequence current is, as seen, obtained in terms of the positive-sequence voltages of the two machines and the circuit constants. The actual current carried by the aggregate grounding impedance is evidently three times the zero-sequence current per phase. If the grounding impedance is Z_n, the voltage above ground to which the grounded side of the transformer windings will rise is given by

$$V_n = I_n Z_n = 3 I_1^0 Z_n \tag{54}$$

EXAMPLE 4

Statement of Problem

In the two-generator system, Fig. 153, Chap. XI, a solid double line-to-ground fault occurs near the high-tension sending-end bus. The high-tension neutral points of the sending transformers are grounded through 10 per cent resistance, while those of the receiving-end bank are solidly grounded. It is desired to calculate the maximum voltage to which the grounded side of the sending-end transformer windings will rise above ground potential.

Details of the circuits, the load conditions, and other information necessary are as given in Example 1, Chap. XI.

Solution

By referring to the solution of Example 1, Chap. XI, part *d*, the expression for the zero-sequence current flowing in the sending-end transformer is

$$I_1^0 = k_1 E_{1a}^+ + k_2 E_{2a}^+ = 0.733/\delta + 140.9^\circ + 0.505/137.2^\circ \text{ per unit} \qquad (a)$$

where δ is the angle between E_{1a}^{+} and E_{2a}^{+}. The initial angle (at the occurrence of the fault) is $\delta_0 = 26.9$ deg. Solving equation (*a*) for different values of δ, the zero-sequence current per phase flowing in the sending-end transformers is obtained as tabulated below:

δ (degrees)	I_1^0 (per unit)	I_1^0 (per unit)
0	−0 941 + j0 807	1 24/139.4°
20	−1 064 + j0 583	1.22/151.3°
40	−1 104 + j0 331	1 15/163 3°
60	−1 056 + j0 081	1.06/175 6°
80	−0 925 − j0 137	0 935\171 6°
100	−0 728 − j0 298	0 785\157 7°
120	−0 487 − j0 380	0 617\142 0°

The values are plotted in Fig. 297, from which it is apparent that maximum zero-sequence current equal to 1.20 per unit flows at the instant the fault

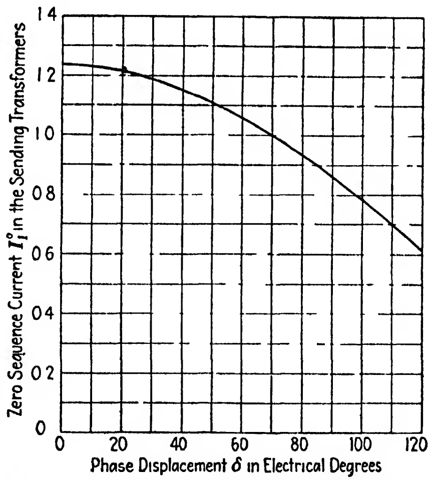


FIG. 297.—Zero-sequence current in the sending-end transformers of the two-generator system shown in Fig. 153 under conditions of a double line-to-ground fault near the high-tension sending-end bus.

occurs, since clearing (assuming a reasonable clearing time) will take place while the angle still increases.

The maximum voltage stress on the grounded side of the sending-end transformer winding is therefore [equation (55)].

$$V_n(\max) = 3 \times 1.20 \times 0.10 = 0.360 \text{ per unit}$$
$$= 45{,}800 \text{ volts.}$$

CHAPTER XIX

DAMPER WINDINGS AND THEIR EFFECT

The question as to whether or not damper windings are desirable, particularly in the generators of a hydroelectric station, is an important one.[1] It has been, and for that matter still is, somewhat debated. A general discussion of the advantages and disadvantages of damper windings will be given below, followed by the presentation of methods by which their effects may be calculated.

Types of Damper Windings; Their Merits and Drawbacks.—In general a distinction may be made between two types of damper windings, *viz.*, *low-resistance* and *high-resistance* damper windings. The operation of either type depends upon induction-motor or generator action, although they become active under essentially different conditions. The low-resistance dampers are active during low-frequency oscillations responding to the (usually small) slip between the damper bars and the positive-sequence air-gap flux. During such low-frequency oscillations high-resistance dampers will be very ineffective.

High-resistance dampers, on the other hand, may be desirable to give *braking* effect under certain conditions. Whenever a dissymmetrical fault occurs so that the generator supplies unbalanced currents containing negative-sequence components, the latter will set up a revolving field in the air gap of the machines traveling at twice synchronous speed with respect to their field structure. This field will, by induction, set up second-harmonic currents in the field windings proper, second-harmonic eddy currents in the pole faces, and second-harmonic currents in the dampers. If the dampers in which these second-harmonic currents flow have sufficient resistance, the torque

[1] BOOTH, R. D., and G. C. DAHL, "Power System Stability—a Non-mathematical Review," *Gen. Elec. Rev.*, Vol. 33, p. 677, December, 1930; Vol. 34, p. 131, February, 1931.

WAGNER, C. F., "Damper Windings for Water Wheel Generators," *Trans. A.I.E.E.*, Vol. 50, p. 140, March, 1931.

produced by them may become fairly appreciable. In cases where large positive differences between power input and power output are established during fault conditions, this additional torque may be advantageous in that a corresponding reduction is effected in the power differential acting on the shaft. Consequently, they add to the stability, and more initial power can be carried by the generators without synchronism being lost when the fault occurs.

The difficulty with high-resistance dampers lies in the fact that it is practically impossible to provide for sufficient heat dissipation to carry currents for any length of time. The short circuits usually last only a fraction of a second, so there is presumably no danger of burning out these damper windings under fault conditions. If, however, the generators have to carry unbalanced loads for longer times, serious trouble may be encountered. These unbalanced loads would produce a steady flow of current in the high-resistance dampers which might cause overheating and ultimate destruction.

No such trouble will be encountered with the low-resistance damper windings under reasonable degrees of unbalance. To be sure, these windings will also carry steadily second-harmonic currents in such cases, but, owing to their low resistance, the power loss in them will be low and should not give rise to dangerous heating.

As already stated, the value of low-resistance dampers is apparent during low-frequency oscillations and more specifically under the following conditions: (1) During the hunting which for several reasons normally occurs among generators in the same station: (2) To a very small extent during the oscillations which take place with respect to the receiving-end system under fault conditions; and (3) During the oscillations which take place subsequent to the clearing of the fault.

A certain amount of hunting is always likely to occur among generators of the same station. There is particularly a chance of such hunting becoming troublesome when high-speed excitation is used, owing to slight differences in the action of the exciters on the individual generators. The reactance of the ties between the machines in the same station is ordinarily low, so that appreciable induction-motor and generator action is obtained from the low-resistance damper windings even for

comparatively low slip velocities. It seems, therefore, that low-resistance damper windings will be quite effective in preventing such hunting from becoming troublesome and that such damper windings, therefore, offer a real advantage in so far as this hunting is concerned.

The effectiveness of low-resistance dampers, during the oscillations which occur when a fault is on the system, is very small and in many cases actually negligible. This is due to the fact that with the fault on the system the effective reactance of the equivalent circuit between sending- and receiving-end machines becomes very high. The amount of power which it is possible to transfer, therefore, by induction-generator and motor action is rather insignificant and will not materially affect the amplitude and duration of the oscillations.

When the fault is cleared, on the other hand, the situation is somewhat different. The reactance of the circuit between sending- and receiving-end machines is now much smaller (equal to the system reactance with the faulty section tripped out), and more damping action will be obtained for the same slip velocities. Furthermore, during the oscillations which occur subsequent to the clearing of a fault, the slip velocities on the average are likely to be higher than during the period when the fault is on. This is a second reason for more effective damping action during this period.

It has been previously stated that when a dissymmetrical fault occurs the shock impressed on the system is reduced if the negative- and zero-sequence reactances are high. The use of low-resistance damper windings will tend to decrease the negative-sequence reactance of the generators to some extent. In a typical hydroelectric-generator design, for instance (see Table 76), introduction of damper windings decreased the negative-sequence reactance from 40 to 27 per cent. Machine designers have stated, however, that by special damper design such a large reduction in this reactance can be avoided. Any reduction will, of course, somewhat increase the shock caused by a line-to-line or a line-to-ground fault. In a system where the negative-sequence reactance of the generators represents only a part of the total negative-sequence reactance, the increase in the shock is quite small. The power output of the generators, while the short circuit is on, will perhaps on the average not be decreased

more than 5 per cent by using generators with low-resistance damper windings.

From the foregoing consideration of the individual effects of low-resistance and high-resistance damper windings, it appears that each has its merits. The use of a combination, therefore, of high-resistance and low-resistance dampers which will produce both effects to a sufficient extent may be advantageous in some instances. Since "braking" action is desired primarily under negative-sequence loads resulting from dissymmetrical faults and "ordinary" induction-generator or motor action is desired only under the low-frequency operation corresponding to slip velocities, it should be possible to provide dampers for each function which will not interfere with the operation of the other. These effects can also be obtained by a combination of a high-resistance damper and a multiple-field winding consisting of an ordinary-field winding and a collar of heavy copper.

General Statements Referring to Calculation of Damper Effects.—Damping in synchronous machines is an asynchronous effect and as already stated, therefore, generally depends upon induction-motor and generator action. It must not be concluded a priori, however, that all damping calculations can be based on conventional induction-machine theory in view of the following points:

1. There is an important difference between an actual induction machine and a machine with damper windings. In the former the rotor circuits are symmetrical, while in the latter these circuits (*i.e.*, essentially the damper windings) are at best only of a semisymmetrical nature.

2. Another aspect in which the damping problem in synchronous machines ordinarily differs from the induction-machine problem is with regard to the slip. In an induction-machine under steady operation (to which the standard theory applies) the slip is *constant*. In a synchronous machine, on the other hand, under oscillatory conditions, the *slip varies all the time.*

The rapid and comprehensive recent development[1] in syn-

[1] Doherty, R. E., and C. A. Nickle, "Synchronous Machines," *Trans. A.I.E.E.;* Part I, p. 912, 1926; Part II, p. 927, 1926; Part III, p. 1, 1927; Part IV, p. 457, 1928.

Park, R. H., "Two-Reaction Theory of Synchronous Machines," Part I, *Trans. A.I.E.E.*, p. 716, July, 1929; Part II, *Trans. A.I.E.E.*, p. 352, June, 1933.

Crary, S. B., and M. L. Waring, "Torque-angle characteristics of

chronous-machine theory has embraced also asynchronous operation. Thus there are available in the literature formulas for damping which, however, usually apply under somewhat restricted conditions and definitely specified assumptions. The system involved consists ordinarily of a synchronous machine connected to an infinite bus through an external circuit. Doherty and Nickle give damping formulas for small sinusoidal low-frequency oscillations about a mean displacement angle. Park develops fundamental (operational) equations for damping under any condition of slip and any number of rotor circuits. Their application beyond the simplest cases is handicapped by serious difficulties of execution primarily due to the necessity of solving complicated determinantal equations. He adapts them for practical use, therefore, predicated on the following assumptions:

1. Zero armature and external resistance.
2. Zero field resistance.
3. Small slip.
4. Principal damping produced in one set of rotor windings only.

The result is an extremely useful equation which will be quoted later. Crary and Waring, using fundamental equations developed by Park, include damping (implicitly) in the point-by-point calculations of the oscillations of a synchronous machine following a disturbance.

In the further discussion of damping and the presentation of methods for practical calculations (to engineering accuracy) of the effect of damper windings the following point of view will be adopted:

1. When the slip is substantially constant and very large, the damping effect is definitely an average one independent of rotor position and may be calculated by conventional induction-machine theory. This covers "negative-sequence" damping and in effect, therefore, the action of high-resistance damper windings.

2. For small slips, constant or variable, the damping torque or power will be considered directly proportional to the slip ("linear damping") but, in general, dependent upon the angular position of the rotor with respect to the voltage impressed on the external circuit.

Synchronous Machines following System Disturbances," *Trans. A.I.E.E.*, p. 764, September, 1932.

3. In the case of a multimachine system the method of superposition may be applied also to the calculations of damping.

Extension of Conventional Induction-machine Theory.—As stated above there are special situations where the conventional induction-machine concepts may be applied also in calculating the effect of damper windings in synchronous machines. The standard equations for internal power and torque in an induction machine are usually developed in terms of the terminal voltage and the slip of the rotor with respect to this voltage. In connection with damping in synchronous machines, however, it is frequently desirable to use a voltage at some other point in the system and similarly to use the slip with respect to this voltage. Extension of induction-machine equations to include the effect of an external network, therefore, is necessary. This will first be done.

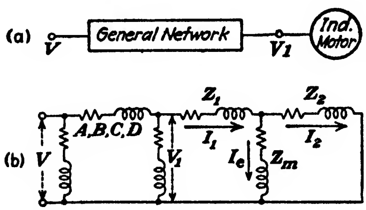


FIG. 298.—(*a*) Induction motor connected to a general network. (*b*) Equivalent circuit of (*a*).

In Fig. 298(*a*) is shown an induction machine connected to a general network whose performance may be specified in terms of general circuit constants A, B, C, and D. The known voltage at the end of this network is V. The induction machine (or symmetrical induction-motor elements of the synchronous machine) is represented by its equivalent circuit as shown in (*b*). In the derivations immediately following, the notation used in induction-motor analyses in Vol. I of this treatise will be adhered to. In order to calculate power (and torque), knowledge of the rotor current is essential.

In terms of the voltage V and the stator current I_1 the terminal voltage V_1 may be expressed by

$$V_1 = \frac{V - BI_1}{D} = I_1\left(Z_1 + \frac{Z_2 Z_m}{Z_2 + Z_m}\right) \tag{1}$$

which solved for the stator current gives

$$I_1 = \frac{V}{D\left(\frac{B}{D} + Z_1 + \frac{Z_2 Z_m}{Z_2 + Z_m}\right)} \tag{2}$$

Using these equations the rotor current may be written

$$I_2 = I_1 \frac{Z_m}{Z_2 + Z_m} = \frac{V \dfrac{Z_m}{Z_1' + Z_m}}{D\left(\dfrac{Z_1' Z_m}{Z_1' + Z_m} + Z_2\right)} = \frac{V_e}{\dfrac{Z_1' Z_m}{Z_1' + Z_m} + Z_2} \tag{3}$$

in which

$$Z_1' = \frac{B}{D} + Z_1 \tag{4}$$

and

$$V_e = \frac{V}{D} \cdot \frac{Z_m}{Z_1' + Z_m} \tag{5}$$

The induction-machine internal power (shaft power plus friction and windage) may now be written

$$P_2 = I_2^2 R_2 \frac{1-s}{s} = \frac{V^2 \left(\dfrac{Z_m}{Z_1' + Z_m}\right)^2 R_2 \dfrac{1-s}{s}}{D^2 \left(\dfrac{Z_1' Z_m}{Z_1' + Z_m} + Z_2\right)^2} = \frac{V_e^2 R_2 \dfrac{1-s}{s}}{\left(\dfrac{Z_1' Z_m}{Z_1' + Z_m} + Z_2\right)^2} \tag{6}$$

This equation is perfectly general and holds for generator action as well as motor action when the induction machine is fed through an external network. It is equally applicable when positive- and negative-sequence voltages are applied, provided the proper constants and slips are used. When the external network consists of a series impedance only, the general circuit constant D is unity, and Z_1' becomes the sum of the external impedance and the stator impedance of the machine.[1] With respect to the use of equation (6) in damper-winding problems, it is well again to emphasize that it is strictly correct only when (1) induction-machine action is confined to a single set of symmetrical rotor windings and when (2) the slip with respect to the applied voltage is constant.

[1] Equation (6) may be compared with equation (60) of Chap. V, the latter including the effect of an external impedance in the symbol Z_1. The difference in the two equations is due to the fact that in Chap. V the exciting current was assumed independent of the slip. When the equations are based on the equivalent circuit, as in the present case, however, the exciting current varies with the slip. This accounts for the presence of the excitation impedance Z_m in the first term of the denominator.

Practical Calculation of Damper Effects in Synchronous Machines.—Consider a synchronous machine with damper windings connected to an infinite bus through an external circuit and assume that for some reason or other—for instance, the occurrence of a fault—the synchronous equilibrium of this system is upset and that the angular velocity of the rotor departs from synchronous speed, *i.e.*, from the angular velocity of the voltage at the infinite bus. The motion of the machine rotor may or may not be oscillatory, although for stability the former is obviously necessary. Under these conditions both positive- and negative-sequence currents may flow in the machine producing two distinctly different values of slip. The practical methods of calculating the damping will be discussed below for each of the two sequences separately.

Positive-sequence Damping.—It will be assumed that the slip is sufficiently small so that the damping torque or power with sufficient accuracy may be considered directly proportional to the slip (linear damping). Under these conditions may be written

$$\text{Damping power} = P_d \frac{d\delta}{dt} = P_d s \tag{7}$$

where δ represents the displacement angle between the direct-axis excitation voltage of the machine and the voltage of the infinite bus, and the damping coefficient P_d is given by

$$P_d = a \sin^2 \delta + b \cos^2 \delta \tag{8}$$

A different form of writing this relationship is

$$P_d = (A - B \cos 2\delta) \tag{9}$$

where

$$A = \frac{a + b}{2} \qquad \text{and} \qquad B = \frac{a - b}{2} \tag{10}$$

With a single symmetrical rotor winding the constants a and b would be equal and the damping independent of rotor position. Were the slip constant (which it is not in ordinary damping problems) an average damping coefficient for motion covering a pair of poles (360 electrical degrees) may be obtained from equation (8) as follows:

$$P_d \text{ (average)} = \frac{2}{\pi}\int^{\frac{\pi}{2}} (a \sin^2 \delta + b \cos^2 \delta) d\delta = \frac{a + b}{2} = A \quad (11)$$

which, as seen, corresponds to the actual damping coefficient for a symmetrical winding.

The values of the constants a and b depend upon the constants of the armature and external circuits, and the number, type, and constants of the rotor circuits. Park has developed some simple expressions useful and sufficiently accurate in most practical situations based on the following assumptions:

1. Zero armature and external resistances.
2. Zero field resistance.
3. Principal damping produced in one set of rotor windings.

The second assumption is equivalent to constant field-flux linkages in the direct axis. Should there actually be changes in these flux linkages—as may well be the case—the effect of these on the action of the damper windings proper is ignored. Changes in flux linkages should, of course, when necessary, be reflected in the computation of synchronous output. They will, strictly speaking, also give rise to a damping action produced by the field winding, but this is seldom allowed for in stability calculations. The expressions for the two constants entering into the equation for linear damping are:

$$a = \frac{e^2(X'_d - X''_d)}{(X'_d + X_e)(X''_d + X_e)} \cdot \frac{1}{\alpha_d} \quad (12)$$

$$b = \frac{e^2(X_q - X''_q)}{(X_q + X_e)(X''_q + X_e)} \cdot \frac{1}{\alpha_q} \quad (13)$$

$$\frac{1}{\alpha_d} = \frac{X''_d + X'_e}{X'_d + X'_e} T''_{do} \quad (14)$$

$$\frac{1}{\alpha_q} = \frac{X'' + X'_e}{X_q + X'_e} T''_{qo} \quad (15)$$

In these the symbols (excepting the machine reactances which have been previously defined) have the following meaning:

e = voltage at infinite bus

α_d, α_q = amortisseur decrement factor in the two axes, respectively

T''_{do}, T''_{qo} = open-circuit, subtransient time constants in the two axes, respectively, expressed in electrical radians

X_e = external transfer reactance

X'_e = external driving-point reactance as viewed from the machine terminals.

In the absence of any shunt-leaks (as introduced, for example, by a fault) in the external circuits $X_e = X'_e$ and equations (12) and (13) combined with equations (14) and (15) may be written

$$a = \frac{e^2(X'_d - X''_d)}{(X'_d + X_e)^2} T''_{do} \tag{16}$$

$$b = \frac{e^2(X_q - X''_q)}{(X_q + X_e)^2} T''_{qo} \tag{17}$$

The open-circuit, subtransient time constants are determinable by computation from design data or by test and can readily be furnished by the manufacturer together with the other pertinent constants.

It should be noted that equations (12) and (13) as given are fully correct only when there *are no external shunt leaks.* With such leaks the *external* transfer reactance is not directly additive as indicated in the equations. Under these circumstances it may be preferable to use in the denominators the actual transfer reactances to back of transient and subtransient reactances, respectively. It is believed, however, that the difference, in general, between the two procedures will be insignificant. The effect of ordinary shunt loads should generally be comparatively minor on account of their relatively high impedance. Under fault conditions having the effect of a low-impedance shunt leak in the equivalent positive-sequence network, the damping power as a rule becomes negligibly small, anyway, and may not have to be considered at all.

As already stated, the given formulas are predicated on the assumption of zero field resistance or constant flux linkages in the direct axis. Going to the extreme in the opposite direction another possible assumption is that of *infinite* field resistance corresponding to *constant field current.*[1] Neglecting the (presumably very small) component of damping power contributed under these conditions by the field winding itself (*i.e.*, still considering the damping a single-rotor-circuit effect, *viz.*, that of the amortisseur-winding proper), the latter assumption may be reflected in the formulas by using therein direct-axis synchronous

[1] Under conditions of *constant* slip the constant-field-current assumption should be more nearly correct than that of constant flux linkages.

reactance X_d instead of direct-axis transient reactance X'_d. In fact the effect of changing flux linkages may be *approximated* in point-by-point calculations of damping by introducing a variable direct-axis reactance adjusted as the analysis progresses. It is very doubtful, however, whether any such attempt at refinement is actually worth while.

When the reactance externally to the machine is small, the two assumptions give nearly the same results. The difference may become appreciable, however, with large external reactances. The curves in Figs. 301, 302, 303, and 304 indicate this effect and also serve to illustrate the general variation in damping torque with time and displacement angle for both constant-slip and variable-slip conditions. The system and calculations on which these curves are based are described in Example 1.

Negative-sequence Damping.—As already stated, since the negative-sequence slip is large and substantially constant, the damping effect is an average one independent of rotor position, and the conventional induction-machine theory applies. Thus, the power expression equation (6) is applicable when the appropriate negative-sequence quantities are inserted. For negative sequence the slip is $2 - s$. The impedances involved are the same as for positive sequence with the exception that the resistance R_2 (the *equivalent* damper-winding resistance) will be somewhat larger than for positive sequence. It represents the effective damper resistance at approximately twice normal frequency and will be designated by R'_2.

In connection with the calculation of negative-sequence damping it is usually more convenient to use the first part of equation (6) involving the negative-sequence rotor current rather than the part determining the damping power in terms of voltage. This is due to the fact that in problems involving damping in synchronous machines the negative-sequence currents are set up by the presence of dissymmetrical faults and are not due to the actual application of any negative-sequence voltage to any part in the system. To be sure, negative-sequence voltages under dissymmetrical fault conditions will appear all along the network except at those specific points where positive-sequence voltages are definitely generated and maintained. It is usually more convenient, however, to determine the negative-sequence currents flowing into the various machines than to determine the

negative-sequence voltages. The negative-sequence damping power, therefore, is usually calculated by

$$P_2^- = -(I_2^-)^2 R_2' \frac{1-s}{2-s} \cong -(I_2^-)^2 \frac{R_2'}{2} \tag{18}$$

or, still more conveniently, by

$$P_2^- = P_0^-(I_2^-)^2 \tag{19}$$

where P_0^- represents the negative-sequence power loss at normal (unity) negative-sequence current, and I_2^- is the actual negative-sequence current expressed in per unit. The negative-sequence slip may be taken as 2 without sensible error even when the machine actually oscillates.

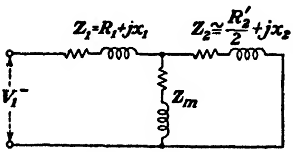


FIG. 299.—Equivalent negative-sequence circuit of induction motor.

It will be noted that knowledge of the resistance R_2' is necessary in equation (18). This, however, may be determined from the negative-sequence resistance of the machine, as shown below. Referring to Fig. 299, which gives the negative-sequence circuit of the machine, the negative-sequence impedance may be developed as follows:

$$Z^- = R_1 + jX_1 + \frac{\left(\frac{R_2'}{2} + jX_2\right)jX_m}{\frac{R_2'}{2} + j(X_2 + X_m)} =$$

$$R_1 + \frac{\frac{R_2'}{2}X_m^2}{\left(\frac{R_2'}{2}\right)^2 + (X_2 + X_m)^2} +$$

$$j\left\{X_1 + \frac{X_m[(R_2'/2)^2 + X_2(X_2 + X_m)]}{(R_2'/2)^2 + (X_2 + X_m)^2}\right. \tag{20}$$

The negative-sequence resistance is, therefore,

$$R^- = R_1 + \frac{\frac{R_2'}{2}X_m^2}{(R_2'/2)^2 + (X_2 + X_m)^2} \tag{21}$$

which when the magnetizing circuit is neglected ($X_m = \infty$) is equal to

$$R^- = R_1 + \frac{R_2'}{2} \tag{22}$$

From the latter two equations the effective double-frequency, damper-winding resistance may be determined. Using equation (21), the correct expression becomes

$$R_2' = \frac{X_m^2}{R^- - R_1} \pm \sqrt{\left(\frac{X_m^2}{R^- - R_1}\right)^2 - 4(X_2 + X_m)^2} \tag{23}$$

From equation (22), on the other hand, a slightly approximate expression is obtained as

$$R_2' = 2(R^- - R_1) = 2(R^- - R^+) \tag{24}$$

The last simple relation is usually accurate enough. It indicates that the effective damper resistance at double frequency is twice the difference between the negative-sequence and the positive-sequence resistances of the synchronous machine.[1] (Note that the positive-sequence resistance of a synchronous machine equals its armature resistance.) When the magnetizing circuit is neglected and equation (24) used, the negative-sequence rotor and stator current become identical (when referred to the same side).

[1] The negative-sequence resistance may be determined by test as follows: With the machine driven at rated speed and with a single-phase short circuit applied between two of its terminals (neutral excluded) the sustained armature current and the shaft power supplied by the driving motor are measured. The negative-sequence resistance equals one-half the ratio of the power input less friction-and-windage and core loss to the square of the line current. This may be shown by equating the shaft input to the aggregate losses in the machine. Thus

$$P_{\text{shaft}} = 3(I^+)^2R_1 + 3(I^-)^2(R_1 + R_2') + P_{F+W} + P_{\text{core}} \tag{a}$$

Remembering that the positive- and negative-sequence-current components in a single-phase, line-to-line short circuit are equal in magnitude to the current actually flowing I divided by the square root of three, equation (a) reduces to

$$P_{\text{shaft}} = I^2[2R^+ + 2(R^- - R^+)] + P_{F+W} + P_{\text{core}} = 2I^2R^- + P_{F+W} + P_{\text{core}} \tag{b}$$

from which the negative-sequence resistance becomes

$$R^- = \frac{P_{\text{shaft}} - P_{F+W} - P_{\text{core}}}{2I^2} \tag{c}$$

High- and Low-resistance Dampers; Double Dampers.—For ordinary operation of an induction motor it is usually desirable that the rotor resistance be as low as practicable. Over the operating range this results in a larger amount of power at a certain slip than would be obtained with larger rotor resistances. For positive-sequence damper effects the same situation holds true in synchronous machines. Since the positive-sequence slip ordinarily is small, the relationship between power and slip is nearly linear, and it is evident from equation (8) taken in conjunction with equations (12) to (17), for instance, that a small rotor resistance (large subtransient time constant) is helpful. Damper windings, therefore, to give maximum effect under low-frequency oscillations, *i.e.*, when positive-sequence action is obtained, should be *low-resistance* damper windings. Such windings usually consist of copper bars embedded in the pole faces and supplied with copper end connections. These end connections may or may not extend across the interpolar spaces resulting in so-called *complete* or *incomplete* damper windings as the case may be. The former have a lower subtransient reactance and a higher subtransient time constant in the quadrature axis than the former. As a consequence, the complete dampers have somewhat better damping characteristics than the incomplete ones at small and moderate displacement angles and also give, therefore, a better average damping effect.

When negative-sequence currents flow, however, the situation is different. The rotor resistance has only a minor effect on the negative-sequence current since the term $R_2'/(2 - s)$ is relatively insignificant as compared with the other terms in the denominator of equation (3). This being the fact, it can be seen from equation (15) for negative-sequence damping power that a high rotor resistance is rather desirable. To make the negative-sequence damping effective, therefore, it is better to have damper windings with high resistance. Such *high-resistance* damper windings may be designed on the same principle as the low-resistance ones. These windings will not be particularly effective during the low-frequency oscillations but will give damping during dissymmetrical fault conditions when negative-sequence currents flow.

In order to obtain the combined benefits of low- and high-resistance dampers, both types may be installed. In other words, the machine may be supplied with a *double* damper winding.

This has lately been done in connection with several large generator installations. A double damper would, in general, consist of high-resistance bars near the pole faces, with air gaps inserted in the magnetic circuits, and low-resistance bars somewhat farther embedded in the iron. With this arrangement the reactance of the low-resistance winding at double system frequency will be considerably higher than that of the high-resistance winding. The major part of the double-frequency current, therefore, will flow in the latter, assuming no appreciable saturation in its local magnetic circuits. At the frequency corresponding to slow oscillations (actual slip frequencies) reactance considerations are unimportant and the current distribution is almost entirely governed by the resistances. Hence under these conditions the low-resistance dampers will carry the predominant part of the current.

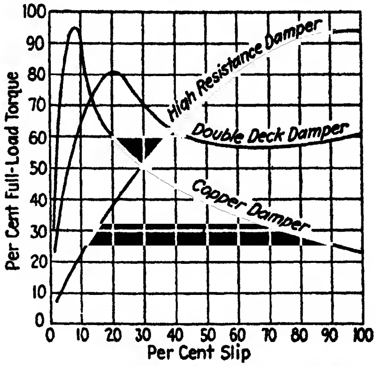


FIG. 300.—Torque-slip curves for three types of damper windings in large salient-pole synchronous machine. Curves are for constant-slip operation at fixed normal terminal voltage.

In Fig. 300 are shown torque-slip curves for a large salient-pole synchronous generator with three types of damper windings: (1) a copper damper (low-resistance damper); (2) a double-deck damper (combined low-resistance and high-resistance damper); and (3) a high-resistance damper. These curves are for *fixed normal voltage at the terminals and for operation at constant slip.*

It will be noted that at low slips the low-resistance damper exhibits the more nearly linear characteristic, the double-deck damper also possessing substantially a linear characteristic below 1 per cent slip. As would be expected, the low-resistance damper gives the largest effect in this slip region while the damping torque of the high-resistance damper is practically negligible. The latter, on the other hand, gives, as seen, large torques at high values of slip. This type of damper, therefore, will produce significant torques when negative-sequence currents flow.

Methods for computing these complete characteristics will not be included here but may be found in the literature.[1]

The formulas previously quoted [equations (7) *et seq.*] for linear damping are applicable to the straight portion of the torque-slip curves of the copper damper at low values of slip and apply also, at least as a first approximation, to the straight portion of the double-damper characteristic.

The presence of external reactance will in general materially reduce the damping torque below the values shown by these curves. Under oscillatory conditions it is evident that the damping performance is affected also by several additional factors (such as variable slip, variations in field current or flux linkages or both, and perchance also variations in the voltage applied to the external circuit) resulting in torque-slip relationships different from those indicated by the curves.

EXAMPLE 1

Statement of Problem

A salient-pole alternator supplies power to an infinite bus over an external circuit consisting of reactance only. It is desired to determine the damping action which may be produced under the conditions given below when this alternator is equipped with low-resistance amortisseurs (complete and incomplete). The generator constants are as follows:

	Incomplete amortisseur	Complete amortisseur
X_d	0.99	0.99
X'_d	0.36	0.36
X''_d	0.26	0.26
X_q	0.72	0.72
X'_q	0.72	0.72
X''_q	0.32	0.28
T''_{d0}	14 rad. (0.037 sec.)	14 rad. (0.037 sec.)
T''_{q0}	11 rad. (0.029 sec.)	28 rad. (0.074 sec.)

With external reactances of 0 and 0.4 per unit, respectively, calculate the damping torque:

[1] LINVILLE, T. M., "Starting Performance of Salient-Pole Synchronous Motors," *Trans. A.I.E.E.*, p. 531, April, 1930.

a. When the slip is constant and equal to 1 per cent.

b. When the machine oscillates sinusoidally with respect to the infinite bus between a minimum angle of 20 deg. and a maximum angle of 100 deg. with a period of 1 sec.

The solutions are to be based on constant field current as well as constant flux linkages in the direct axis. Plot the results so as to illustrate the variation in damping torque with rotor angular position. Also, plot angle, slip, and damping torque as functions of time.

Solution

Damping torque and damping power are, for all practical purposes, interchangeable when expressed on a per unit basis since the actual speed of the rotor does not differ materially from synchronous speed under the slip conditions usually encountered. Using equations (7) and (8) the damping torque becomes

$$T_D = (a \sin^2 \delta + b \cos^2 \delta)s \tag{a}$$

From equations (15) and (16)

$$a = \frac{e^2(X'_d - X''_d)}{(X'_d + X_e)^2}T''_{d0} \tag{b}$$

$$b = \frac{e^2(X_q - X''_q)}{(X_q + X_e)^2}T''_{q0} \tag{c}$$

The values of a and b are tabulated below for the various conditions imposed.

Constant Flux Linkages:

COMPLETE AMORTISSEUR

$X_e = 0$

$$a = \frac{(1.0)^2(0.36 - 0.26)14}{(0.36)^2} = 10.80$$

$$b = \frac{(1.0)^2(0.72 - 0.28)28}{(0.72)^2} = 23.77$$

$X_e = 0.4$

$$a = \frac{(1.0)^2(0.36 - 0.26)14}{(0.36 + 0.4)^2} = 2.42$$

$$b = \frac{(1.0)^2(0.72 - 0.28)28}{(0.72 + 0.4)^2} = 9.82$$

INCOMPLETE AMORTISSEUR

The coefficient a remains unaltered since only the subtransient reactance and time constant in the quadrature axis are affected.

$X_e = 0$

$$b = \frac{(1.0)^2(0.72 - 0.32)11}{(0.72)^2} = 8.49$$

$X_e = 0.4$

$$b = \frac{(1.0)^2(0.72 - 0.32)11}{(0.72 + 0.4)^2} = 3.51$$

Constant Field Current:

$X_e = 0$

$$a = \frac{(1.0)^2(0.99 - 0.26)}{(0.99)^2} = 10.43$$

$X_e = 0.4$

$$a = \frac{(1.0)^2(0.99 - 0.26)}{(0.99 + 0.4)^2} = 5.29$$

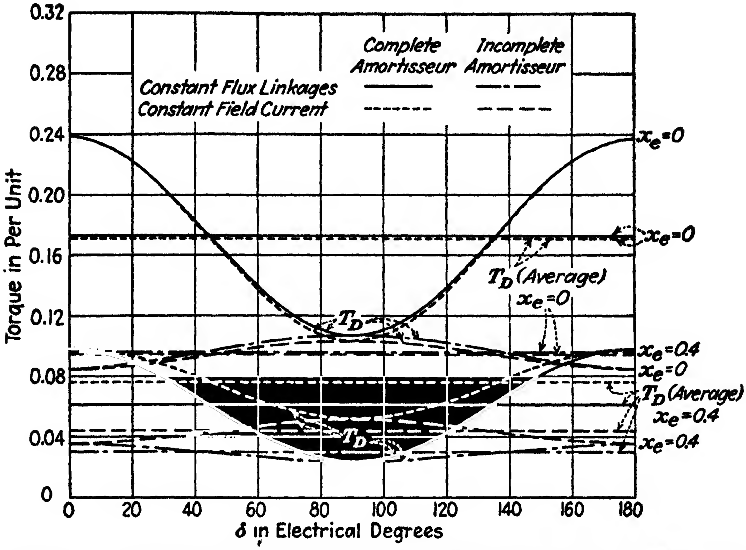


Fig. 301.—Curves of damping torque versus displacement angle; constant slip (Example 1).

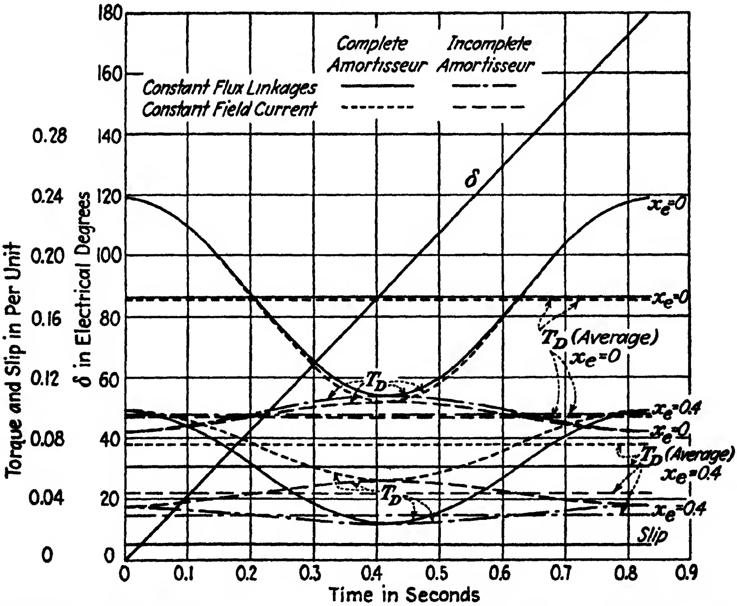


Fig. 302.—Curves of damping torque and displacement angle versus time; constant slip (Example 1).

Again the coefficient a is independent of complete or incomplete dampers as before. The coefficient b has the same values as for the corresponding conditions in the constant flux-linkage solution.

Part (*a*).—When the slip is constant the angular displacement is given by

$$\delta = s\omega_s t \tag{d}$$

where ω_s represents synchronous speed. In a 60-cycle system and with 1 per cent slip, equation (*d*) becomes

$$\delta = 0.01 \times 60 \times 360t = 216t \text{ elec. deg.} \tag{e}$$

The calculations have been carried out in Table 73 by assuming convenient values of δ and obtaining the corresponding values of t from the above equation and the values of T_D by means of equation (*a*). The results are plotted in Figs. 301 and 302. The average torque which for constant slip is given by [see equation (11)]

$$T_{D(\text{avg})} = T_{d(\text{avg})}s = \frac{a+b}{2}s \tag{f}$$

is also shown. The maxima and minima of these curves occur either at $\delta = 0$ deg. or $\delta = 90$ deg. and are repetitive thereafter.

Part (*b*).—In this case the displacement angle is given by

$$\delta = 60 - 40 \cos mt = 60 - 40 \cos \frac{2\pi}{T}t \text{ elec. deg.} \tag{g}$$

and the slip by

$$s = \frac{\omega}{\omega_s} = \frac{1}{\omega_s}\frac{d\delta}{dt} = \frac{80}{\omega_s T} \sin \frac{2\pi}{T}t \text{ per unit} \tag{h}$$

With the period $T = 1.0$ sec. and $\omega_s = 2\pi f = 120\pi$ rad. per sec. $= 21{,}600$ elec. deg. per sec.

$$\delta = (60 - 40 \cos 2\pi t) \text{ elec. deg.} \tag{i}$$

$$s = \frac{80\pi}{21{,}600} \sin 2\pi t = 0.0116 \sin 2\pi t \text{ per unit} \tag{j}$$

The calculations will be found in Table 74. The angle δ was considered the independent variable and the corresponding values of t and s computed from the above formulas, whereafter the damping torque was obtained by means of equation (*a*). The results are plotted in Figs. 303 and 304. It is of interest to observe that with a variable slip the maximum and minimum values of torque do not occur at intervals of 90 deg.

EXAMPLE 2

Statement of Problem

In the system shown in Fig. 153 representing a hydroelectric generating station supplying power over a 220-kv., double-circuit, transmission line to a

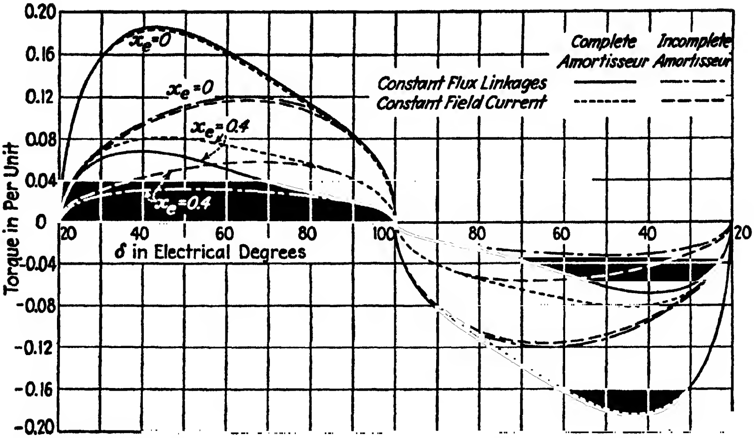


FIG. 303.—Curves of damping torque versus displacement angle; machine oscillating sinusoidally (Example 1).

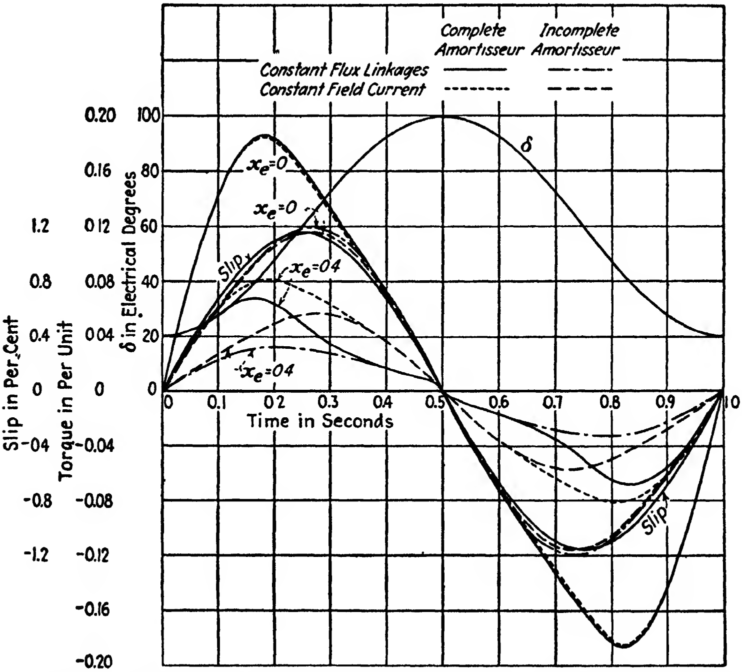


FIG. 304.—Curves of damping torque, slip and displacement angle versus time; machine oscillating sinusoidally (Example 1).

TABLE 73.—CALCULATION OF DAMPING TORQUE
Constant slip = 0.01
(Example 1)

	X_e per unit	δ elec. deg.	t sec.	Complete amortisseur					Incomplete amortisseur				
				$a \sin^2 \delta$	$b \cos^2 \delta$	T_d per unit	s per unit	T_D per unit	$a \sin^2 \delta$	$b \cos^2 \delta$	T_d per unit	s per unit	T_D per unit
Constant flux linkages in direct axis	0	0	0	0	23.77	23.77	0.01	0.238	0	8.49	8.49	0.01	0.085
	0	30	0.139	2.70	17.83	20.53	0.01	0.205	2.70	6.37	9.07	0.01	0.091
	0	60	0.278	8.10	5.94	14.04	0.01	0.140	8.10	2.12	10.22	0.01	0.102
	0	90	0.417	10.80	0	10.80	0.01	0.108	10.80	0	10.80	0.01	0.108
	0	120	0.556	8.10	5.94	14.04	0.01	0.140	8.10	2.12	10.22	0.01	0.102
	0	150	0.695	2.70	17.83	20.53	0.01	0.205	2.70	6.37	9.07	0.01	0.091
	0	180	0.834	0	23.77	23.77	0.01	0.238	0	8.49	8.49	0.01	0.085
	0.4	0	0	0	9.820	9.82	0.01	0.098	0	3.510	3.510	0.01	0.035
	0.4	30	0.139	0.605	7.365	7.97	0.01	0.080	0.605	2.633	3.238	0.01	0.032
	0.4	60	0.278	1.815	2.455	4.27	0.01	0.043	1.815	0.878	2.693	0.01	0.027
	0.4	90	0.417	2.420	0	2.42	0.01	0.024	2.420	0	2.420	0.01	0.024
	0.4	120	0.556	1.815	2.455	4.27	0.01	0.043	1.815	0.878	2.693	0.01	0.027
	0.4	150	0.695	0.605	7.365	7.97	0.01	0.080	0.605	2.633	3.238	0.01	0.032
	0.4	180	0.834	0	9.820	9.82	0.01	0.098	0	3.510	3.510	0.01	0.035
Constant field current	0	0	0	0	23.77	23.77	0.01	0.238	0	8.49	8.49	0.01	0.085
	0	30	0.139	2.61	17.83	20.44	0.01	0.204	2.61	6.37	8.98	0.01	0.090
	0	60	0.278	7.82	5.94	13.76	0.01	0.138	7.82	2.12	9.94	0.01	0.099
	0	90	0.417	10.43	0	10.43	0.01	0.104	10.43	0	10.43	0.01	0.104
	0	120	0.556	7.82	5.94	13.76	0.01	0.138	7.82	2.12	9.94	0.01	0.099
	0	150	0.695	2.61	17.83	20.44	0.01	0.204	2.61	6.37	8.98	0.01	0.090
	0	180	0.834	0	23.77	23.77	0.01	0.238	0	8.49	8.49	0.01	0.085
	0.4	0	0	0	9.820	9.820	0.01	0.098	0	3.510	3.510	0.01	0.035
	0.4	30	0.139	1.32	7.365	8.685	0.01	0.087	1.32	2.633	3.953	0.01	0.040
	0.4	60	0.278	3.97	2.455	6.425	0.01	0.064	3.97	0.878	4.848	0.01	0.048
	0.4	90	0.417	5.29	0	5.290	0.01	0.053	5.29	0	5.290	0.01	0.053
	0.4	120	0.556	3.97	2.455	6.425	0.01	0.064	3.97	0.878	4.848	0.01	0.048
	0.4	150	0.695	1.32	7.365	8.685	0.01	0.087	1.32	2.633	3.953	0.01	0.040
	0.4	180	0.834	0	9.820	9.820	0.01	0.098	0	3.510	3.510	0.01	0.035

TABLE 74.—CALCULATION OF DAMPING TORQUE
Sinusoidal Oscillations
(Example 1)

	X_e per unit	δ elec. deg.	t sec.		Complete amortisseur					Incomplete amortisseur				
					$a \sin^2 \delta$	$b \cos^2 \delta$	T_d per unit	s per unit	T_D per unit	$a \sin^2 \delta$	$b \cos^2 \delta$	T_d per unit	s per unit	T_D per unit
Constant flux linkages in direct axis	0	20	0	1.0	1.264	20.989	22.25	0	0	1.264	7.505	8.77	0	0
	0	25	0.081	0.919	1.929	19.525	21.45	±0.00562	±0.121	1.929	6.974	8.90	±0.00562	±0.050
	0	30	0.115	0.885	2.700	17 828	20 53	±0.00767	±0.157	2.700	6.368	9.07	±0.00767	±0.070
	0	40	0.167	0.833	4.463	13.948	18.41	±0.01005	±0.185	4.463	4.982	9.45	±0.01005	±0.095
	0	50	0 210	0.790	6.337	9.822	16.16	±0.01123	±0.181	6.337	3.508	9.85	±0.01123	±0.111
	0	60	0.250	0.750	8.100	5.943	14 04	±0.01160	±0.163	8.100	2.123	10.22	±0.01160	±0.119
	0	70	0.290	0.710	9.536	2.781	12.32	±0 01123	±0.138	9.536	0.993	10.53	±0.01123	±0.118
	0	80	0.333	0.667	10.474	0.715	11.19	±0.01005	±0.112	10.474	0.256	10.73	±0.01005	±0.108
	0	90	0.385	0.615	10.800	0	10.80	±0.00767	±0.083	10.800	0	10.80	±0.00767	±0.083
	0	95	0.419	0.581	10.718	0.181	10.90	±0.00562	±0.062	10.718	0.065	10.78	±0.00562	±0.061
	0	100	0.500		10.474	0.715	11.19	0	0	10.474	0.256	10.73	0	0
	0.4	20	0	1.0	0.283	8.671	8.95	0	0	0.283	3.099	3.38	0	0
	0.4	25	0.081	0.919	0.432	8 066	8.50	±0.00562	±0.048	0.432	2.883	3.32	±0.00562	±0.019
	0.4	30	0.115	0.885	0.605	7.365	7.97	±0 00767	±0.061	0.605	2.633	3.24	±0.00767	±0.025
	0.4	40	0.167	0.833	1.000	5.762	6.76	±0.01005	±0.068	1.000	2.060	3.06	±0.01005	±0.031
	0.4	50	0.210	0.790	1.420	4.058	5.48	±0.01123	±0.062	1.420	1.450	2.87	±0.01123	±0.032
	0.4	60	0.250	0.750	1.815	2.455	4.27	±0.01160	±0.050	1.815	0.878	2.69	±0.01160	±0.031
	0.4	70	0.290	0.710	2.137	1.149	3.29	±0.01123	±0.037	2.137	0.411	2.55	±0.01123	±0.029
	0.4	80	0.330	0.667	2.347	0.296	2.64	±0.01005	±0.027	2.347	0.106	2.45	±0.01005	±0.025
	0.4	90	0.385	0.615	2.420	0	2.42	±0.00767	±0.019	2.420	0	2.42	±0.00767	±0.019
	0.4	95	0.419	0.581	2.402	0.075	2.48	±0.00562	±0.014	2.402	0.027	2.43	±0.00562	±0.014
	0.4	100	0.500		2.347	0.296	2.64	0	0	2 347	0.106	2.45	0	0

	X_e per unit	δ elec deg	t sec	Complete amortisseur					Incomplete amortisseur				
				$a \sin^2 \delta$	$b \cos^2 \delta$	T_d per unit	s per unit	T_D per unit	$a \sin^2 \delta$	$b \cos^2 \delta$	T_d per unit	s per unit	T_D per unit
Constant field current	0	20	0 1 0	1 220	20 989	22 21	0	0	1 220	7 505	8 73	0	0
	0	25	0 081 0 919	1 863	19 525	21 39	±0 00562	±0 120	1 863	6 974	8 84	±0 00562	±0 050
	0	30	0 115 0 885	2 608	17 828	20 44	±0 00767	±0 157	2 608	6 368	8 98	±0 00767	±0 069
	0	40	0 167 0 833	4 310	13 948	18 26	±0 01005	±0 183	4 310	4 982	9 29	±0 01005	±0 093
	0	50	0 210 0 790	6 120	9 822	15 94	±0 01123	±0 179	6 120	3 508	9 63	±0 01123	±0 108
	0	60	0 250 0 750	7 823	5 943	13 77	±0 01160	±0 160	7 823	2 123	9 95	±0 01160	±0 115
	0	70	0 290 0 710	9 210	2 781	11 99	±0 01123	±0 135	9 210	0 993	10 20	±0 01123	±0 115
	0	80	0 330 0 667	10 115	0 715	10 83	±0 01005	±0 109	10 115	0 256	10 37	±0 01005	±0 104
	0	90	0 385 0 615	10 430	0	10 43	±0 00767	±0 080	10 430	0	10 43	±0 00767	±0 080
	0	95	0 419 0 581	10 351	0 181	10 53	±0 00562	±0 059	10 351	0 065	10 42	±0 00562	±0 059
	0	100	0 500	10 115	0 715	10 83	0	0	10 115	0 256	10 37	0	0
	0 4	20	0 1 0	0 619	8 671	9 29	0	0	0 619	3 099	3 72	0	0
	0 4	25	0 081 0 919	0 945	8 066	9 01	±0 00562	±0 051	0 945	2 883	3 83	±0 00562	±0 022
	0 4	30	0 115 0 885	1 323	7 365	8 69	±0 00767	±0 067	1 323	2 633	3 96	±0 00767	±0 030
	0 4	40	0 167 0 833	2 186	5 762	7 95	±0 01005	±0 080	2 186	2 060	4 25	±0 01005	±0 043
	0 4	50	0 210 0 790	3 104	4 058	7 16	±0 01123	±0 080	3 104	1 450	4 55	±0 01123	±0 051
	0 4	60	0 250 0 750	3 968	2 455	6 42	±0 01160	±0 074	3 968	0 878	4 85	±0 01160	±0 056
	0 4	70	0 290 0 710	4 671	1 149	5 82	±0 01123	±0 065	4 671	0 411	5 08	±0 01123	±0 057
	0 4	80	0 330 0 667	5 130	0 296	5 43	±0 01005	±0 055	5 130	0 106	5 24	±0 01005	±0 053
	0 4	90	0 385 0 615	5 290	0	5 29	±0 00767	±0 041	5 290	0	5 29	±0 00767	±0 041
	0 4	95	0 419 0 581	5 250	0 075	5 33	±0 00562	±0 030	5 250	0 027	5 28	±0 00562	±0 030
	0 4	100	0 500	5 130	0 296	5 43	0	0	5 130	0 106	5 24	0	0

receiving-end system, a solid double-line-to-ground fault occurs near the high-tension sending-end bus. The generators are equipped with high-resistance damper windings giving them 12 per cent negative-sequence resistance.

Determine the damping power produced by these windings during the period the fault is on the system, and plot the values versus displacement angle.

Details of the circuits, the load conditions, and other necessary data are as specified in Example 1, Chap. XI.

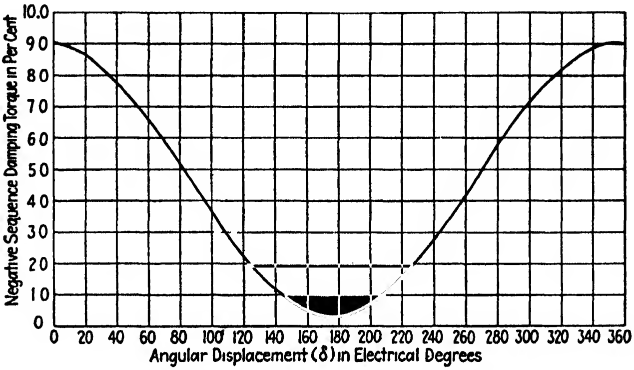


FIG. 305.—Negative-sequence damping torque of hydroelectric generators in system shown in Fig. 153 under conditions of a double line-to-ground fault near the high-tension sending-end bus (Example 2).

Solution

Only the negative-sequence damping action will be considered. From the solution of Example 1, Chap. XI, the negative-sequence current supplied by the generator is as a function of the displacement angle

$$I^- = 0.522\underline{/\delta + 85.0^\circ} + 0.360\underline{/81.3^\circ} \text{ per unit}$$

Using equations (18) and (24) the damping power is

$$\begin{aligned} P_2^- &= -(I^-)^2(R^- - R^+) \\ &= -(I^-)^2(0.12 - 0.004) \\ &= -0.116(I^-)^2 \text{ per unit} \end{aligned}$$

Computations of P_2^- for every 20-deg. angle are summarized in Table 75. A curve of the damping power in per cent versus displacement angle is plotted in Fig. 305. This damping power is always braking or decelerating power.

Benefits of Damper Windings during Faults.—In order to get an idea of the character of the damper effects during dissymmetrical fault conditions, consider the system in Fig. 306. This

TABLE 75.—CALCULATION OF NEGATIVE-SEQUENCE DAMPING POWER (Example 2)

δ deg	I^- per unit	$P_2^- = -0\ 116(I^-)^2$ per unit
0	0 882	−0 0902
20	0 865	−0 0868
40	0 813	−0 0767
60	0 754	−0 0660
80	0 665	−0 0513
100	0 560	−0 0364
120	0 441	−0 0226
140	0 315	−0 0115
160	0 204	−0 0048
180	0 165	−0 0032
200	0 242	−0 0068
220	0 377	−0 0165
240	0 486	−0 0274
260	0 600	−0 0418
280	0 701	−0 0571
300	0 782	−0 0709
320	0 840	−0 0819
340	0 874	−0 0886
360	0 882	−0 0902

consists, as seen, of a generating station supplying power to a large system (infinite bus) over a transmission line. Let a dissymmetrical fault occur at some place along this line. The generating station will then first speed up and, assuming stability

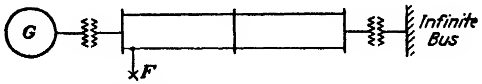


FIG. 306.—Generator supplying a large system (infinite bus) over a two-circuit transmission line. A dissymmetrical fault occurs near the high-tension sending-end bus.

is maintained, will oscillate with respect to the infinite bus. In Fig. 307 are indicated the angle-time as well as the slip-time curve with respect to the infinite bus.

The total damping power is the algebraic sum of the positive-sequence and the negative-sequence damping power. The actual slip s is always small. Adhering to the conventional induction-machine conception, the slip is negative when the

machine speeds away from the infinite bus. During these conditions positive-sequence *induction-generator action* is obtained, *i.e.*, the electromagnetic torque and power are negative and the machine supplies power to the line. When the machine is approaching the infinite bus, the slip is *positive*. This results in *induction-motor action, i.e.*, power is supplied to the machine. The negative-sequence slip $2 - s$ is evidently *always positive*. This means that the negative-sequence damping is always *induction-motor action*, *i.e.*, it always supplies negative (retarding) electromagnetic torque and power. (Note negative sign in equation (18).) The effects just discussed may be summarized as follows:

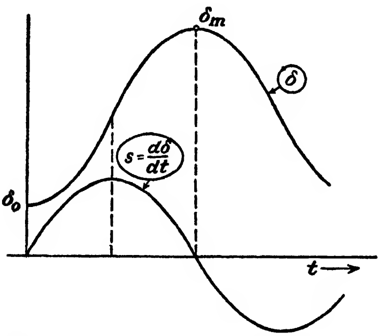


FIG. 307.—Angle- and slip-time curves for the system in Fig. 306.

Positive-sequence (or *actual*) slip (s)
$= \text{negative} \longrightarrow$ induction-generator action, *i.e.* $P_2^+ = \text{negative}$
Positive-sequence (or *actual*) slip (s)
$= \text{positive} \longrightarrow$ induction-motor action, *i.e.* $P_2^+ = \text{positive}$
Negative-sequence slip $(2 - s)$
$= \text{always positive} \longrightarrow$ induction-motor action, *i.e.* $P_2^- = \text{negative}$

On the basis of the above it may now be concluded that for this particular system (1) low-resistance dampers act in the right direction all the time and (2) high-resistance dampers acting on negative sequence give the proper effect up to the maximum angle. Thereafter their effect would be detrimental rather than beneficial. However, in practically all commercial systems the fault will have been cleared before the maximum angle is reached. Clearing the fault eliminates the negative-sequence current and hence the negative-sequence damper action. Under such conditions, therefore, high-resistance dampers are nothing but advantageous.

It is clear, then, that high-resistance damper windings are to be considered only when the machine in question has a tendency to speed up. This limits their possible use to generators. High-resistance dampers would never be desirable in synchronous

motors and condensers since these types of machine have a general tendency to slow down upon the occurrence of a fault and in perhaps the majority of cases will continue to slow down for at least an appreciable part of the duration of the fault.

Effect of Damper Windings on Machine Reactances.—The presence of damper windings does affect the machine reactances although the variation can be controlled by design. From the standpoint of stability under dissymmetrical fault conditions, it is usually desirable that the direct-axis transient reactance be low and the negative-sequence reactance relatively high. The presence of damper windings, however, reduces the latter even though the former is kept constant. In order to get an idea of the changes in circuit reactances, a number of values for a generator with various kinds of damper windings have been collected in Table 76. The figures relate to a salient-pole machine where the design is assumed to be such that the synchronous reactances as well as the transient reactances in both axes remain constant for the various types of damper windings considered. Under these conditions changes will occur in the subtransient reactances as well as in the negative-sequence

TABLE 76.—TYPICAL PER-UNIT REACTANCES OF A LARGE SALIENT-POLE GENERATOR

Short-circuit ratio = 1.25

	x_d	x_q	x_d'	x_q'	x_d''	x_q''	x^-
No dampers; normal field winding	0.780	0.510	0.300	0.510	0.300	0.510	0.400
High-resistance dampers	....	....	....	....	0.261	0.407	0.334
Low-resistance dampers	....	....	....	....	0.228	0.318	0.273
Low-resistance dampers and multiple field	....	....	....	....	0.213	0.318	0.265
Low-resistance and high-resistance dampers	....	....	....	....	0.245	0.362	0.303
High-resistance dampers and multiple field	....	....	....	....	0.240	0.407	0.323
Multiple field winding	....	....	....	....	0.252	0.510	0.376

NOTE: x_d = synchronous reactance; direct axis.
x_q = synchronous reactance; quadrature axis.
x_d' = positive-sequence transient reactance; direct axis.
x_d'' = positive-sequence subtransient reactance; direct axis.
x_q' = positive-sequence transient reactance; quadrature axis.
x_q'' = positive-sequence subtransient reactance; quadrature axis.
x^- = negative-sequence reactance (transient and sustained).

reactances. In regard to the latter, the table indicates a drop from 40 per cent without damper windings to 26.5 per cent when a combination of low-resistance dampers and a field collar is used. These figures evidently should merely be considered as *indicative* of the general trend which may be expected when no particular effort is made to prevent the negative-sequence reactance from decreasing. It is undoubtedly possible, however, by attention to design to counteract and avoid such a substantial decrease.

Reduction in the negative-sequence reactance will reduce the synchronous output of a machine under fault conditions. The question is then whether the damping effect secured is sufficiently material to offset this decrease in synchronous output. The decrease in synchronous output depends upon the external reactance between the machine in question and the other machine or machines with respect to which it oscillates. If the external reactance is large, as it is when an isolated hydroelectric station transmits power over a long transmission line, the decrease in maximum synchronous output will not be large. In Example 3 the maximum amplitude of the synchronous output is calculated for a generating installation having machines with damper windings in connection with what may be considered a representative transmission system. It will be noted that the reduction in synchronous output due to the decrease in negative-sequence reactance, in this case considered, is comparatively insignificant. In Example 4 similar calculations are based on a simple system for a range of values of negative-sequence reactance. The effect of a reduction in the latter is somewhat larger for this system than for the one used in the first example.

EXAMPLE 3

Statement of Problem

Consider a transmission system of the type shown in Fig. 153. Without damper windings the transient reactance of the proposed generators for the hydroelectric station is 30 per cent, and the negative-sequence reactance is 36 per cent. With damper windings the transient reactance and the negative-sequence reactance would be 27 per cent and 25 per cent, respectively. Data on the remainder of the system are not given here. These are the same in both cases, however. It is desired to compare the maximum amplitude of the power-angle curve for the two cases when a solid line-to-line short circuit occurs near the high-tension bus of the hydroelectric station.

Solution

The comparison will be made by calculating the chart constants for the output of the hydroelectric generators assuming constant flux-linkage conditions in these machines as well as in the equivalent generator at the receiving end of the system. The constants of the performance charts, *i.e.*, horizontal and vertical displacements of the center A_s and B_s and the radius of the unity-ratio voltage circle C_s giving the hydroelectric station output under the given fault conditions, have been calculated to be for the two cases, respectively:

Case 1	Case 2
$A_s = 20{,}580$ kw.	$A_s = 18{,}870$ kw.
$B_s = -436{,}500$ kva.	$B_s = -479{,}000$ kva.
$C_s = 108{,}200$ kva.	$C_s = 105{,}000$ kva.

Hence

$$P_m(1) = 108{,}200 + 20{,}580 = 128{,}780 \text{ kw.}$$
$$P_m(2) = 105{,}000 + 18{,}870 = 123{,}870 \text{ kw.}$$

$$P_m(1) - P_m(2) \cong 5{,}000 \text{ kw.}$$

The figures show that there is a difference of approximately 5,000 kw. in the maximum synchronous output for the two cases. A 10 per cent reduction in the direct-axis transient reactance associated with a 30 per cent reduction in the negative-sequence reactance will reduce the maximum output by about 3.9 per cent. This result may also be approximately stated as follows, *viz.*, that 1 per cent reduction in negative-sequence reactance causes somewhat over $\frac{1}{10}$ of 1 per cent decrease in the maximum power output during a line-to-line fault.

EXAMPLE 4

Statement of Problem

A generating station supplies power to an infinite bus, as indicated in Fig. 308(a). The transient reactance of the generators is 30 per cent, the reactance of each of the transformer banks 10 per cent, and the reactance of the line 40 per cent. Considering conditions during a sustained single-phase line-to-line fault on the line side of the sending-end transformers, calculate and compare the maximum generator outputs for values of negative-sequence reactance ranging from 10 to 40 per cent. Assume that the voltage behind the transient reactance in the generator is 110 per cent and that of the infinite bus exactly 100 per cent.

Solution

Referring to Fig. 308(b) and using per-unit quantities, the symmetrical shunt replacing the fault in the positive-sequence system is given by

$$X_F = \frac{(0.1 + X_1^-)0.5}{0.6 + X^-} \qquad (a)$$

and the transfer reactance by

$$X_{12} = \frac{0.4X_F + 0.5X_F + 0.4 \times 0.5}{X_F} = 0.9 + \frac{0.2}{X_F} \qquad (b)$$

The maximum power output is, therefore,

$$P_1\ (\max) = \frac{E_1E_2}{X_{12}} = \frac{1.2}{0.9 + \dfrac{0.2}{X_F}} = \frac{1.2}{D} \qquad (c)$$

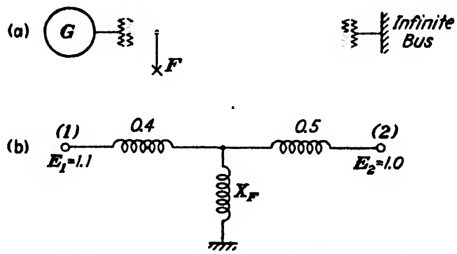


FIG. 308.—(*a*) Generator supplying a large system (infinite bus) over a transmission line with sending- and receiving-end transformers. (*b*) Equivalent positive-sequence circuit of (*a*) under fault conditions (Example 4).

The calculated per-unit values for different negative-sequence reactances are tabulated below:

X^-	X_F	D	P_1 (max)
0.1	0.143	2.30	0.522
0.2	0.188	2.06	0.582
0.3	0.222	1.80	0.667
0.4	0.250	1.70	0.706

It will be noted that the maximum power output decreases as the negative-sequence reactance is lowered. Using the output at 40 per cent negative-sequence reactance as basis, the reduction in output for 1 per cent decrease in negative-sequence reactance is approximately ¼ of 1 per cent.

Inclusion of Damping in the Swing Equation of the Machine.— For any machine the general relation holds that the inertia power plus the output power equals the input power, as indicated in

$$P_{\text{inertia}} + P_{\text{output}} = P_{\text{input}} \qquad (25)$$

As previously discussed in Chap. XII, the output power represents the *total* output and is, when damping is present, the sum of the synchronous output and the power generated by induction-machine action. Equation (25) may then be written

$$P_{\text{inertia}} + P_{\text{damping}} + P_{\text{synchronous}} = P_{\text{input}} \qquad (26)$$

In solving this equation (as, for instance, by a point-by-point analysis) the damping power may be calculated as described above. Damping power caused by positive- and negative-sequence currents must be determined separately. The positive-sequence damping power may be calculated in terms of any constant positive-sequence voltage in the system, provided the slip of the machine with respect to this same voltage be used. The negative-sequence damping power is essentially independent of the slip since for all small slips the negative-sequence slip is very nearly equal to 2. Whether or not a linear relationship actually exists between the positive-sequence damping power and the slip depends, as already stated, on the problem at hand. In most practical problems, however, it is usually assumed sufficiently accurate to consider the positive-sequence damping as directly proportional to the slip. This simplifies the calculations and, in fact eliminates very material—perhaps almost insurmountable—difficulties. It is evident that such a linear relationship can never exist for the negative-sequence damping. Since, however [as indicated by equation (18)], this power is essentially independent of the slip, it may be combined with the input. In other words, the presence of this type of damping may be looked upon as equivalent to a decrease in the input, and it may, therefore, be subtracted from the latter for each step in the point-by-point solution.[1]

[1] The analysis is usually based on the positive-sequence network in which the effect of the dissymmetrical fault is simulated by a symmetrical shunt impedance. Under these conditions positive- as well as negative-sequence currents flow, and positive- as well as negative-sequence power exists at the terminals of the machine. Omitting the inertia power and disregarding rotational losses, the power balance between input and output is given by the following equation:

$$
\begin{aligned}
P_{\text{input}} &= (1)\left[\text{terminal output}\begin{pmatrix}\text{positive sequence}\\ \text{negative sequence}\end{pmatrix}\right] \\
&\quad + (2)\left[\text{stator copper loss}\begin{pmatrix}\text{positive sequence}\\ \text{negative sequence}\end{pmatrix}\right] \\
&\quad + (3)\ (\text{negative-sequence rotor copper loss}) \\
&= [(1) + (2) + \tfrac{1}{2}\,(3)] + [\tfrac{1}{2}\,(3)] \\
&= \begin{bmatrix}\text{positive-sequence power}\\ \text{supplied to equivalent circuit}\end{bmatrix} + \left[\tfrac{1}{2}\begin{pmatrix}\text{negative-sequence}\\ \text{rotor copper loss}\end{pmatrix}\right] \\
&= P^{+} + (I^{-})^{2}\frac{R_2'}{2} = P^{+} + (I^{-})^{2}(R^{-} - R^{+})
\end{aligned}
$$

It will be noted that, when treated in this manner, the positive-sequence

If the damping is to be calculated on the basis of symmetrical-rotor theory, the damping power is merely a constant times the slip. At a given slip it is independent of the angular position of the machine relative to the voltage vector with respect to which the slip is considered. With ordinary (essentially semi-symmetrical) damper windings a correct treatment should take cognizance of the fact that damper action, even when varying linearly with the slip, is also a function of the *above-mentioned angular position of the machine.* This has already been discussed and formulas for the proportionality factor (or damping coefficient) given [equation (11), *et seq.*]. In handling any particular problem it is advisable not to lose sight of the limiting assumptions, also previously stated, on which these formulas are predicated. Evidently these assumptions have an important bearing on the accuracy with which the damping effect is reflected in a particular case.

In the following, damping will be considered for various systems. With the damping coefficients properly interpreted this discussion is equally applicable to non-salient- and salient-pole machines without reference to the question of symmetry of rotor circuits.

One Machine Connected to an Infinite Bus.—In this case it is obviously proper to consider the slip with respect to the infinite bus, since the synchronous output also is given by the angular displacement δ between the field structure of the machine and the voltage vector at the infinite bus. The slip is then $s = d\delta/dt$ and the general machine equation becomes

$$M \frac{d^2\delta}{dt^2} + P_d \frac{d\delta}{dt} + P_s(\delta) = P_i \tag{27}$$

---

power supplied to the equivalent circuit includes the negative-sequence terminal output as well as the stator copper loss and one-half the rotor copper loss due to the flow of negative-sequence currents. These items, therefore, representing a degeneration of corresponding amounts of positive-sequence power caused by circuit dissymmetry, are automatically taken into account by the use of the equivalent positive-sequence circuit, which is a convenient situation indeed. The other half of the negative-sequence rotor copper loss is supplied directly from the shaft, *i.e.*, by the prime mover. Hence this part may be subtracted from the input before calculating the power differentials (difference between the shaft input and the output to the equivalent circuit) which cause the machine to swing.

General Two-machine System.—Two synchronous machines considered as generators are connected by a general network as shown in Fig. 309. The network is assumed to be linear and may or may not include the effect of loads, the latter being considered as impedances. Since the synchronous output of either machine depends upon the angular displacement between them, it is convenient to consider the slip of one machine with respect to the other and calculate the damping coefficient accordingly. The appropriate internal voltage of the machine is then used for calculation of the latter. The general machine equations may be written

FIG. 309.—General two-machine system.

$$M_1 \frac{d^2\delta_1}{dt^2} + P_{d1} \frac{d(\delta_1 - \delta_2)}{dt} + P_{s1}(\delta_1 - \delta_2) = P_{i1} \tag{28}$$

$$M_2 \frac{d^2\delta_2}{dt^2} + P_{d2} \frac{d(\delta_2 - \delta_1)}{dt} + P_{s2}(\delta_2 - \delta_1) = P_{i2} \tag{29}$$

The individual angles δ_1 and δ_2 represent displacements with respect to a standard arbitrary reference axis. The synchronous outputs then depend upon the difference between these angles, and the general output expression may or may not involve a constant term. The damping power is proportional to the first derivative of the difference between these angles with respect to time, this derivative representing the relative slip of the two machines. Dividing equations (28) and (29) by the inertia constants M_1 and M_2, respectively, and introducing $\delta_1 - \delta_2 = \delta$, the following relationship results:

$$\frac{d^2\delta}{dt^2} + \left(\frac{P_{d1}}{M_1} + \frac{P_{d2}}{M_2}\right)\frac{d\delta}{dt} + \frac{P_{s1}}{M_1}(\delta) - \frac{P_{s2}}{M_2}(-\delta) = \frac{P_{i1}}{M_1} - \frac{P_{i2}}{M_2} \tag{30}$$

It is evident that by combining terms this equation immediately reduces to the form given by equation (27). Hence the general two-machine system including linear positive-sequence damping can be solved with the same ease as the system consisting of only one machine connected to an infinite bus.

Referring to the treatment of the general two-machine system in Chap. XIV, it is evident that equation (30) may also be written

$$M \frac{d^2\delta}{dt^2} + P_d \frac{d\delta}{dt} + P' = P'_i \tag{31}$$

in which

$$P_d = \frac{M_2 P_{d1} + M_1 P_{d2}}{M_1 + M_2} \tag{32}$$

The quantities P_i', P', and M, representing equivalent input, output, and inertia, are all defined by equations (14), (15), and (16) in Chap. XIV. Equation (31) may be the most convenient form for actual calculations.

The synchronous outputs of the machines in a general non-salient pole, two-machine system as indicated above are given by

$$P_1 = \frac{E_1^2}{Z_{11}} \sin \alpha_{11} + \frac{E_1 E_2}{Z_{12}} \sin (\delta_1 - \delta_2 - \alpha_{12}) = \frac{E_1^2}{Z_{11}} \sin \alpha_{11} + \frac{E_1 E_2}{Z_{12}} \sin (\delta - \alpha_{12}) \tag{33}$$

$$P_2 = \frac{E_2^2}{Z_{22}} \sin \alpha_{22} + \frac{E_1 E_2}{Z_{12}} \sin (\delta_1 - \delta_2 - \alpha_{12}) = \frac{E_2^2}{Z_{22}} \sin \alpha_{22} - \frac{E_1 E_2}{Z_{12}} \sin (\delta + \alpha_{12}) \tag{34}$$

With these output relations equation (31) assumes the following form:

$$M \frac{d^2\delta}{dt^2} + P_d \frac{d\delta}{dt} + P_C' - P_m' \cos (\delta + \psi) = P_i' \tag{35}$$

where the additional symbols P_C', P_m', and ψ, all relating to the equivalent power-angle curves, are given by equations (19), (20), and (21) in Chap. XIV.

If the two-machine system contains reactance only, the outputs of the two machines are equal and opposite and are given by

$$P_1 = -P_2 = \frac{E_1 E_2}{X_{12}} \sin \delta \tag{36}$$

In this case one of the machines obviously operates as a motor. Assuming further that the inputs (generator action assumed) all the time are equal and opposite ($P_{i1} = -P_{i2} = P_i$), the swing equation may be written

$$M \frac{d^2\delta}{dt^2} + P_d \frac{d\delta}{dt} + \frac{E_1 E_2}{X_{12}} \sin \delta = P_i \tag{37}$$

in which M is defined as before, and P_d by equation (32).

Multimachine Systems.—When the machines in a multimachine system oscillate, each machine, in general, has a distinct slip with respect to each of the other machines. It is assumed that the principle of superposition may be applied to the calculation of damping just as it applies to calculations of synchronous output and that the damping power of any one machine may be obtained to engineering accuracy by this process. On this basis the swing equation for the nth machine may be written as follows:

$$M_n \frac{d^2\delta_n}{dt^2} + P_{dn1}\frac{d(\delta_n - \delta_1)}{dt} + P_{dn2}\frac{d(\delta_n - \delta_2)}{dt} + P_{dn3}\frac{d(\delta_n - \delta_3)}{dt}$$
$$+ \cdot \cdot \cdot + P_n[(\delta_n - \delta_1), (\delta_n - \delta_2), (\delta_n - \delta_3), \cdot \cdot \cdot] = P_{in} \quad (38)$$

The angle δ_n is taken with respect to the common reference axis for all the machines. The synchronous output is a function of the difference between this angle and that of every other machine supplying power to the system; in other words, of all the relative angles. The damping power is considered proportional to the rate of change of each of these relative angles, or to the relative slips, a distinct damping coefficient, in general, applying to each slip. The damping coefficients may be evaluated by the formulas previously presented. Thus the coefficient for machine n to be associated with the slip of this machine with respect to machine 1 becomes

$$P_{dn1} = a_{n1} \sin^2 \delta_{n1} + b_{n1} \cos^2 \delta_{n1} \quad (39)$$

where

$$a_{n1} = \frac{e_1^2(X'_{dn} - X''_{dn})}{(X'_{dn} + X_{en1})(X''_{dn} + X_{en1})} \cdot \frac{X''_{dn} + X_{enn}}{X'_{dn} + X_{enn}} T''_{don} \quad (40)$$

$$b_{n1} = \frac{e_1^2(X_{qn} - X''_{qn})}{(X_{qn} + X_{en1})(X''_{qn} + X_{en1})} \cdot \frac{X''_{qn} + X_{enn}}{X'_{qn} + X_{enn}} T''_{qon} \quad (41)$$

In these

e_1 = internal voltage of machine 1
T''_{don}, T''_{qon} = open-circuit, subtransient time constants in the two axes, respectively, of machine n, expressed in electrical radians
X_{en1} = external transfer reactance between machine n and 1, including the internal reactance of the latter
X_{enn} = external driving-point reactance as viewed from the terminals of machine n

The comments previously made in this chapter (on page 656) regarding the significance and use of transfer reactances apply also here and should be noted.

The internal voltage in each of the other machines (such as e_1 above) should:

1. Strictly speaking be constant in magnitude, since the formulas for damping coefficients assume an infinite bus external to the machine in question.

2. Preferably be a direct-axis voltage (the vector actually coinciding with the quadrature axis) in order to line up with the use of the actual relative slip.

It is believed, however, that practically the internal voltage:

1. May be adjusted to allow for changes in flux linkages (as caused by exciter action, for instance) and that the effect of this, if desired, may be reflected in the damping calculations by adjusting the damping coefficient as the point-by-point analysis progresses.

2. Need not necessarily be a direct-axis voltage but may be taken, as often is the case,[1] as the voltage behind direct-axis transient reactance. In this case the slip to be used is that with respect to this voltage rather than the actual relative slip between the field structures of the two machines.

It will undoubtedly be appreciated from the above that whatever course is pursued, the inclusion of damping in a multimachine system (and for that matter also on a smaller scale in a two-machine system) involves several uncertainties and empiricisms. This is unavoidable, at least for the present, since the problem must be reduced to a basis where it becomes practicable and practical to handle. It is believed, however, that provided the limiting assumptions in any particular case are not too severely violated, the damping will be obtained to engineering accuracy.

Schedules of Point-by-point Analysis When Damping Is Considered.—Provision is made in Schedules D and E, presented in Chap. XIV, for taking into account the effect of damper windings. The former includes high-resistance damping, the latter high-resistance and low-resistance damping. Referring to the discussion of these schedules in Chap. XIV, it is believed that additional comments are necessary only on Schedule E.

[1] See Chap. XII.

The particular arrangement adopted in this schedule for the calculation of low-resistance damping applies to *generator* 1 in a *three-machine* system, the procedure being in accordance with equations (38) to (41), inclusive. With the swing calculations based on Method II, the relative (per unit) slips of the machine considered with respect to the other two are determined at the mid-point of each interval by means of differences of actual angular velocities above or below synchronous speed. Multiplying each relative slip by the corresponding damping-power coefficient gives the components of damping power, the algebraic sum of which constitutes the net low-resistance (positive-sequence) damping power for the machine under consideration. The accelerating or decelerating power differential is obtained by subtracting the synchronous output, as well as the high- and low-resistance damping power from the input. Note that this involves a departure from, and in fact a reversal of, conventional induction-machine conceptions under which motor action and slip behind the speed corresponding to impressed frequency are considered positive, but it is distinctly more logical and convenient in dealing with damping in synchronous machines (especially generators).

Since the damping-power coefficients depend on the relative displacement angles, they vary from interval to interval and are generally computed from the angles at the beginning of the interval in question. The constants [see equations (40) and (41)] entering into the expression for these coefficients depend on an internal voltage in the "other" machine. Should any of these internal voltages change as a result of flux decay or exciter action, this change, if deemed necessary, may be reflected in the corresponding constants. Being functions of the circuit parameters the above constants are also subject to readjustment at the instant of fault clearing or the occurrence of any other circuit discontinuity.

Of the quantities entering into the calculation of the power differential (in Method II considered constant from the mid-point of the previous interval to the mid-point of the interval in question), input, synchronous output, and high-resistance damping power are those at the beginning of an interval, while the low-resistance damping power is based on relative slips corresponding to average angular velocities during the interval for

which the swing calculation is made. These velocities are not known until the displacement angles at the end of the interval are determined. However, the problem is fundamentally one of successive approximations, the angular velocities involved being first assumed (or extrapolated from curves plotted as the analysis progresses). Having thus determined the angles, a better value of the velocities can be obtained and a recomputation made if required. Since the damping power as a rule is small, any adjustment is ordinarily of the second order and can be made without much inconvenience and additional effort.

With a two-machine system (actual or equivalent) certain simplifications are possible. Unless it is of specific interest to keep track of the absolute rotational velocities of the machines (or the "system drift") all point-by-point computations may be reduced to a relative basis corresponding, in fact, to considering the rotor position of one of the machines as standard phase. Under these conditions a single, appropriately modified schedule suffices. If, on the other hand, the system contains more than three machines necessitating further break up of the calculation of low-resistance damping in any one machine, Schedule *E* requires two more rows for this purpose for each additional machine.

Handling of Damping on Network Analyzer.—The great utility of a network analyzer as an adjunct in transient-stability calculations by the point-by-point method has been stressed already (particularly in Chap. XIV). A network analyzer being a static device permits correct measurement of power quantities which depend on voltage values, displacement angles, and circuit constants (including with the latter the effect of faults and discontinuities). It is inherently unable, however, to yield information on quantities which are also functions of relative motion, such as damping.

The consideration of negative-sequence (high-resistance) damping causes no difficulty on the analyzer. This is due to the fact that for all practical purposes (even when the machines actually oscillate) the negative-sequence slip is constant (equal to 2). Thus this kind of damping is essentially independent of any relative motion, the effect of the negative-sequence currents under dissymmetrical fault conditions being properly reflected in the positive-sequence circuit set up on the analyzer.[1]

[1] See footnote on p. 677.

For the positive-sequence (low-resistance) damping the situation is totally different. In the actual system this damping power, established by virtue of relative motion, actually flows between machines over the external network thus influencing the electrical quantities in magnitude as well as phase. On the network analyzer, on the other hand, the flow of damping power is not physically reproducible, *except by special means.*

Unless such special means are provided, inclusion of damping power in the point-by-point analysis (Schedule *E*) introduces it merely as an algebraic modification of the electric power flowing. This is undoubtedly sufficiently accurate in the majority of ordinary cases where the damping power is in the neighborhood of 2 or 3 per cent of the machine output. Should the damping power be of the order of 10 per cent or more, however, account should be taken of its physical presence also on the analyzer.

For a particular machine (phase shifter) this may be accomplished by introducing the damping power at the *air gap, i.e.,* back of armature leakage reactance, by means of an additional phase shifter connected at this point. The power factor of the damping power with respect to the air-gap voltage should be close to unity, and it will generally be sufficiently accurate to supply it at this power factor, although any other desired power factor can be used with equal facility. It is evident that handling the damping power in this manner on the network analyzer introduces a manipulative factor influencing the successive-approximation nature of the problem in addition to the computational one previously discussed in connection with Schedule *E*.

The driving-point and transfer reactances required for the determination of the damping coefficients can, of course, be directly measured on the analyzer. These measurements may be performed with all loads connected for both fault and fault-cleared conditions.

EXAMPLE 5

Statement of Problem

An hydroelectric station supplies power to a large metropolitan area over a long double-circuit transmission line, as shown in Fig. 310(*a*). The generators have high-resistance damper windings giving negligible damping torque at ordinary slip frequencies but a damping torque worth considering at negative-sequence slips.

Assume that the generators are delivering 80 per cent of rated power at unity power factor at the receiving end. The voltage at the receiving-end bus (which may be assumed infinite) has its normal (100 per cent) value. A single-phase, line-to-line short circuit occurs on the transmission line directly outside the sending-end, high-tension bus. Compute the power differential which will tend to accelerate the generators immediately after

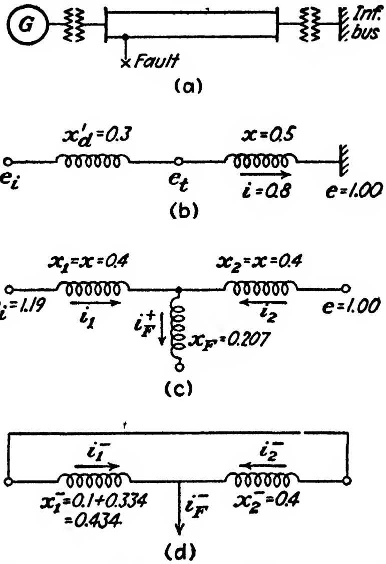


FIG. 310.—(*a*) Hydroelectric generating station supplying a large system (infinite bus) over a double-circuit transmission line. (*b*) Equivalent circuit of (*a*) under normal operating conditions. (*c*) Equivalent positive-sequence circuit of (*a*) with a single-phase line-to-line short circuit on one of the transmission lines near the sending-end high-tension bus. (*d*) Equivalent negative-sequence circuit of (*a*) for the same fault condition as in (*c*).

the short circuit occurs, including herein the effect of the high-resistance dampers.

Line and Transformer Data:

Line reactance = 0.3 (two circuits in parallel)

Transformer reactance = 0.1 (sending- and receiving-end banks equal)

Generator Data:

$$X_d = 0.78 \qquad X''_q = 0.407$$
$$X_q = 0.51 \qquad X^- = 0.334$$
$$X'_d = 0.30 \qquad X_a = 0.20$$
$$X''_d = 0.261 \qquad R^+ = 0.005$$
$$R^- = 0.15$$

All constants are in per unit based on the station rating.

Neglect resistances in the positive- and negative-sequence networks used for voltage, current, and angle computations. Ignore friction and windage losses and core loss in the generators. In the positive-sequence network represent the generators by their direct-axis transient reactance only, and consider the voltage behind this transient reactance to remain constant.

Solution

To determine the voltage behind transient reactance and the initial rotor angle, consider the system before the fault, Fig. 310(*b*). Taking the infinite-bus voltage as the reference,

$$\begin{aligned} e_i &= e + ji(X + X'_d) \\ &= 1.00 + j0.80(0.5 + 0.3) \\ &= 1.00 + j0.64 = 1.19\underline{/32.6°} \end{aligned} \qquad (a)$$

and

$$\delta = 32.6 \text{ deg.}$$

After the fault occurs, the equivalent circuit becomes that of Fig. 310(*c*). The reactance of the fault shunt X_F is the driving-point impedance of the negative-sequence network [Fig. 310 (*d*)], viewed from the point of fault and is

$$X_F = \frac{(0.334 + 0.1)0.4}{0.334 + 0.1 + 0.4} = 0.207$$

The transfer impedance between the infinite bus and the voltage e_i, during the fault [equal to the architrave reactance of the π-circuit equivalent to the T-circuit in Fig. 310(c)] is

$$X_t = 0.40 + 0.40 + \frac{0.40 \times 0.40}{0.207} = 1.572$$

Hence the synchronous power output immediately after fault is

$$\begin{aligned} P_s &= \frac{e_i e}{X_t} \sin \delta \\ &= \frac{1.19 \times 1.00}{1.572} \sin 32.6° \\ &= 0.408 \end{aligned} \qquad (b)$$

To compute the negative-sequence damping power, the negative-sequence generator current must first be found. From Fig. 310(*c*)

$$e_i = ji_1X + ji_F^{\ddagger}X_F \qquad (c)$$

$$e = ji_2X + ji_F^{\ddagger}X_F \qquad (d)$$

Addition of these two equations and use of the relation

$$i_1 + i_2 = i_F \qquad (e)$$

give

$$e_i + e = j(i_1 + i_2)X + j2i_F^{\ddagger}X_F = j(X + 2X_F)i_F^{\ddagger} \qquad (f)$$

or

$$i_F^+ = \frac{e_i + e}{j(X + 2X_F)} \tag{g}$$

This is the positive-sequence fault current. Assuming that the sequence analysis applies to the unfaulted phase, the negative-sequence fault current may be written [see equation (21), Chap. XI]

$$i_F^- = i_F^+ = \frac{e_i + e}{j(X + 2X_F)} \tag{h}$$

Referring to the diagram of the negative-sequence network [Fig. 310(*d*)], the portion of i_F^- contributed by the generator i_1^- is obviously

$$\begin{aligned} i_1^- &= i_F^- \frac{X_2^-}{X_1^- + X_2^-} \\ &= \left(\frac{X_2^-}{X_1^- + X_2^-}\right)\frac{e_i + e}{X + 2X_F}\diagdown\!\underline{90^\circ} \end{aligned} \tag{i}$$

Substituting numerical values,

$$i_1^- = \left(\frac{0.40}{0.40 + 0.434}\right)\frac{1.19\underline{/32.6^\circ} + 1.00}{0.4 + 2 \times 0.207}\diagdown\!\underline{90^\circ} = 1.24\diagdown\!\underline{72.3^\circ}$$

From equations (17) and (23) the negative-sequence damping power is

$$\begin{aligned} P_2^- &= P_{\text{rated}}\,(I^-)^2(R^- - R^+) \\ &= (1.24)^2(0.150 - 0.005) \\ &= 0.223 \text{ per unit} \end{aligned} \tag{j}$$

The initial accelerating power differential is, therefore,

$$\begin{aligned} \Delta P &= P_i - P_{\text{output}} = P_i - P_s - P_2^- \\ &= 0.800 - 0.408 - 0.223 \\ &= 0.169 \text{ per unit} \end{aligned}$$

The initial power differential without negative-sequence damping included would be $0.800 - 0.408 = 0.392$, or more than twice the above value.

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